Topic A

Real Numbers

N-RN.A.1, N-RN.A.2, N-Q.A.2, F-IF.B.6, F-BF.A.1a, F-LE.A.2

Focus Standards:	N-RN.A.1	Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents. *For example, we define* $5^{\frac{1}{3}}$ *to be the cube root of* 5 *because we want* $\left(5^{\frac{1}{3}}\right)^3 = 5^{\left(\frac{1}{3}\right)^3}$ *to hold, so* $\left(5^{\frac{1}{3}}\right)^3$ *must equal* 5.
	N-RN.A.2	Rewrite expressions involving radicals and rational exponents using the properties of exponents.
	N-Q.A.2	Define appropriate quantities for the purpose of descriptive modeling.★
	F-IF.6	Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.★
	F-BF.A.1a	Write a function that describes a relationship between two quantities.★ Determine an explicit expression, a recursive process, or steps for calculation from a context
	F-LE.A.2	Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table).★
Instructional Days:	6	
Lesson 1:	Integer Exponents (E)[1]	
Lesson 2:	Base 10 and Scientific Notation (P)	
Lesson 3:	Rational Exponents—What are $2^{\frac{1}{2}}$ and $2^{\frac{1}{3}}$? (S)	
Lesson 4:	Properties of Exponents and Radicals (P)	
Lesson 5:	Irrational Exponents—What are $2^{\sqrt{2}}$ and 2^{π}? (S)	
Lesson 6:	Euler's Number, *e* (P)	

[1]Lesson Structure Key: **P**-Problem Set Lesson, **M**-Modeling Cycle Lesson, **E**-Exploration Lesson, **S**-Socratic Lesson

In Topic A, students prepare to generalize what they know about various function families by examining the behavior of exponential functions. One goal of the module is to show that the domain of the exponential function, $f(x) = b^x$, where b is a positive number not equal to 1, is all real numbers. In Lesson 1, students review and practice applying the laws of exponents to expressions in which the exponents are integers. Students first tackle a challenge problem on paper folding that is related to exponential growth and then apply and practice applying the laws of exponents to rewriting algebraic expressions. They experiment, create a table of values, observe patterns, and then generalize a formula to represent different measurements in the folded stack of paper as specified in **F-LE.A.2**. They also use the laws of exponents to work with very large and very small numbers.

Lesson 2 sets the stage for the introduction of base-10 logarithms in Topic B of the module by reviewing how to express numbers using scientific notation, how to compute using scientific notation, and how to use the laws of exponents to simplify those computations in accordance with **N-RN.A.2**. Students should gain a sense of the change in magnitude when different powers of 10 are compared. The activities in these lessons prepare students for working with quantities that increase in magnitude by powers of 10 and by showing them the usefulness of exponent properties when performing arithmetic operations. Similar work is done in later lessons relating to logarithms. Exercises on distances between planets in the solar system and on comparing magnitudes in other real-world contexts provide additional practice with arithmetic operations on numbers written using scientific notation.

Lesson 3 begins with students examining the graph of $y = 2^x$ and estimating values as a means of extending their understanding of integer exponents to rational exponents. The examples are generalized to $2^{\frac{1}{n}}$ before generalizing further to $2^{\frac{m}{n}}$. As the domain of the identities involving exponents is expanded, it is important to maintain consistency with the properties already developed. Students work specifically to make sense that $2^{\frac{1}{2}} = \sqrt{2}$ and $2^{\frac{1}{3}} = \sqrt[3]{2}$ to develop the more general concept that $2^{\frac{1}{n}} = \sqrt[n]{2}$. The lesson demonstrates how people develop mathematics (1) to be consistent with what is already known and (2) to make additional progress. Additionally, students practice MP.7 as they extend the rules for integer exponents to rules for rational exponents (**N-RN.A.1**).

Lesson 4 continues the discussion of properties of exponents and radicals, and students continue to practice MP.7 as they extend their understanding of exponents to all rational numbers and for all positive real bases as specified in **N-RN.A.1**. Students rewrite expressions involving radicals and rational exponents using the properties of exponents (**N-RN.A.2**). The notation $x^{\frac{1}{n}}$ specifically indicates the principal root of x: the positive root when n is even and the real-valued root when n is odd. To avoid inconsistencies in the later work with logarithms, x is required to be positive.

Lesson 5 revisits the work of Lesson 3 and extends student understanding of the domain of the exponential function $f(x) = b^x$, where b is a positive real number, from the rational numbers to all real numbers through the process of considering what it means to raise a number to an irrational exponent (such as $2^{\sqrt{2}}$). In many ways, this lesson parallels the work students did in Lesson 3 to make a solid case for why the laws of exponents hold for all rational number exponents. The recursive procedure that students employ in this lesson aligns with **F-BF.A.1a**. This lesson is important both because it helps to portray mathematics as a coherent body of knowledge that makes sense and because it is necessary to make sure that students understand that logarithms can be irrational numbers. Essentially, it is necessary to guarantee that exponential and logarithmic functions are continuous functions. Students take away from these lessons an understanding that the domain of exponents in the laws of exponents does indeed extend to all real numbers rather than just to the integers, as defined previously in Grade 8.

Lesson 6 is a modeling lesson in which students practice MP.4 when they find an exponential function to model the amount of water in a tank after t seconds when the height of the water is constantly doubling or tripling and apply **F-IF.B.6** as they explore the average rate of change of the height of the water over smaller and smaller intervals. If the height of the water in the tank at time t seconds is denoted by $H(t) = b^t$, then the average rate of change of the height of the water on an interval $[T, T+\varepsilon]$ is approximated by $\frac{H(T+\varepsilon)-H(t)}{\varepsilon} \approx c \cdot H(T)$. Students calculate that if the height of the water is doubling each second, then $c \approx 0.69$, and if the height of the water is tripling each second, then $c \approx 1.1$. Students discover Euler's number, e, by applying repeated reasoning (MP.8) and numerically approximating the base b for which the constant c is equal to 1. Euler's number is used extensively in the future and occurs in many different applications.

Topic B

Logarithms

N-Q.A.2, A-CED.A.1, F-BF.A.1a, F-LE.A.4

Focus Standards:	N-Q.A.2	Define appropriate quantities for the purpose of descriptive modeling.★
	A-CED.A.1	Create equations and inequalities in one variable and use them to solve problems. Include equations arising from linear and quadratic functions, and simple rational and exponential functions.★
	F-BF.A.1a	Write a function that describes a relationship between two quantities.★ a. Determine an explicit expression, a recursive process, or steps for calculation from a context.
	F-LE.A.4	For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology.★
Instructional Days:	9	
Lesson 7:	Bacteria and Exponential Growth (S)[1]	
Lesson 8:	The "WhatPower" Function (P)	
Lesson 9:	Logarithms—How Many Digits Do You Need? (E)	
Lesson 10:	Building Logarithmic Tables (P)	
Lesson 11:	The Most Important Property of Logarithms (P)	
Lesson 12:	Properties of Logarithms (P)	
Lesson 13:	Changing the Base (P)	
Lesson 14:	Solving Logarithmic Equations (P)	
Lesson 15:	Why Were Logarithms Developed? (P)	

The lessons covered in Topic A familiarize students with the laws and properties of real-valued exponents. In Topic B, students extend their work with exponential functions to include solving exponential equations numerically and developing an understanding of the relationship between logarithms and exponentials. In Lesson 7, students use an algorithmic numerical approach to solve simple exponential equations that arise from modeling the growth of bacteria and other populations (**F-BF.A.1a**). Students work to develop

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progressively better approximations for the solutions to equations whose solutions are irrational numbers. In doing this, students increase their understanding of the real number system and truly begin to understand what it means for a number to be irrational. Students learn that some simple exponential equations can be solved exactly without much difficulty but that mathematical tools are lacking to solve other equations whose solutions must be approximated numerically.

Lesson 8 begins with the logarithmic function disguised as the more intuitive "WhatPower" function, whose behavior is studied as a means of introducing how the function works and what it does to expressions. Students find the power needed to raise a base b in order to produce a given number. The lesson ends with students defining the term *logarithm base* b. Lesson 8 is just a first introduction to logarithms in preparation for solving exponential equations per **F-LE.A.4**; students neither use tables nor look at graphs in this lesson. Instead, they simply develop the ideas and notation of logarithmic expressions, leaving many ideas to be explored later in the module.

Just as population growth is a natural example that gives context to exponential growth, Lesson 9 gives context to logarithmic calculation through the example of assigning unique identification numbers to a group of people. In this lesson, students consider the meaning of the logarithm in the context of calculating the number of digits needed to create student ID numbers, phone numbers, and social security numbers, in accordance with **N-Q.A.2**. This gives students a real-world context for the abstract idea of a logarithm; in particular, students observe that a base-10 logarithm provides a way to keep track of the number of digits used in a number in the base-10 system.

Lessons 10–15 develop both the theory of logarithms and procedures for solving various forms of exponential and logarithmic equations. In Lessons 10 and 11, students discover the logarithmic properties by completing carefully structured logarithmic tables and answering sets of directed questions. Throughout these two lessons, students look for structure in the table and use that structure to extract logarithmic properties (MP.7). Using the structure of the logarithmic expression together with the logarithmic properties to rewrite an expression aligns with the foundational standard **A-SSE.A.2**. While the logarithmic properties are not themselves explicitly listed in the standards, standard **F-LE.A.4** cannot be adequately met without an understanding of how to apply logarithms to solve exponential equations, and the seemingly odd behavior of graphs of logarithmic functions (**F-IF.C.7e**) cannot be adequately explained without an understanding of the properties of logarithms. In particular, in Lesson 11, students discover the "most important property of logarithms": For positive real numbers x and y, $\log(xy) = \log(x) + \log(y)$. Students also discover the pattern $\log_b\left(\frac{1}{x}\right) = -\log_b(x)$ that leads to conjectures about additional properties of logarithms.

Lesson 12 continues the consideration of properties of the logarithm function, while remaining focused solely on base-10 logarithms. Its centerpiece is the demonstration of basic properties of logarithms such as the power, product, and quotient properties, which allows students to practice MP.3 and **A-SSE.A.2**, providing justification in terms of the definition of logarithm and the properties already developed. In this lesson, students begin to learn how to solve exponential equations, beginning with base-10 exponential equations that can be solved by taking the common logarithm of both sides of the equation.

Lesson 13 again focuses on the structure of expressions (**A-SSE.A.2**), as students change logarithms from one base to another. It begins by showing students how they can make that change and then develops properties of logarithms for the general base b. Students are introduced to the use of a calculator instead of a table in finding logarithms, and then *natural logarithms* are defined: $\ln(x) = \log_e(x)$. One goal of the lesson, in addition to introducing the base e for logarithms, is to explain why, for finding logarithms to any base, the calculator has only LOG and LN keys. In this lesson, students learn to solve exponential equations with any base by the application of an appropriate logarithm. Lessons 12 and 13 both address **F-LE.A.4**, solving equations of the form $ab^{ct} = d$, as do later lessons in the module.

Lesson 14 includes the first introduction to solving logarithmic equations. In this lesson, students apply the definition of the logarithm to rewrite logarithmic equations in exponential form, so the equations must first be rewritten in the form $\log_b(X) = c$, for an algebraic expression X and some constant c. Solving equations in this way requires that students think deeply about the definition of the logarithm and how logarithms interact with exponential expressions. Although solving logarithmic equations is not listed explicitly in the standards, this skill is implicit in standard **A-REI.D.11**, which has students solve equations of the form $f(x) = g(x)$ where f and g can be logarithmic functions. Additionally, logarithmic equations provide a greater context in which to study both the properties of logarithms and the definition, both of which are needed to solve the equations listed in **F-LE.A.4**.

Topic B concludes with Lesson 15, in which students learn a bit of the history of how and why logarithms first appeared. The materials for this lesson contain a base-10 logarithm table. Although modern technology has made logarithm tables functionally obsolete, there is still value in understanding the historical development of logarithms. Logarithms were critical to the development of astronomy and navigation in the days before computing machines, and this lesson presents a rationale for the pre-technological advantage afforded to scholars by the use of logarithms. In this lesson, the case is finally made that logarithm functions are one-to-one (without explicitly using that terminology): If $\log_b(X) = \log_b(Y)$, then $X = Y$. In alignment with **A-SSE.A.2**, this fact not only validates the use of tables to look up anti-logarithms but also allows exponential equations to be solved with logarithms on both sides of the equation.

Topic C

Exponential and Logarithmic Functions and their Graphs

F-IF.B.4, F-IF.B.5, F-IF.C.7e, F-BF.A.1a, F-BF.B.3, F-BF.B.4a, F-LE.A.2, F-LE.A.4

Focus Standards:	F-IF.B.4	For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.★
	F-IF.B.5	Relate the domain of a function to its graph and, where applicable, to the quantitative relationship is describes. For example, if the function $h(n)$ gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.★
	F-IF.C.7e	Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★ Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.
	F-BF.A.1a	Write a function that describes a relationship between two quantities.★ Determine an explicit expression, a recursive process, or steps for calculation from a context.
	F-BF.B.3	Identify the effect on the graph of replacing $f(x)$ by $f(x)+k, k\,f(x)$, $f(kx)$, and $f(x+k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.
	F-BF.B.4a	Find inverse functions. Solve an equation of the form $f(x)=c$ for a simple function f that has an inverse and write an expression for the inverse. For example, $f(x)=2x^3$ or $f(x)=(x+1)/(x-1)$ for $x \neq 1$.
	F-LE.A.2	Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table).

	F-LE.A.4	For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where a, c, and d are numbers, and the base b is 2, 10, or e; evaluate the logarithm using technology.
Instructional Days:	7	
Lesson 16:	Rational and Irrational Numbers (S)[1]	
Lesson 17:	Graphing the Logarithm Function (P)	
Lesson 18:	Graphs of Exponential Functions and Logarithmic Functions (P)	
Lesson 19:	The Inverse Relationship between Logarithmic and Exponential Functions (P)	
Lesson 20:	Transformations of the Graphs of Logarithmic and Exponential Functions (E)	
Lesson 21:	The Graph of the Natural Logarithm Function (E)	
Lesson 22:	Choosing a Model (P)	

The lessons covered in Topic A and Topic B build upon students' prior knowledge of the properties of exponents, exponential expression, and solving equations by extending the properties of exponents to all real number exponents and positive real number bases before introducing logarithms. This topic reintroduces exponential functions, introduces logarithmic functions, explains their inverse relationship, and explores the features of their graphs and how they can be used to model data.

Lesson 16 ties back to work in Topic A by helping students to further extend their understanding of the properties of real numbers, both rational and irrational (**N-RN.B.3**). This Algebra I standard is revisited in Algebra II so that students know and understand that the exponential functions are defined for all real numbers, and, thus, the graphs of the exponential functions can be represented by a smooth curve. Another consequence is that the logarithm functions are also defined for all positive real numbers. Lessons 17 and 18 introduce the graphs of logarithmic functions and exponential functions. Students compare the properties of graphs of logarithm functions for different bases and identify common features, which align with standards **F-IF.B.4**, **F-IF.B.5**, and **F-IF.C.7**. Students understand that because the range of this function is all real numbers, then some logarithms must be irrational. Students notice that the graphs of $f(x) = b^x$ and $g(x) = \log_b(x)$ appear to be related via a reflection across the graph of the equation $y = x$.

Lesson 19 addresses standards **F-BF.B.4a** and **F-LE.A.4** while continuing the ideas introduced graphically in Lesson 18 to help students make the connection that the logarithmic function base b and the exponential function base b are inverses of each other. Inverses are introduced first by discussing operations and functions that can "undo" each other; then, students look at the graphs of pairs of these functions. The lesson ties the ideas back to reflections in the plane from Geometry and illuminates why the graphs of inverse functions are reflections of each other across the line given by $y = x$, developing these ideas intuitively without formalizing what it means for two functions to be inverses. Inverse functions will be addressed in greater detail in Precalculus.

During all of these lessons, connections are made to the properties of logarithms and exponents. The relationship between graphs of these functions, the process of sketching a graph by transforming a parent function, and the properties associated with these functions are linked in Lessons 20 and 21, showcasing standards **F-IF.C.7e** and **F-BF.B.3**. Students use properties and their knowledge of transformations to explain why two seemingly different functions such as $f(x) = \log(10x)$ and $g(x) = 1 + \log(x)$ have the same

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graph. Lesson 21 revisits the natural logarithm function, and students see how the change of base property of logarithms implies that we can write a logarithm function of any base b as a vertical scaling of the natural logarithm function (or any other base logarithm function we choose).

Finally, in Lesson 22, students must synthesize knowledge across both Algebra I and Algebra II to decide whether a linear, quadratic, sinusoidal, or exponential function will best model a real-world scenario by analyzing the way in which we expect the quantity in question to change. For example, students need to determine whether or not to model daylight hours in Oslo, Norway, with a linear or a sinusoidal function because the data appears to be linear, but, in context, the choice is clear. They model the outbreak of a flu epidemic with an exponential function and a falling body with a quadratic function. In this lesson, the majority of the scenarios that require modeling are described verbally, and students determine an explicit expression for many of the functions in accordance with **F-BF.A.1a**, **F-LE.A.1**, and **F-LE.A.2**.

Topic D

Using Logarithms in Modeling Situations

A-SSE.B.3c, A-CED.A.1, A-REI.D.11, F-IF.B.3, F-IF.B.6, F-IF.C.8b, F-IF.C.9, F-BF.A.1a, F-BF.1b, F-BF.A.2, F-BF.B.4a, F-LE.A.4, F-LE.B.5

Focus Standards:	A-SSE.B.3c	Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression. c. Use the properties of exponents to transform expressions for exponential functions. *For example the expression* 1.15^t *can be rewritten as* $\left(1.15^{\frac{1}{12}}\right)^{12t} \approx 1.012^{12t}$ *to reveal the approximate equivalent monthly interest rate if the annual rate is* 15%.
	A-CED.A.1	Create equations and inequalities in one variable and use them to solve problems. Include equations arising from linear and quadratic functions, and simple rational and exponential functions.
	A-REI.D.11	Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.★
	F-IF.B.3	Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers. *For example, the Fibonacci sequence is defined recursively by* $f(0) = f(1) = 1$, $f(n+1) = f(n) + f(n-1)$ *for* $n \geq 1$.
	F-IF.B.6	Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.★
	F-IF.C.8b	Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function. b. Use the properties of exponents to interpret expressions for exponential functions. *For example, identify percent rate of change in functions such as* $y = (1.02)^t$, $y = (0.97)^t$, $y = (1.01)^{12t}$, $y = (1.2)^{\frac{t}{10}}$, *and classify them as representing exponential growth or decay.*
	F-IF.C.9	Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.*

	F-BF.A.1	Write a function that describes a relationship between two quantities.★ a. Determine an explicit expression, a recursive process, or steps for calculation from a context. b. Combine standard function types using arithmetic operations. *For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.*
	F-BF.A.2	Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms.★
	F-BF.B.4a	Find inverse functions. a. Solve an equation of the form $f(x) = c$ for a simple function f that has an inverse and write an expression for the inverse. *For example,* $f(x) = 2x^3$ *or* $f(x) = (x+1)/(x-1)$ *for* $x \neq 1$.
	F-LE.A.4	For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology.
	F-LE.B.5	Interpret the parameters in a linear or exponential function in terms of a context.
Instructional Days:	6	
Lesson 23:	Bean Counting (M)[1]	
Lesson 24:	Solving Exponential Equations (P)	
Lesson 25:	Geometric Sequences and Exponential Growth and Decay (P)	
Lesson 26:	Percent Rate of Change (P)	
Lesson 27:	Modeling with Exponential Functions (M)	
Lesson 28:	Newton's Law of Cooling, Revisited (M)	

This topic opens with a simulation and modeling activity where students start with one bean, roll it out of a cup onto the table, and add more beans each time the marked side is up. The lesson unfolds by having students discover an exponential relationship by examining patterns when the data is presented numerically and graphically. Students blend what they know about probability and exponential functions to interpret the parameters a and b in the functions $f(t) = a(b^t)$ that they find to model their experimental data (**F-LE.B.5**, **A-CED.A.2**).

In both Algebra I and Lesson 6 in this module, students had to solve exponential equations when modeling real-world situations numerically or graphically. Lesson 24 shows students how to use logarithms to solve these types of equations analytically and makes the connections between numeric, graphical, and analytical approaches explicit, invoking the related standards **F-LE.A.4**, **F-BF.B.4a**, and **A-REI.D.11**. Students are encouraged to use multiple approaches to solve equations generated in the next several lessons.

In Lessons 25 to 27, a general growth/decay rate formula is presented to students to help construct models from data and descriptions of situations. Students must use properties of exponents to rewrite exponential expressions in order to interpret the properties of the function (**F-IF.C.8b**). For example, in Lesson 27, students compare the initial populations and annual growth rates of population functions given in the forms

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$E(t) = 281.4(1.0093)^{t-100}$, $f(t) = 81.1(1.0126)^{t}$, and $g(t) = 76.2(13.6)^{t/10}$. Many of the situations and problems presented here were first encountered in Module 3 of Algebra I; students are now able to solve equations involving exponents that they could only estimate previously, such as finding the time when the population of the United States is expected to surpass a half-billion people. Students answer application questions in the context of the situation and use technology to evaluate logarithms of base 10 and e. Additionally, Lesson 25 begins to develop geometric sequences that are needed for the financial content in the next topic (**F-BF.A.2**). Lesson 26 continues developing the skills of distinguishing between situations that require exponential or linear models (**F-LE.A.1**), and Lesson 27 continues the work with geometric sequences that started in Lesson 25 (**F-IF.B.3**, **F-BF.A.1a**).

Lesson 28 closes this topic and addresses **F-BF.A.1b** by revisiting Newton's law of cooling, a formula that involves the sum of an exponential function and a constant function. Students first learned about this formula in Algebra I, but now that they are armed with logarithms and have more experience understanding how transformations affect the graph of a function, they can find the precise value of the decay constant using logarithms and, thus, can solve problems related to this formula more precisely and with greater depth of understanding.

Topic E

Geometric Series and Finance

A-SSE.B.4, F-IF.C.7e, F-IF.C.8b, F-IF.C.9, F-BF.A.1b, F-BF.A.2, F-LE.B.5

Focus Standards:	A-SSE.B.4	Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems. *For example, calculate mortgage payments*.
	F-IF.C.7e	Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★ e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.
	F-IF.C.8b	Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function. b. Use the properties of exponents to interpret expressions for exponential functions. *For example, identify percent rate of change in functions such as* $y = (1.02)^t$, $y = (0.97)^t$, $y = (1.01)^{12t}$, $y = (1.2)^{\frac{t}{10}}$, *and classify them as representing exponential growth or decay.*
	F-IF.C.9	Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.*
	F-BF.A.1b	Write a function that describes a relationship between two quantities.★ b. Combine standard function types using arithmetic operations. *For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.*
	F-BF.A.2	Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms.★
	F-LE.B.5	Interpret the parameters in a linear or exponential function in terms of a context.
Instructional Days:	5	
Lesson 29:	The Mathematics Behind a Structured Savings Plan (M)[1]	
Lesson 30:	Buying a Car (M)	

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Topic E is a culminating series of lessons driven by MP.4, Modeling with Mathematics. Students apply what they have learned about mathematical models and exponential growth to financial literacy, while developing and practicing the formula for the sum of a finite geometric series. Lesson 29 develops the future value formula for a structured savings plan and, in the process, develops the formula for the sum of a finite geometric series (**A-SSE.B.4**). The summation symbol, Σ, is introduced in this lesson.

Lesson 30 introduces loans through the context of purchasing a car. To develop the formula for the present value of an annuity, students combine two formulas for the future value of the annuity (**F-BF.A.1b**) and apply the sum of a finite geometric series formula. Throughout the remaining lessons, various forms of the present value of an annuity formula are used to calculate monthly payments and loan balances. The comparison of the effects of various interest rates and repayment schedules requires that students translate between symbolic and numerical representations of functions (**F-IF.C.9**). Lesson 31 addresses the issue of revolving credit such as credit cards, for which the borrower can choose how much of the debt to pay each cycle. Students again sum a geometric series to develop a formula for this scenario, and it turns out to be equivalent to the formula used for car loans. Key features of tables and graphs are used to answer questions about finances (**F-IF.C.7e**).

Lessons 32 and 33 are modeling lessons in which students apply what they have learned in earlier lessons to new financial situations (MP.4). Lesson 32 may be extended to an open-ended project in which students research buying a home and justify its affordability. Lesson 33, the final lesson of the module, is primarily a summative lesson in which students formulate a plan to have $\$1{,}000{,}000$ in assets within a fixed time frame, using the formulas developed in the prior lessons in the topic. Students graph the present value function and compare that with an amortization table, in accordance with **F-IF.C.9**. In both of these lessons, students need to combine functions using standard arithmetic operations (**F-IF.A.1b**).

Lesson 1: Integer Exponents

Student Outcomes

- Students review and practice applying the properties of exponents for integer exponents.
- Students model a real-world scenario involving exponential growth and decay.

Lesson Notes

To fully understand exponential functions and their use in modeling real-world situations, students must be able to extend the properties of integer exponents to rational and real numbers. In previous grades, students established the properties of exponents for integer exponents and worked with radical expressions and irrational numbers such as $\sqrt{2}$.

In this module, the properties of exponents are used to show that for any positive real number b, the domain of the exponential function $f(x) = b^x$ is all real numbers. In Algebra I, students primarily worked with exponential functions where the domain was limited to a set of integers. In the latter part of this module, students are introduced to logarithms, which allow them to find real number solutions to exponential equations. Students come to understand how logarithms simplify computation, particularly with different measuring scales.

Much of the work in this module relies on the ability to reason quantitatively and abstractly (MP.2), to make use of structure (MP.7), and to model with mathematics (MP.4). Lesson 1 begins with a challenge problem where students are asked to fold a piece of paper in half 10 times and construct exponential functions based on their experience (**F-LE.A.2**). It is physically impossible to fold a sheet of notebook paper in half more than seven or eight times; the difficulty lies in the thickness of the paper compared to the resulting area when the paper is folded. To fold a piece of paper in half more than seven or eight times requires either a larger piece of paper, a very thin piece of paper, or a different folding scheme, such as accordion folding. In 2001, a high school student, Britney Gallivan, successfully folded a very large piece of paper in half 12 times and derived a mathematical formula to determine how large a piece of paper would be required to successfully accomplish this task (http://pomonahistorical.org/12times.htm). Others have tried the problem as well, including the hosts of the television show "MythBusters" (https://www.youtube.com/watch?v=kRAEBbotuIE) and students at St. Mark's High School in Massachusetts (http://www.newscientist.com/blogs/nstv/2012/01/paper-folding-limits-pushed.html). Consider sharing one of these resources with the class at the close of the lesson or after they have completed the Exploratory Challenge. After this challenge, which reintroduces students to exponential growth and decay, students review the properties of exponents for integer exponents and apply them to the rewriting of algebraic expressions (**N-RN.A.2**). The lesson concludes with fluency practice where students apply properties of exponents to rewrite expressions in a specified form.

Consider having the following materials on hand in case students want to explore this problem in more detail: access to the Internet, chart paper, cash register tape, a roll of toilet paper, origami paper, tissue paper or facial tissues, and rulers.

Classwork

Students begin this lesson by predicting whether they can fold a piece of paper in half 10 times, how tall the folded paper will be, and whether or not the area of paper showing on top is smaller or larger than a postage stamp. They explore the validity of their predictions in the Exploratory Challenge that follows.

Opening Exercise (3 minutes)

Give students a short amount of time to think about and write individual responses. Lead a short discussion with the entire class to poll students on their responses. Record solutions on the board or chart paper for later reference. At this point, most students will probably say that they *can* fold the paper in half 10 times. The sample responses shown below are *not* correct and represent possible initial student responses. Some students may be familiar with this challenge, having seen it discussed on a television program or on the Internet and, consequently, say that a piece of notebook paper cannot be folded in half 10 times. Accept all responses, and avoid excessive explanation or justification of answers at this point.

Scaffolding:

- Demonstrate folding the paper once or twice during the discussion to illustrate what the questions are asking and what is meant by *how thick* the folded paper will be.
- Ask advanced learners to provide justification for their claims about the thickness of the folded paper.

Opening Exercise

Can you fold a piece of notebook paper in half 10 times?

Answers will vary. Although incorrect, many students may initially answer "Yes."

How thick will the folded paper be?

Answers will vary. The following is a typical student guess: It will be about 1 cm.

Will the area of the paper on the top of the folded stack be larger or smaller than a postage stamp?

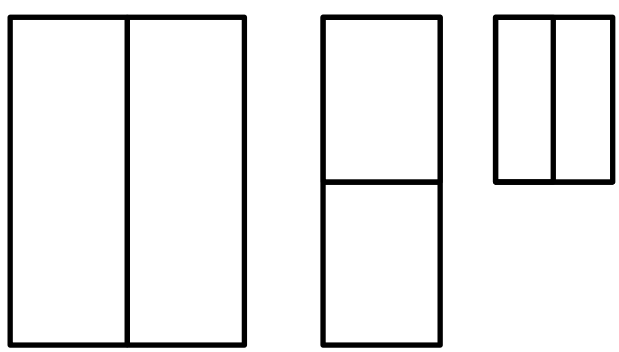
It will be smaller because I will be dividing the rectangle in half 10 ***times, and since a piece of paper is about*** 8.5 in. ***by*** 11 in.***, it will be very small when divided in half that many times.***

Discussion (2 minutes)

MP.1

Students should brainstorm ideas for further exploring these questions to come up with a more precise answer to these questions. At this point, some students are likely folding a piece of notebook paper. On chart paper, record ideas for additional information needed to answer the original questions more precisely.

- How can you be sure of your answers?
  - *We could actually fold a piece of paper and measure the height and area on top of the stack. We could determine the thickness of a sheet of notebook paper and then multiply it by the number of folds. We could find the area of the original paper and divide it by* 2 *successively for each fold.*



MP.1

- What additional information is needed to solve this problem?
  - *We need to know the thickness of the paper, the dimensions of the original piece of paper, and a consistent way to fold the paper.*
  - *We need to know the size of a postage stamp.*
- How will you organize your work?
  - *We can make a table.*

Exploratory Challenge (20 minutes)

Students should work on this challenge problem in small groups. Groups can use the suggested scaffolding questions on the student pages, or simply give each group a piece of chart paper and access to appropriate tools such as a ruler, different types of paper, a computer for researching the thickness of a sheet of paper, etc., and start them on the task. Have them report their results on their chart paper. Student solutions will vary from group to group. Sample responses have been provided below for a standard 8.5 in. by 11 in. piece of paper that has been folded in half as shown below.

The size of a small postage stamp is $\frac{7}{8}$ in. by 1 in.

Exploratory Challenge

a. What are the dimensions of your paper?

The dimensions are 8.5 in. by 11 in.

b. How thick is one sheet of paper? Explain how you decided on your answer.

A ream of paper is 500 sheets. It is about 2 in. high. Dividing 2 by 500 would give a thickness of a piece of paper to be approximately 0.004 in.

c. Describe how you folded the paper.

First, we folded the paper in half so that it was 8.5 in. by 5.5 in.; then, we rotated the paper and folded it again so that it was 5.5 in. by 4.25 in.; then, we rotated the paper and folded it again, and so on.

d. Record data in the following table based on the size and thickness of your paper.

Number of Folds	0	1	2	3	4
Thickness of the Stack (in.)	0.004	0.008	0.016	0.032	0.064
Area of the Top of the Stack (sq. in.)	93.5	46.75	23.375	11.6875	5.84375

Number of Folds	5	6	7	8	9	10
Thickness of the Stack (in.)	0.128	0.256	0.512	1.024	2.048	4.096
Area of the Top of the Stack (sq. in.)	2.921875	1.461	0.730	0.365	0.183	0.091

Answers are rounded to three decimal places after the fifth fold.

e. Were you able to fold a piece of notebook paper in half 10 times? Why or why not?

No. It got too small and too thick for us to continue folding it.

Debrief after part (e) by having groups present their solutions so far. At this point, students should realize that it is impossible to fold a sheet of notebook paper in half 10 times. If groups wish to try folding a larger piece of paper, such as a piece of chart paper, or a different thickness of paper, such as a facial tissue, or using a different folding technique, such as an accordion fold, then allow them to alter their exploration. Consider having tissue paper, facial tissues, a roll of toilet paper, chart paper, or cash register tape on hand for student groups to use in their experiments.

After students have made adjustments to their models and tested them, have them write formulas to predict the height and area after 10 folds and explain how these answers compare to their original predictions. Students worked with exponential functions and geometric sequences in Algebra I. Since this situation involves doubling or halving, most groups should be able to write a formula. When debriefing this next section with the entire class, help students to write a well-defined formula. Did they specify the meaning of their variables? Did they specify a domain if they used function notation? They may not have used the same variables shown in the solutions below and should be using specific values for the thickness and area of the paper based on their assumptions during modeling. Students are likely surprised by these results.

Scaffolding:

If students are struggling to develop the formulas in their groups, complete the rest of this challenge as a whole class. Students have many opportunities to model using exponential functions later in this module. Write the height and thickness as products of repeated twos to help students see the pattern. For example, after three folds, the height would be $T \cdot 2 \cdot 2 \cdot 2$, where T is the thickness of the paper.

- How thick would the stack be if you could fold it 10 times?
- Is the area of the top of the stack smaller or larger than a postage stamp?
- How do these answers compare to your prediction?

MP.4

f. **Create a formula that approximates the height of the stack after n folds.**

Our formula is $H(n) = T \cdot 2^n$, where T is the thickness of the paper and $H(n)$ is the height after n folds. In this case, $T = 0.004$ in.

g. **Create a formula that will give you the approximate area of the top after n folds.**

Our formula is $A(n) = A_0\left(\frac{1}{2}\right)^n$, where A_0 is the area of the original piece of paper and $A(n)$ is the area of the top after n folds. In this case, $A_0 = 93.5$ sq. in.

h. **Answer the original questions from the Opening Exercise. How do the actual answers compare to your original predictions?**

It was impossible to fold the paper more than 7 times. Using our model, if we could fold the paper 10 times, it would be just over 4 in. thick and less than $\frac{1}{10}$ sq. in., which is much smaller than the area of a postage stamp. Our predictions were inaccurate because we did not consider how drastically the sizes change when successively doubling or halving measurements.

Student groups should present their solutions again. If it did not come up earlier, ask students to consider how they might increase the likelihood that they could fold a piece of paper in half more than seven or eight times.

- What are some ways to increase the likelihood that you could successfully fold a piece of paper in half more than seven or eight times?
  - *You could use a thinner piece of paper. You could use a larger piece of paper. You could try different ways of folding the paper.*

Brittney Gallivan, the high school student who solved this problem in 2001, first folded a very thin sheet of gold foil in half over seven times and then successfully folded an extremely large piece of paper in half 12 times at a local shopping mall. In 2011, students at St. Mark's High School in Massachusetts folded miles of taped-together toilet paper in half 13 times.

Example 1 (5 minutes): Using the Properties of Exponents to Rewrite Expressions

In this example, show students how to represent their expressions using powers of 2. Model this using the folding of a 10 in. by 10 in. square sheet of gold foil. The thickness of gold foil is 0.28 millionth of a meter. The information in this problem is based on the first task Britney accomplished: folding a sheet of gold foil in half twelve times. Of course, her teacher then modified the task and required her to actually use paper (http://www.abc.net.au/science/articles/2005/12/21/1523497.htm). The goal of this example is to remind students of the meaning of integer exponents.

> *Scaffolding:*
>
> Use a vocabulary notebook or an anchor chart posted on the wall to remind students of vocabulary words associated with exponents.
>
> b^n — n is the exponent; b is the base

Many students likely see the area sequence as successive divisions by two.
For students who are struggling to make sense of the meaning of a negative exponent, model rewriting the expressions in the last column as follows:

$$\frac{100}{2} = 100\left(\frac{1}{2}\right) = 100\left(\frac{1}{2}\right)^1 = 100 \cdot 2^{-1};$$

$$\frac{100}{4} = 100\left(\frac{1}{4}\right) = 100\left(\frac{1}{2}\right)^2 = 100 \cdot 2^{-2};$$

$$\frac{100}{8} = 100\left(\frac{1}{8}\right) = 100\left(\frac{1}{2}\right)^3 = 100 \cdot 2^{-3}.$$

Example 1: Using the Properties of Exponents to Rewrite Expressions

The table below displays the thickness and area of a folded square sheet of gold foil. In 2001, Britney Gallivan, a California high school junior, successfully folded a 100-square-inch sheet of gold foil in half 12 times to earn extra credit in her mathematics class.

Rewrite each of the table entries as a multiple of a power of 2.

Number of Folds	Thickness of the Stack (millionths of a meter)	Thickness Using a Power of 2	Area of the Top (square inches)	Area Using a Power of 2
0	0.28	$0.28 \cdot 2^0$	100	$100 \cdot 2^0$
1	0.56	$0.28 \cdot 2^1$	50	$100 \cdot 2^{-1}$
2	1.12	$0.28 \cdot 2^2$	25	$100 \cdot 2^{-2}$
3	2.24	$0.28 \cdot 2^3$	12.5	$100 \cdot 2^{-3}$
4	4.48	$0.28 \cdot 2^4$	6.25	$100 \cdot 2^{-4}$
5	8.96	$0.28 \cdot 2^5$	3.125	$100 \cdot 2^{-5}$
6	17.92	$0.28 \cdot 2^6$	1.5625	$100 \cdot 2^{-6}$

While modeling this example with students, take the opportunity to discuss the fact that exponentiation with positive integers can be thought of as repeated multiplication by the base, whereas exponentiation with negative integers can be thought of as repeated division by the base. For example,

$$4^{24} = \underbrace{4 \cdot 4 \cdot 4 \cdot \cdots \cdot 4}_{24 \text{ times}} \quad \text{and} \quad 4^{-24} = \frac{1}{\underbrace{4 \cdot 4 \cdot 4 \cdot \cdots \cdot 4}_{24 \text{ times}}}.$$

Alternatively, the meaning of a negative exponent when the exponent is an integer can be described as repeated multiplication by the reciprocal of the base. For example,

$$4^{-24} = \underbrace{\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\cdots\left(\frac{1}{4}\right)}_{24 \text{ times}}.$$

Interpreting exponents as repeated multiplication or division only makes sense for integer exponents. However, the properties of exponents do apply for any real number exponent.

Example 2 (5 minutes): Applying the Properties of Exponents to Rewrite Expressions

Transition into this example by explaining that many times when working with algebraic and numeric expressions that contain exponents, it is advantageous to rewrite them in different forms (MP.7). One obvious advantage of exponents is that they shorten the length of an expression involving repeated multiplication. Ask students to imagine always having to write out the product of ten 2's if they wanted to express the number $1{,}024$ as a power of 2. While the exponent notation gives a way to express repeated multiplication succinctly, the properties of exponents also provide a way to make computations with exponents more efficient. Share the properties of exponents below. Have students record these properties in their math notebooks.

> The Properties of Exponents
>
> For nonzero real numbers x and y and all integers a and b, the following properties hold.
>
> $$x^a \cdot x^b = x^{a+b}$$
> $$(x^a)^b = x^{ab}$$
> $$(xy)^a = x^a y^a$$
> $$\frac{x^a}{x^b} = x^{a-b}$$

(Note: Most cases of the properties listed above hold when $x = 0$ or $y = 0$. The only cases that cause problems are when values of the variables result in the expression 0^0 or division by 0.) Ask students to discuss with a partner different ways to rewrite the following expressions in the form kx^n, where k is a real number and n is an integer. Have students share out their responses with the entire class. For each problem, model both approaches. While modeling, make sure to have students verbalize the connections between the methods. Ask volunteers to explain why the rules hold.

Scaffolding:

Students needing additional practice with exponents can use the following numeric examples that mirror the algebraic expressions in Example 2. Have students work them side by side to help see the structure of the expressions.

- Write each expression in the form kb^n, where k is a real number and b and n are integers.

$$(5 \cdot 2^7)(-3 \cdot 2^2)$$

$$\frac{3 \cdot 4^5}{(2 \cdot 4)^4}$$

$$\frac{3}{(5^2)^{-3}}$$

$$\frac{3^{-3} \cdot 3^4}{3^8}$$

Example 2: Applying the Properties of Exponents to Rewrite Expressions

Rewrite each expression in the form kx^n, where k is a real number, n is an integer, and x is a nonzero real number.

a. $(5x^5)\cdot(-3x^2)$

Method 1: Apply the definition of an exponent and properties of algebra.

$$(5x^5)\cdot(-3x^2) = 5\cdot -3\cdot x^5\cdot x^2 = -15\cdot(x\cdot x\cdot x\cdot x\cdot x)\cdot(x\cdot x) = -15x^7$$

Method 2: Apply the rules of exponents and the properties of algebra.

$$5x^5\cdot -3x^2 = 5\cdot -3\cdot x^5\cdot x^2 = -15\cdot x^{5+2} = -15x^7$$

b. $\dfrac{3x^5}{(2x)^4}$

Method 1: Apply the definition of an exponent and properties of algebra.

$$\frac{3x^5}{(2x)^4} = \frac{3\cdot x\cdot x\cdot x\cdot x\cdot x}{2x\cdot 2x\cdot 2x\cdot 2x} = \frac{3\cdot x\cdot x\cdot x\cdot x\cdot x}{2\cdot 2\cdot 2\cdot 2\cdot x\cdot x\cdot x\cdot x} = \frac{3x}{16}\cdot\frac{x}{x}\cdot\frac{x}{x}\cdot\frac{x}{x}\cdot\frac{x}{x} = \frac{3x}{16}\cdot 1\cdot 1\cdot 1\cdot 1 = \frac{3}{16}x$$

Method 2: Apply the rules of exponents and the properties of algebra.

$$\frac{3x^5}{2^4x^4} = \frac{3}{16}x^{5-4} = \frac{3}{16}x$$

c. $\dfrac{3}{(x^2)^{-3}}$

Method 1: Apply the definition of an exponent and properties of algebra.

$$\frac{3}{(x^2)^{-3}} = \frac{3}{\left(\frac{1}{x^2}\right)\left(\frac{1}{x^2}\right)\left(\frac{1}{x^2}\right)} = \frac{3}{\frac{1}{x}\cdot\frac{1}{x}\cdot\frac{1}{x}\cdot\frac{1}{x}\cdot\frac{1}{x}\cdot\frac{1}{x}} = \frac{3}{\frac{1}{x^6}} = 3\cdot x^6 = 3x^6$$

Method 2: Apply the rules of exponents and the properties of algebra.

$$\frac{3}{(x^2)^{-3}} = \frac{3}{x^{2(-3)}} = \frac{3}{x^{-6}} = 3x^6$$

d. $\dfrac{x^{-3}x^4}{x^8}$

Method 1: Apply the definition of an exponent and properties of algebra.

$$\frac{x^{-3}x^4}{x^8} = \frac{\frac{1}{x}\cdot\frac{1}{x}\cdot\frac{1}{x}\cdot x\cdot x\cdot x\cdot x}{x\cdot x\cdot x\cdot x\cdot x\cdot x\cdot x\cdot x} = \frac{\frac{x}{x}\cdot\frac{x}{x}\cdot\frac{x}{x}\cdot x}{x\cdot x\cdot x\cdot x\cdot x\cdot x\cdot x\cdot x} = \frac{x}{x}\cdot\frac{1}{x\cdot x\cdot x\cdot x\cdot x\cdot x\cdot x} = x^{-7}$$

Method 2: Apply the rules of exponents and the properties of algebra.

$$\frac{x^{-3}x^4}{x^8} = \frac{x^{-3+4}}{x^8} = \frac{x^1}{x^8} = x^{1-8} = x^{-7}$$

After seeing these examples, students should begin to understand why the properties of exponents are so useful for working with exponential expressions. Show both methods as many times as necessary in order to reinforce the properties with students so that they ultimately rewrite most expressions like these by inspection, using the properties rather than expanding exponential expressions.

Exercises 1–5 (5 minutes)

The point of these exercises is to force students to use the properties of exponents to rewrite expressions in the form kx^n. Typical high school textbooks ask students to write expressions with nonnegative exponents. However, in advanced mathematics classes, students need to be able to fluently rewrite expressions in different forms. Students who continue on to study higher-level mathematics such as calculus need to rewrite expressions in this form in order to quickly apply a common derivative rule. The last two exercises are not feasible to work out by expanding the exponent. Use them to assess whether or not students are able to apply the rules fluently even when larger numbers or variables are involved.

Exercises 1–5

Rewrite each expression in the form kx^n, where k is a real number and n is an integer. Assume $x \neq 0$.

1. $2x^5 \cdot x^{10}$

   $2x^{15}$

2. $\frac{1}{3x^8}$

   $\frac{1}{3}x^{-8}$

3. $\frac{6x^{-5}}{x^{-3}}$

   $6x^{-5-(-3)} = 6x^{-2}$

4. $\left(\frac{3}{x^{-22}}\right)^{-3}$

   $(3x^{22})^{-3} = 3^{-3}x^{-66} = \frac{1}{27}x^{-66}$

5. $(x^2)^n \cdot x^3$

   $x^{2n} \cdot x^3 = x^{2n+3}$

Closing (2 minutes)

Have students respond individually in writing or with a partner to the following questions.

- How can the properties of exponents help us to rewrite expressions?
  - *They make the process less tedious, especially when the exponents are very large or very small integers.*
- Why are the properties of exponents useful when working with large or small numbers?
  - *You can quickly rewrite expressions without having to rewrite each power of the base in expanded form.*

Lesson Summary

The Properties of Exponents
For real numbers x and y with $x \neq 0$, $y \neq 0$, and all integers a and b, the following properties hold.

1. $x^a \cdot x^b = x^{a+b}$
2. $(x^a)^b = x^{ab}$
3. $(xy)^a = x^a y^a$
4. $\frac{1}{x^a} = x^{-a}$
5. $\frac{x^a}{x^b} = x^{a-b}$
6. $\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}$
7. $x^0 = 1$

Exit Ticket (3 minutes)

Name ______________________________ Date ____________

Lesson 1: Integer Exponents

Exit Ticket

The following formulas for paper folding were discovered by Britney Gallivan in 2001 when she was a high school junior. The first formula determines the minimum width, W, of a square piece of paper of thickness T needed to fold it in half n times, alternating horizontal and vertical folds. The second formula determines the minimum length, L, of a long rectangular piece of paper of thickness T needed to fold it in half n times, always folding perpendicular to the long side.

$$W = \pi \cdot T \cdot 2^{\frac{3(n-1)}{2}} \qquad L = \frac{\pi T}{6}(2^n + 4)(2^n - 1)$$

1. Notebook paper is approximately 0.004 in. thick. Using the formula for the width W, determine how wide a square piece of notebook paper would need to be to successfully fold it in half 13 times, alternating horizontal and vertical folds.

2. Toilet paper is approximately 0.002 in. thick. Using the formula for the length L, how long would a continuous sheet of toilet paper have to be to fold it in half 12 times, folding perpendicular to the long edge each time?

3. Use the properties of exponents to rewrite each expression in the form kx^n. Then, evaluate the expression for the given value of x.

   a. $2x^3 \cdot \frac{5}{4}x^{-1}$; $x = 2$

   b. $\frac{9}{(2x)^{-3}}$; $x = -\frac{1}{3}$

Exit Ticket Sample Solutions

The following formulas for paper folding were discovered by Britney Gallivan in 2001 when she was a high school junior. The first formula determines the minimum width, W, of a square piece of paper of thickness T needed to fold it in half n times, alternating horizontal and vertical folds. The second formula determines the minimum length, L, of a long rectangular piece of paper of thickness T needed to fold it in half n times, always folding perpendicular to the long side.

$$W = \pi \cdot T \cdot 2^{\frac{3(n-1)}{2}} \qquad L = \frac{\pi T}{6}(2^n + 4)(2^n - 1)$$

1. **Notebook paper is approximately 0.004 in. thick. Using the formula for the width W, determine how wide a square piece of notebook paper would need to be to successfully fold it in half 13 times, alternating horizontal and vertical folds.**

   The paper would need to be approximately $3{,}294.2$ in. *wide:* $W = \pi T\, 2^{\frac{3(13-1)}{2}} = \pi(0.004)2^{18} \approx 3294.199.$

2. **Toilet paper is approximately 0.002 in. thick. Using the formula for the length L, how long would a continuous sheet of toilet paper have to be to fold it in half 12 times, folding perpendicular to the long edge each time?**

   The paper would have to be approximately $17{,}581.92$ in. *long, which is approximately* 0.277 mi.*:*

   $$L = \left(\frac{\pi(0.002)}{6}\right)(2^{12} + 4)(2^{12} - 1) = \pi\left(\frac{1}{3000}\right)(4100)(4095) = 5596.5\,\pi \approx 17{,}581.92.$$

3. **Use the properties of exponents to rewrite each expression in the form kx^n. Then, evaluate the expression for the given value of x.**

   a. $2x^3 \cdot \frac{5}{4}x^{-1};\ x = 2$

   $2\left(\frac{5}{4}\right)x^3x^{-1} = \frac{5}{2}x^2$

   When $x = 2$, $\frac{5}{2}x^2 = \frac{5}{2}(2)^2 = 10.$

   b. $\frac{9}{(2x)^{-3}};\ x = -\frac{1}{3}$

   $\frac{9}{2^{-3}x^{-3}} = 72x^3$

   When $x = -\frac{1}{3}$, $72x^3 = 72\left(-\frac{1}{3}\right)^3 = -\frac{8}{3}.$

Problem Set Sample Solutions

Note to the teacher: Problem 3, part (g), is important for the financial lessons that occur near the end of this module.

1. **Suppose your class tried to fold an unrolled roll of toilet paper. It was originally 4 in. wide and 30 ft. long. Toilet paper is approximately 0.002 in. thick.**

   a. **Complete each table, and represent the area and thickness using powers of 2.**

Number of Folds n	Thickness After n Folds (in.)
0	$0.002 = 0.002 \cdot 2^0$
1	$0.004 = 0.002 \cdot 2^1$
2	$0.008 = 0.002 \cdot 2^2$
3	$0.016 = 0.002 \cdot 2^3$
4	$0.032 = 0.002 \cdot 2^4$
5	$0.064 = 0.002 \cdot 2^5$
6	$0.128 = 0.002 \cdot 2^6$

Number of Folds n	Area on Top After n Folds (in^2)
0	$1440 = 1440 \cdot 2^0$
1	$720 = 1440 \cdot 2^{-1}$
2	$360 = 1440 \cdot 2^{-2}$
3	$180 = 1440 \cdot 2^{-3}$
4	$90 = 1440 \cdot 2^{-4}$
5	$45 = 1440 \cdot 2^{-5}$
6	$22.5 = 1440 \cdot 2^{-6}$

   b. **Create an algebraic function that describes the area in square inches after n folds.**

   $A(n) = 1440 \cdot 2^{-n}$, where n is a nonnegative integer.

   c. **Create an algebraic function that describes the thickness in inches after n folds.**

   $T(n) = 0.002 \cdot 2^n$, where n is a nonnegative integer.

2. **In the Exit Ticket, we saw the formulas below. The first formula determines the minimum width, W, of a square piece of paper of thickness T needed to fold it in half n times, alternating horizontal and vertical folds. The second formula determines the minimum length, L, of a long rectangular piece of paper of thickness T needed to fold it in half n times, always folding perpendicular to the long side.**

$$W = \pi \cdot T \cdot 2^{\frac{3(n-1)}{2}} \qquad L = \frac{\pi T}{6}(2^n + 4)(2^n - 1)$$

   Use the appropriate formula to verify why it is possible to fold a 10 inch by 10 inch sheet of gold foil in half 13 times. Use 0.28 millionth of a meter for the thickness of gold foil.

   Given that the thickness of the gold foil is 0.28 millionth of a meter, we have

$$\frac{0.28}{1{,}000{,}000}\,\text{m} \cdot \frac{100\text{ cm}}{1\text{ m}} = 0.000028\text{ cm} \cdot \frac{1\text{ in}}{2.54\text{ cm}} = 0.00001102\text{ in}.$$

   Using the formula

$$W = \pi\, T\, 2^{\frac{3(n-1)}{2}}$$

   with $n = 13$ and $T = 0.00001102$, we get

$$W = \pi(0.00001102)2^{\frac{3(13-1)}{2}} \approx 9.1$$

   Thus, any square sheet of gold foil larger than 9.1 inches by 9.1 inches can be folded in half 13 times, so a 10 inch by 10 inch sheet of gold foil can be folded in half 13 times.

3. Use the formula from Problem 2 to determine if you can fold an unrolled roll of toilet paper in half more than 10 times. Assume that the thickness of a sheet of toilet paper is approximately 0.002 in. and that one roll is 102 ft. long.

First, convert feet to inches. $102 \text{ ft.} = 1224 \text{ in.}$

Then, substitute 0.002 and 10 into the formula for T and n, respectively.

$$L = \frac{\pi(0.002)}{6}(2^{10}+4)(2^{10}-1) = 1101.3$$

The roll is just long enough to fold in half 10 times.

4. Apply the properties of exponents to rewrite each expression in the form kx^n, where n is an integer and $x \neq 0$.

a. $(2x^3)(3x^5)(6x)^2$

$2 \cdot 3 \cdot 36x^{3+5+2} = 216x^{10}$

b. $\dfrac{3x^4}{(-6x)^{-2}}$

$3x^4 \cdot 36x^2 = 108x^6$

c. $\dfrac{x^{-3}x^5}{3x^4}$

$\dfrac{1}{3}x^{-3+5-4} = \dfrac{1}{3}x^{-2}$

d. $5(x^3)^{-3}(2x)^{-4}$

$\dfrac{5}{16}x^{-9+(-4)} = \dfrac{5}{16}x^{-13}$

e. $\left(\dfrac{x^2}{4x^{-1}}\right)^{-3}$

$\dfrac{x^{-6}}{4^{-3}x^3} = 64x^{-6-3} = 64x^{-9}$

5. Apply the properties of exponents to verify that each statement is an identity.

a. $\dfrac{2^{n+1}}{3^n} = 2\left(\dfrac{2}{3}\right)^n$ for integer values of n

$\dfrac{2^{n+1}}{3^n} = \dfrac{2^n 2^1}{3^n} = \dfrac{2 \cdot 2^n}{3^n} = 2 \cdot \left(\dfrac{2}{3}\right)^n$

b. $3^{n+1} - 3^n = 2 \cdot 3^n$ for integer values of n

$3^{n+1} - 3^n = 3^n \cdot 3^1 - 3^n = 3^n(3-1) = 3^n \cdot 2 = 2 \cdot 3^n$

c. $\dfrac{1}{(3^n)^2} \cdot \dfrac{4^n}{3} = \dfrac{1}{3}\left(\dfrac{2}{3}\right)^{2n}$ for integer values of n

$\dfrac{1}{(3^n)^2} \cdot \dfrac{4^n}{3} = \dfrac{1}{3^{2n}} \cdot \dfrac{(2^2)^n}{3} = \dfrac{1 \cdot 2^{2n}}{3 \cdot 3^{2n}} = \dfrac{1}{3} \cdot \left(\dfrac{2}{3}\right)^{2n}$

MP.3

6. **Jonah was trying to rewrite expressions using the properties of exponents and properties of algebra for nonzero values of x. In each problem, he made a mistake. Explain where he made a mistake in each part, and provide a correct solution.**

Jonah's Incorrect Work

a. $(3x^2)^{-3} = -9x^{-6}$

b. $\frac{2}{3x^{-5}} = 6x^5$

c. $\frac{2x - x^3}{3x} = \frac{2}{3} - x^3$

In part (a), he multiplied 3 by the exponent -3. The correct solution is $3^{-3}x^{-6} = \frac{1}{27}x^{-6}$.

In part (b), he multiplied 2 by 3 when he rewrote x^{-5}. The 3 should remain in the denominator of the expression. The correct solution is $\frac{2}{3}x^5$.

In part (c), he only divided the first term by $3x$, but he should have divided both terms by $3x$. The correct solution is $\frac{2x}{3x} - \frac{x^3}{3x} = \frac{2}{3} - \frac{x^2}{3}$.

7. **If $x = 5a^4$ and $a = 2b^3$, express x in terms of b.**

By the substitution property, if $x = 5a^4$ and $a = 2b^3$, then $x = 5(2b^3)^4$. Rewriting the right side in an equivalent form gives $x = 80b^{12}$.

8. If $a = 2b^3$ and $b = -\frac{1}{2}c^{-2}$, express a in terms of c.

By the substitution property, if $a = 2b^3$ and $b = -\frac{1}{2}c^{-2}$, then $a = 2\left(-\frac{1}{2}c^{-2}\right)^3$. Rewriting the right side in an equivalent form gives $a = -\frac{1}{4}c^{-6}$.

9. If $x = 3y^4$ and $y = \frac{s}{2x^3}$, show that $s = 54y^{13}$.

Rewrite the equation $y = \frac{s}{2x^3}$ to isolate the variable s.

$$y = \frac{s}{2x^3}$$
$$2x^3y = s$$

By the substitution property, if $s = 2x^3y$ and $x = 3y^4$, then $s = 2(3y^4)^3 \cdot y$. Rewriting the right side in an equivalent form gives $s = 2 \cdot 27y^{12} \cdot y = 54y^{13}$.

10. **Do the following tasks without a calculator.**

a. **Express 8^3 as a power of 2.**

$8^3 = (2^3)^3 = 2^9$

b. **Divide 4^{15} by 2^{10}.**

$\frac{4^{15}}{2^{10}} = \frac{2^{30}}{2^{10}} = 2^{20}$ *or* $\frac{4^{15}}{2^{10}} = \frac{4^{15}}{4^5} = 4^{10}$

11. Use powers of 2 to perform each calculation without a calculator or other technology.

a. $\frac{2^7 \cdot 2^5}{16}$

$$\frac{2^7 \cdot 2^5}{16} = \frac{2^7 \cdot 2^5}{2^4} = 2^{7+5-4} = 2^8 = 256$$

b. $\frac{512000}{320}$

$$\frac{512000}{320} = \frac{512 \cdot 1000}{32 \cdot 10} = \frac{2^9}{2^5} \cdot 100 = 2^4 \cdot 100 = 1600$$

12. Write the first five terms of each of the following recursively defined sequences:

a. $a_{n+1} = 2a_n$, $a_1 = 3$

$3, 6, 12, 24, 48$

b. $a_{n+1} = (a_n)^2$, $a_1 = 3$

$3, 9, 81, 6561, 43\,046\,721$

c. $a_{n+1} = 2(a_n)^2$, $a_1 = x$, where x is a real number Write each term in the form kx^n.

$x, 2x^2, 8x^4, 128x^8, 32768x^{16}$

d. $a_{n+1} = 2(a_n)^{-1}$, $a_1 = y$, $(y \neq 0)$ Write each term in the form kx^n.

$y, 2y^{-1}, y, 2y^{-1}, y$

13. **In Module 1, you established the identity $(1-r)(1+r+r^2+\cdots+r^{n-1}) = 1-r^n$, where r is a real number and n is a positive integer.**

Use this identity to respond to parts (a)–(g) below.

a. **Rewrite the given identity to isolate the sum $1+r+r^2+\cdots+r^{n-1}$ for $r \neq 1$.**

$$(1+r+r^2+\cdots+r^{n-1}) = \frac{1-r^n}{1-r}$$

b. **Find an explicit formula for $1+2+2^2+2^3+\cdots+2^{10}$.**

$$\frac{1-2^{11}}{1-2} = 2^{11}-1$$

c. **Find an explicit formula for $1+a+a^2+a^3+\cdots+a^{10}$ in terms of powers of a.**

$$\frac{1-a^{11}}{1-a}$$

d. **Jerry simplified the sum $1+a+a^2+a^3+a^4+a^5$ by writing $1+a^{15}$. What did he do wrong?**

He assumed that when you add terms with the same base, you also add the exponents. You only add the exponents when you multiply terms with the same base.

e. Find an explicit formula for $1 + 2a + (2a)^2 + (2a)^3 + \cdots + (2a)^{12}$ in terms of powers of a.

$$\frac{1-(2a)^{13}}{1-2a}$$

f. Find an explicit formula for $3 + 3(2a) + 3(2a)^2 + 3(2a)^3 + \cdots + 3(2a)^{12}$ in terms of powers of a.
Hint: Use part (e).

$$3 \cdot \left(\frac{1-(2a)^{13}}{1-2a}\right)$$

g. Find an explicit formula for $P + P(1+r) + P(1+r)^2 + P(1+r)^3 + \cdots + P(1+r)^{n-1}$ in terms of powers of $(1+r)$.

$$P \cdot \left(\frac{1-(1+r)^n}{1-(1+r)}\right) = P \cdot \left(\frac{1-(1+r)^n}{-r}\right)$$

Lesson 2: Base 10 and Scientific Notation

Student Outcomes

- Students review place value and scientific notation.
- Students use scientific notation to compute with large numbers.

Lesson Notes

This lesson reviews how to express numbers using scientific notation. Students first learned about scientific notation in Grade 8 where they expressed numbers using scientific notation (**8.EE.A.3**) and computed with and compared numbers expressed using scientific notation (**8.EE.A.4**). Refer to Grade 8, Module 1, Topic B to review the approach to introducing scientific notation and its use in calculations with very large and very small numbers. This lesson sets the stage for the introduction of base-10 logarithms later in this module by focusing on the fact that every real number can be expressed as the product of a number between 1 and 10 and a power of 10. In the paper-folding activity in the last lesson, students worked with some very small and very large numbers. This lesson opens with these numbers to connect these lessons. Students also compute with numbers using scientific notation and discuss how the properties of exponents can simplify these computations (**N-RN.A.2**). In both the lesson and the Problem Set, students define appropriate quantities for the purpose of descriptive modeling (**N-Q.A.2**). The lesson includes a demonstration that reinforces the point that using scientific notation is convenient when working with very large or very small numbers and helps students gain some sense of the change in magnitude when different powers of 10 are compared. This is an excellent time to watch the 9-minute classic film "Powers of 10" by Charles and Ray Eames, available at https://www.youtube.com/watch?v=0fKBhvDjuy0, which clearly illustrates the effect of adding another zero. The definition of scientific notation from Grade 8, Module 1, Lesson 9 is included after Example 1. Consider allowing students to use a calculator for this lesson.

Classwork

Opening (2 minutes)

In the last lesson, Example 1 gave the thickness of a sheet of gold foil as 0.28 millionth of a meter. In the Exit Ticket, students calculated the size of a square sheet of paper that could be folded in half thirteen times, and this area was very large. These numbers are used in the Opening Exercise of this lesson. Before students begin the Opening Exercise, briefly remind them of these numbers that they saw in the previous lesson, and tell them that this lesson provides them with a way to conveniently represent very small or very large numbers. If there was not an opportunity to share one of the news stories in Lesson 1, that could be done at this time as well.

Opening Exercise (5 minutes)

Students should work these exercises independently and then share their responses with a partner. Ask one or two students to explain their calculations to the rest of the class. Be sure to draw out the meaning of 0.28 millionth of a meter and how that can be expressed as a fraction of a meter. Check to make sure students know the place value for large numbers such as billions. Students should be able to work the first exercise without a calculator, but on the second exercise, students should definitely use a calculator. If they do not have access to a calculator, give them the number squared, and simply have them write the rounded value.

Opening Exercise

In the last lesson, you worked with the thickness of a sheet of gold foil (a very small number) and some very large numbers that gave the size of a piece of paper that actually could be folded in half more than 13 times.

a. **Convert 0.28 millionth of a meter to centimeters, and express your answer as a decimal number.**

$$\frac{0.28}{1,000,000}\text{ m}\cdot\frac{100\text{ cm}}{1\text{ m}}=\frac{28}{1,000,000}\text{cm}=0.000028\text{ cm}$$

b. **The length of a piece of square notebook paper that can be folded in half 13 times is $3,294.2$ in. Use this number to calculate the area of a square piece of paper that can be folded in half 14 times. Round your answer to the nearest million.**

$$(2\cdot 3294.2)^2=43,407,014.56$$

Rounded to the nearest million, the area is $43,000,000$ *square inches.*

c. **Match the equivalent expressions without using a calculator.**

3.5×10^5	-6	-6×10^0	0.6	3.5×10^{-6}
$3,500,000$	$350,000$	6×10^{-1}	0.0000035	3.5×10^6

3.5×10^5 *is equal to* $350,000$.
-6×10^0 *is equal to* -6.
6×10^{-1} *is equal to* 0.6.
3.5×10^6 *is equal to* $3,500,000$.
3.5×10^{-6} *is equal to* 0.0000035.

While reviewing these solutions, point out that very large and very small numbers require students to include many digits to indicate the place value as shown in parts (a) and (b). Also, based on part (c), it appears that integer powers of 10 can be used to express a number as a product.

Example 1 (7 minutes)

Write the following statement on the board, and ask students to consider whether or not they believe it is true. Have them discuss their thoughts with a partner, and then ask for volunteers to explain their thinking.

> Every positive decimal number can be expressed as the product of a number between 1 and 10 and a power of 10.

Students should explain that every positive decimal number can be expressed as the product of a number between 1 and 10 and a power of 10. Consider using the following prompts to direct student thinking:

MP.3

- Think of an example of a decimal number between 1 and 10. Write it down.
  - 2.5
- Think of a power of 10. Write it down.
  - 100 *or* 10^2
- What does the word *product* mean?
  - *It means the result of multiplying two numbers together.*
- Compute the product of the two numbers you wrote down.
  - $2.5 \cdot 10^2 = 2500$

Scaffolding:

- Have students working above grade level write the number 245 as the product of a number and a power of 10 three different ways.
- To challenge students working above grade level, have them make a convincing argument regarding the truth of a statement such as the following:

  Every decimal number can be expressed as the product of another decimal number and a power of 10.

First, have students share their answers with a partner. Then, put a few of the examples on the board. Finally, demonstrate how to reverse the process to express the following numbers as the product of a number between 1 and 10 and a power of 10. At this point, explain to students that when numbers are written in this fashion, it is said that they are written using *scientific notation*. This is an especially convenient way to represent extremely large or extremely small numbers that otherwise would have many zero digits as placeholders. Make sure to emphasize that using this notation is simply rewriting a numerical expression in a different form, which helps students to quickly determine the size of the number. Students may need to be reminded of place value for numbers less than 1 and writing equivalent fractions whose denominators are powers of 10. The solutions demonstrate how numbers can be expressed using scientific notation.

Example 1

Write each number as a product of a decimal number between 1 and 10 and a power of 10.

a. $234{,}000$

$\mathbf{2.34 \cdot 100{,}000 = 2.34 \times 10^5}$

b. 0.0035

$\mathbf{\frac{35}{10\,000} = \frac{3.5}{1000} = 3.5 \cdot \frac{1}{1000} = 3.5 \times 10^{-3}}$

c. $532{,}100{,}000$

$\mathbf{5.321 \cdot 100{,}000{,}000 = 5.321 \times 10^8}$

d. 0.0000000012

$$\frac{12}{10,000,000,000} = \frac{1.2}{1,000,000,000} = 1.2 \cdot \frac{1}{1,000,000,000} = 1.2 \times 10^{-9}$$

e. 3.331

$3.331 \cdot 1 = 3.331 \times 10^0$

- Our knowledge of the integer powers of 10 enables us to understand the concept of scientific notation.
- Consider the estimated number of stars in the universe: 6×10^{22}. This is a 23-digit *whole number* with the *leading digit* (the leftmost digit) 6 followed by 22 zeros. When it is written in the form 6×10^{22}, it is said to be expressed in *scientific notation*.

Students may recall scientific notation from previous grades. Take time to review the definition of scientific notation provided below.

A positive, finite decimal s is said to be written in scientific notation if it is expressed as a product $d \times 10^n$, where
d is a finite decimal number so that $1 \leq d < 10$, and n is an integer.

The integer n is called the *order of magnitude* of the decimal $d \times 10^n$.

Exercises 1–6 (4 minutes)

Students should work these exercises independently as their progress is monitored. Encourage students to work quickly and begin generalizing a process for quickly writing numbers using scientific notation (such as counting the number of digits between the leading digit and the ones digit). After a few minutes, share the solutions with students so they can check their work.

Exercises 1–6

For Exercises 1–6, write each number in scientific notation.

1. $532,000,000$

   5.32×10^8

2. 0.00000000000000000123 **(16 zeros after the decimal place)**

   1.23×10^{-17}

3. $8,900,000,000,000,000$ **(14 zeros after the 9)**

   8.9×10^{15}

4. 0.00003382

   3.382×10^{-5}

5. $34,000,000,000,000,000,000,000,000$ (24 zeros after the 4)
   3.4×10^{25}

6. 0.0000000000000000000004 (21 zeros after the decimal place)
   4×10^{-22}

To help students quickly write these problems using scientific notation, the number of zeros is written above for each problem. Be very careful that students are not using this number as the exponent on the base 10. Lead a discussion to clarify that difference for all students who make this careless mistake.

Exercises 7–8 (5 minutes)

After students practice writing numbers in scientific notation, emphasize the usefulness of scientific notation for working with very large or very small numbers by showing this demonstration: http://joshworth.com/dev/pixelspace/pixelspace_solarsystem.html, which illustrates just how far the planets in our solar system are from each other. After the demonstration, write down the distances between Earth and the sun, between Jupiter and the sun, and between Pluto and the sun on the board, and have students work with a partner to answer Exercise 7. Be sure to mention that these distances are averages; the distances between the planets and the sun are constantly changing as the planets complete their orbits. The average distance from the sun to Earth is $151{,}268{,}468$ km. The average distance from the sun to Jupiter is $780{,}179{,}470$ km. The average distance between the sun and Pluto is $5{,}908{,}039{,}124$ km. In these exercises, students round the distances to the nearest tenth to minimize all the writing and help them focus more readily on the magnitude of the numbers relative to one another.

Exercises 7–8

7. Use the fact that the average distance between the sun and Earth is $151,268,468$ km, the average distance between the sun and Jupiter is $780,179,470$ km, and the average distance between the sun and Pluto is $5,908,039,124$ km to approximate the following distances. Express your answers in scientific notation ($d \times 10^n$), where d is rounded to the nearest tenth.

   a. Distance from the sun to Earth:

      1.5×10^8 km

   b. Distance from the sun to Jupiter:

      7.8×10^8 km

   c. Distance from the sun to Pluto:

      5.9×10^9 km

   d. How much farther Jupiter is from the sun than Earth is from the sun:

      $780,179,470 \text{ km} - 151,268,468 \text{ km} = 628,911,002 \text{ km}$

      6.3×10^8 km

e. How much farther Pluto is from the sun than Jupiter is from the sun:

$5,908,039,124 \text{ km} - 780,179,470 \text{ km} = 5,127,859,654 \text{ km}$

$5.1 \times 10^9 \text{ km}$

MP.7

8. Order the numbers in Exercise 7 from smallest to largest. Explain how writing the numbers in scientific notation helps you to quickly compare and order them.

The numbers from smallest to largest are 1.5×10^8, 6.3×10^8, 7.8×10^8, 5.1×10^9, and 5.9×10^9. The power of 10 helps to quickly sort the numbers by their order of magnitude, and then it is easy to quickly compare the numbers with the same order of magnitude because they are only written as a number between one and ten.

Example 2 (10 minutes): Arithmetic Operations with Numbers Written Using Scientific Notation

Model the solutions to the following example problems. Be sure to emphasize that final answers should be expressed using the scientific notation convention of a number between 1 and 10 and a power of 10. On part (a), it may be necessary to provide some additional scaffolding if students are struggling to rewrite the numbers using the same order of magnitude. Have students practice writing a number as a product of a power of 10 in three different ways. For example, $15\,000 = 1.5 \times 10^4 = 15 \times 10^3 = 150 \times 10^2$. The lessons of Algebra I, Module 1, Topic B provide some suggestions and fluency exercises if students need additional practice on arithmetic operations with numbers in scientific notation. Be sure that students understand that the properties of exponents allow them to quickly perform the indicated operations.

Example 2: Arithmetic Operations with Numbers Written Using Scientific Notation

a. $(2.4 \times 10^{20}) + (4.5 \times 10^{21})$

$(2.4 \times 10^{20}) + (45 \times 10^{20}) = 47.4 \times 10^{20} = 4.74 \times 10^{21}$

b. $(7 \times 10^{-9})(5 \times 10^{5})$

$(7 \cdot 5) \times (10^{-9} \cdot 10^{5}) = 35 \times 10^{-4} = 3.5 \times 10^{-3}$

c. $\dfrac{1.2 \times 10^{15}}{3 \times 10^{7}}$

$\dfrac{1.2}{3} \times 10^{15-7} = 0.4 \times 10^{8} = 4 \times 10^{7}$

Scaffolding:

For Example 2, part (a), students may want to add the exponents as they do when multiplying numbers written using scientific notation. Take time to discuss the differences in the three expressions if students are making this type of mistake.

Debrief with the questions designed to help students see that the order of magnitude and the properties of exponents greatly simplify calculations.

- How do the properties of exponents help to simplify these calculations?
- How can you quickly estimate the size of your answer?

Exercises 9–10 (5 minutes)

Exercises 9–10

9. Perform the following calculations without rewriting the numbers in decimal form.

a. $(1.42 \times 10^{15}) - (2 \times 10^{13})$

$142 \times 10^{13} - 2 \times 10^{13} = (142 - 2) \times 10^{13} = 140 \times 10^{13} = 1.4 \times 10^{15}$

b. $(1.42 \times 10^{15})(2.4 \times 10^{13})$

$(1.42 \cdot 2.4) \times (10^{15} \cdot 10^{13}) = 3.408 \times 10^{28}$

c. $\dfrac{1.42 \times 10^{-5}}{2 \times 10^{13}}$

$\dfrac{1.42 \times 10^{-5}}{2 \times 10^{13}} = 0.71 \times 10^{-5-13} = 0.71 \times 10^{-18} = 7.1 \times 10^{-19}$

10. Estimate how many times farther Jupiter is from the sun than Earth is from the sun. Estimate how many times farther Pluto is from the sun than Earth is from the sun.

Earth is approximately 1.5×10^{8} km *from the sun, and Jupiter is approximately* 7.8×10^{8} km *from the sun. Therefore, Jupiter is about* 5 *times as far from the sun as Earth is from the sun. Pluto is approximately* 5.9×10^{9} km *from the sun. Therefore, since*

$$\frac{59 \times 10^{8}}{1.5 \times 10^{8}} = \frac{59}{1.5} \approx 39.33,$$

Pluto is approximately 39 *times as far from the sun as Earth is from the sun.*

Closing (3 minutes)

Have students discuss the following question with a partner and record the definition of scientific notation in their mathematics notebooks. Debrief by asking a few students to share their responses with the entire class.

- List two advantages of writing numbers using scientific notation.
  - *You do not have to write as many zeros when working with very large or very small numbers, and you can quickly multiply and divide numbers using the properties of exponents.*

Exit Ticket (4 minutes)

Name ____ Date ____________________

Lesson 2: Base 10 and Scientific Notation

Exit Ticket

1. A sheet of gold foil is 0.28 millionth of a meter thick. Write the thickness of a gold foil sheet measured in centimeters using scientific notation.

2. Without performing the calculation, estimate which expression is larger. Explain how you know.

$$(4 \times 10^{10})(2 \times 10^{5}) \quad \text{and} \quad \frac{4 \times 10^{12}}{2 \times 10^{-4}}$$

Exit Ticket Sample Solutions

1. **A sheet of gold foil is 0.28 millionth of a meter thick. Write the thickness of a gold foil sheet measured in centimeters using scientific notation.**

   The thickness is 0.28×10^{-6} m. In scientific notation, the thickness of a gold foil sheet is 2.8×10^{-7} m, which is 2.8×10^{-5} cm.

2. **Without performing the calculation, estimate which expression is larger. Explain how you know.**

   $$(4 \times 10^{10})(2 \times 10^{5}) \quad \text{and} \quad \frac{4 \times 10^{12}}{2 \times 10^{-4}}$$

   The order of magnitude on the first expression is 15, and the order of magnitude on the second expression is 16. The product and quotient of the number between 1 and 10 in each expression is a number between 1 and 10. Therefore, the second expression is larger than the first one.

Problem Set Sample Solutions

1. **Write the following numbers used in these statements in scientific notation. (Note: Some of these numbers have been rounded.)**

   a. **The density of helium is 0.0001785 gram per cubic centimeter.**

      1.785×10^{-4}

   b. **The boiling point of gold is $5{,}200$°F.**

      5.2×10^{3}

   c. **The speed of light is $186{,}000$ miles per second.**

      1.86×10^{5}

   d. **One second is 0.000278 hour.**

      2.78×10^{-4}

   e. **The acceleration due to gravity on the sun is $900\ \text{ft/s}^2$.**

      9×10^{2}

   f. **One cubic inch is 0.0000214 cubic yard.**

      2.14×10^{-5}

   g. **Earth's population in 2012 was $7{,}046{,}000{,}000$ people.**

      7.046×10^{9}

   h. **Earth's distance from the sun is $93{,}000{,}000$ miles.**

      9.3×10^{7}

i. Earth's radius is $4,000$ miles.

4×10^3

j. The diameter of a water molecule is 0.000000028 cm.

2.8×10^{-8}

2. Write the following numbers in decimal form. (Note: Some of these numbers have been rounded.)

a. A light year is 9.46×10^{15} m.

$9,460,000,000,000,000$

b. Avogadro's number is $6.02 \times 10^{23}\ \text{mol}^{-1}$.

$602,000,000,000,000,000,000,000$

c. The universal gravitational constant is $6.674 \times 10^{-11}\ \text{N}\left(\frac{\text{m}}{\text{kg}}\right)^2$.

0.00000000006674

d. Earth's age is 4.54×10^9 years.

$4,540,000,000$

e. Earth's mass is 5.97×10^{24} kg.

$5,970,000,000,000,000,000,000,000$

f. A foot is 1.9×10^{-4} mile.

0.00019

g. The population of China in 2014 was 1.354×10^9 people.

$1,354,000,000$

h. The density of oxygen is 1.429×10^{-4} grams per liter.

0.0001429

i. The width of a pixel on a smartphone is 7.8×10^{-2} mm.

0.078

j. The wavelength of light used in optic fibers is 1.55×10^{-6} m.

0.00000155

3. **State the necessary value of n that will make each statement true.**

   a. $0.000\,027 = 2.7 \times 10^n$

      -5

   b. $-3.125 = -3.125 \times 10^n$

      0

   c. $7{,}540{,}000{,}000 = 7.54 \times 10^n$

      9

   d. $0.033 = 3.3 \times 10^n$

      -2

   e. $15 = 1.5 \times 10^n$

      1

   f. $26{,}000 \times 200 = 5.2 \times 10^n$

      6

   g. $3000 \times 0.0003 = 9 \times 10^n$

      -1

   h. $0.0004 \times 0.002 = 8 \times 10^n$

      -7

   i. $\frac{16000}{80} = 2 \times 10^n$

      2

   j. $\frac{500}{0.002} = 2.5 \times 10^n$

      5

4. **Perform the following calculations without rewriting the numbers in decimal form.**

   a. $(2.5 \times 10^4) + (3.7 \times 10^3)$

      2.87×10^4

   b. $(6.9 \times 10^{-3}) - (8.1 \times 10^{-3})$

      -1.2×10^{-3}

c. $(6 \times 10^{11})(2.5 \times 10^{-5})$

1.5×10^{7}

d. $\dfrac{4.5 \times 10^{8}}{2 \times 10^{10}}$

2.25×10^{-2}

5. The wavelength of visible light ranges from 650 nanometers to 850 nanometers, where $1 \text{ nm} = 1 \times 10^{-7} \text{ cm}$. Express the range of wavelengths of visible light in centimeters.

Convert 650 nanometers to centimeters: $(6.5 \times 10^{2})(1 \times 10^{-7}) = 6.5 \times 10^{-5}$
Convert 850 nanometers to centimeters: $(8.5 \times 10^{2})(1 \times 10^{-7}) = 8.5 \times 10^{-5}$

The wavelength of visible light in centimeters is 6.5×10^{-5} cm to 8.5×10^{-5} cm.

6. In 1694, the Dutch scientist Antonie van Leeuwenhoek was one of the first scientists to see a red blood cell in a microscope. He approximated that a red blood cell was "$25,000$ times as small as a grain of sand." Assume a grain of sand is $\frac{1}{2}$ mm wide, and a red blood cell is approximately 7 micrometers wide. One micrometer is 1×10^{-6} m. Support or refute Leeuwenhoek's claim. Use scientific notation in your calculations.

Convert millimeters to meters: $(5 \times 10^{-1})(1 \times 10^{-3}) = 5 \times 10^{-4}$. A medium-size grain of sand measures 5×10^{-4} m across. Similarly, a red blood cell is approximately 7×10^{-6} m across. Dividing these numbers produces

$$\frac{5 \times 10^{-4}}{7 \times 10^{-6}} = 0.714 \times 10^{2} = 7.14 \times 10^{1}.$$

So, a red blood cell is 71.4 times as small as a grain of sand. Leeuwenhoek's claim was off by approximately a factor of 350.

7. When the Mars Curiosity Rover entered the atmosphere of Mars on its descent in 2012, it was traveling roughly $13,200$ mph. On the surface of Mars, its speed averaged 0.00073 mph. How many times faster was the speed when it entered the atmosphere than its typical speed on the planet's surface? Use scientific notation in your calculations.

$$\frac{1.32 \times 10^{4}}{7.3 \times 10^{-4}} = 0.18 \times 10^{8} = 1.8 \times 10^{7}$$

The speed when it entered the atmosphere is greater than its surface speed by an order of magnitude of 7.

8. Earth's surface is approximately 70% water. There is no water on the surface of Mars, and its diameter is roughly half of Earth's diameter. Assume both planets are spherical. The radius of Earth is approximately $4,000$ miles. The surface area of a sphere is given by the formula $SA = 4\pi r^{2}$, where r is the radius of the sphere. Which has more land mass, Earth or Mars? Use scientific notation in your calculations.

The surface area of Earth: $4\pi(4000 \text{ mi})^{2} \approx 2 \times 10^{8} \text{ mi}^{2}$

The surface area of Mars: $4\pi(2000 \text{ mi})^{2} \approx 5 \times 10^{7} \text{mi}^{2}$

Thirty percent of Earth's surface area is approximately 6×10^{7} square miles. Earth has more land mass by approximately 20%.

9. There are approximately 25 trillion (2.5×10^{13}) red blood cells in the human body at any one time. A red blood cell is approximately 7×10^{-6} m wide. Imagine if you could line up all your red blood cells end to end. How long would the line of cells be? Use scientific notation in your calculations.

Because $(2.5 \times 10^{13})(7 \times 10^{-6}) = 1.75 \times 10^{8}$, the line of cells would be 1.75×10^{8} m long, which is 1.75×10^{5} km. One mile is equivalent to 1.6 km, so the line of blood cells measures $\frac{1.75 \times 10^{5}}{1.6}$ km, which is approximately $109{,}375$ mi, which is almost halfway to the moon!

10. Assume each person needs approximately 100 square feet of living space. Now imagine that we are going to build a giant apartment building that will be 1 mile wide and 1 mile long to house all the people in the United States, estimated to be 313.9 million people in 2012. If each floor of the apartment building is 10 feet high, how tall will the apartment building be?

Since $(3.139 \times 10^{8})(100) = 3.139 \times 10^{10}$, we need 3.139×10^{10} ft^2 of living space.

Next, divide the total number of square feet by the number of square feet per floor to get the number of needed floors. Remember that $1 \text{ mi}^2 = 5280^2 \text{ ft}^2 = 27\,878\,400 \text{ ft}^2$.

$$\frac{3.139 \times 10^{10} \text{ ft}^2}{2.78784 \times 10^{7} \text{ ft}^2} \approx 1.126 \times 10^{3}$$

Multiplying the number of floors by 10 feet per floor gives a height of $11{,}260$ feet, which is approximately 2.13 miles.

Lesson 3: Rational Exponents—What are $2^{\frac{1}{2}}$ and $2^{\frac{1}{3}}$?

Student Outcomes

- Students calculate quantities that involve positive and negative rational exponents.

Lesson Notes

Students extend their understanding of integer exponents to rational exponents by examining the graph of $f(x) = 2^x$ and estimating the values of $2^{\frac{1}{2}}$ and $2^{\frac{1}{3}}$. The lesson establishes the meaning of these numbers in terms of radical expressions, and these form the basis of how expressions of the form $b^{\frac{1}{n}}$ are defined before generalizing further to expressions of the form $b^{\frac{m}{n}}$, where b is a positive real number and m and n are integers, with $n \neq 0$ (**N-RN.A.1**). The lesson and Problem Set provide fluency practice in applying the properties of exponents to expressions containing rational exponents and radicals (**N-RN.A.2**). In the following lesson, students verify that the definition of an expression with rational exponents, $b^{\frac{m}{n}} = \sqrt[n]{b^m}$, is consistent with the remaining exponential properties. The lesson begins with students creating a graph of a simple exponential function, $f(x) = 2^x$, which they studied in Module 3 of Algebra I. In that module, students also learned about geometric sequences and their relationship to exponential functions, which is a concept that is revisited at the end of this module. In Algebra I, students worked with exponential functions with integer domains. This lesson, together with the subsequent Lessons 4 and 5, helps students understand why the domain of an exponential function $f(x) = b^x$, where b is a positive number and $b \neq 1$, is the set of real numbers. To do so, it is necessary to establish what it means to raise b to a rational power and, in Lesson 5, to any real number power.

Classwork

Opening (1 minute)

In Algebra I, students worked with geometric sequences and simple exponential functions. Remind them that in Lesson 1, they created formulas based on repeatedly doubling and halving a number when they modeled folding a piece of paper. They reviewed how to use the properties of exponents for expressions that had integer exponents.

Scaffolding:

- Encourage students to create a table of values to help them construct the graph.
- For students working above grade level, have them repeat these exercises with the function $f(x) = 3^x$, and ask them to estimate $3^{\frac{1}{2}}$ and $3^{\frac{1}{3}}$.

Opening Exercise (5 minutes)

Have students graph $f(x) = 2^x$ for each integer x from $x = -2$ to $x = 5$ on the axes provided without using a graphing utility or calculator. Discuss the pattern of points, and ask students to connect the points in a way that produces a smooth curve. Students should work these two problems independently. If time permits, have them check their solutions with a partner before leading a whole-class discussion to review the solutions.

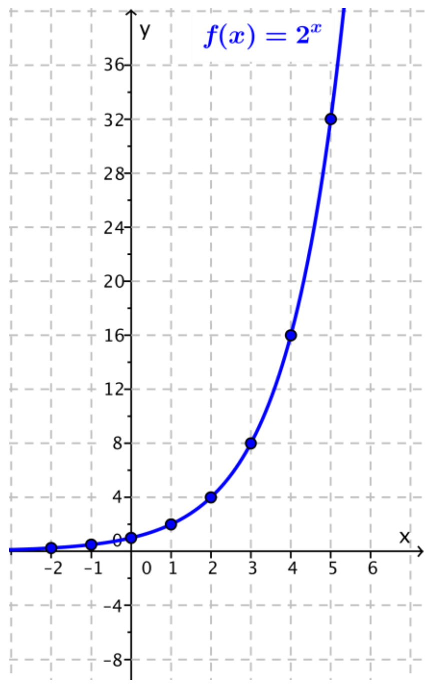
MP.3

Opening Exercise

a. **What is the value of $2^{\frac{1}{2}}$? Justify your answer.**

A possible student response follows: I think it will be around 1.5 because $2^0 = 1$ and $2^1 = 2$.

b. **Graph $f(x) = 2^x$ for each integer x from $x = -2$ to $x = 5$. Connect the points on your graph with a smooth curve.**



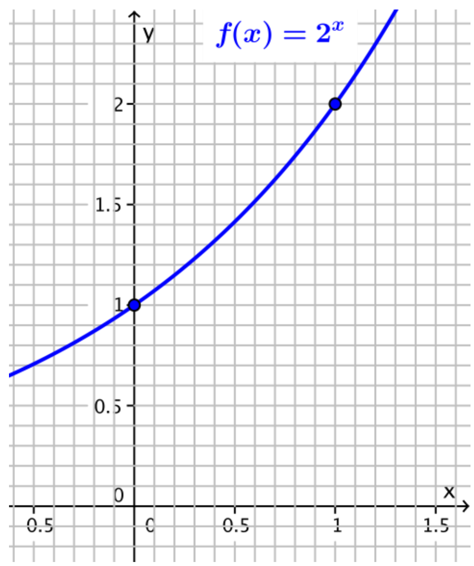
Ask for a few volunteers to explain their reasoning for their answers to Opening Exercise, part (a). Then, debrief these two exercises by leading a short discussion.

- The directions in the Opening Exercise said to connect the points with a smooth curve. What does it imply about the domain of a function when we connect points that we have plotted?
  - *That the domain of the function includes the points between the points that we plotted.*
- How does your graph support or refute your answer to the first exercise? Do you need to modify your answer to the question: What is the value of $2^{\frac{1}{2}}$? Why or why not?
  - *If the domain is all real numbers, then the value of $2^{\frac{1}{2}}$ will be the y-coordinate of the point on the graph where $x = \frac{1}{2}$. From the graph, it looks like the value is somewhere between 1 and 2. The scaling on this graph is not detailed enough for me to accurately refine my answer yet.*

Transition to the next set of exercises by telling students that they can better estimate the value of $2^{\frac{1}{2}}$ by looking at a graph that is scaled more precisely.

The graph of $f(x) = 2^x$ shown below for the next exercises appears in the student materials, but the graph could also be displayed using technology. Have students estimate the value of $2^{\frac{1}{2}}$ and $2^{\frac{1}{3}}$ from the graph. Students should work by themselves or in pairs to work through the following questions without using a calculator.

The graph on the right shows a close-up view of $f(x) = 2^x$ for $-0.5 < x < 1.5$.



c. **Find two consecutive integers that are under and over estimates of the value of $2^{\frac{1}{2}}$.**

$2^0 < 2^{\frac{1}{2}} < 2^1$

$1 < 2^{\frac{1}{2}} < 2$

d. **Does it appear that $2^{\frac{1}{2}}$ is halfway between the integers you specified in Exercise 1?**

No. It looks like $2^{\frac{1}{2}}$ is a little less than halfway between 1 and 2.

e. **Use the graph of $f(x) = 2^x$ to estimate the value of $2^{\frac{1}{2}}$.**

$2^{\frac{1}{2}} \approx 1.4$

f. **Use the graph of $f(x) = 2^x$ to estimate the value of $2^{\frac{1}{3}}$.**

$2^{\frac{1}{3}} \approx 1.25$

Discussion (9 minutes)

Before getting into the point of this lesson, which is to connect rational exponents to radical expressions, revisit the initial question with students.

MP.6

- What is the value of $2^{\frac{1}{2}}$? Does anyone want to adjust his or her initial guess?
  - *Our initial guess was a little too big. It seems like 1.4 might be a better answer.*
- How could we make a better guess?
  - *We could look at the graph with even smaller increments for the scale using technology.*

> *Scaffolding:*
>
> If needed, demonstrate the argument using perfect squares. For example, use a base of 4 instead of a base of 2.
>
> Show that $\left(4^{\frac{1}{2}}\right)^2 = 4$ and $\left(\sqrt{4}\right)^2 = 4$.

If time permits, zoom in further on the graph of $f(x) = 2^x$ using a graphing calculator or other technology either by examining a graph or a table of values of x closer and closer to $\frac{1}{2}$.

Next, make the connection that $2^{\frac{1}{2}} = \sqrt{2}$. Walk students through the following questions, providing guidance as needed. Students proved that there was only one positive number that squared to 2 in Geometry, Module 2. It may be necessary to remind them of this with a bit more detail if they are struggling to follow this argument.

MP.7

- Assume for the moment that whatever $2^{\frac{1}{2}}$ means, it satisfies our known rule for integer exponents $b^m \cdot b^n = b^{m+n}$. Working with this assumption, what is the value of $2^{\frac{1}{2}} \cdot 2^{\frac{1}{2}}$?
  - *It would be 2 because $2^{\frac{1}{2}} \cdot 2^{\frac{1}{2}} = 2^{\frac{1}{2}+\frac{1}{2}} = 2^1 = 2$.*
- What unique positive number squares to 2? That is, what is the only positive number that when multiplied by itself is equal to 2?
  - *By definition, we call the unique positive number that squares to 2 the square root of 2, and we write $\sqrt{2}$.*

Write the following statements on the board, and ask students to compare them and think about what the statements must tell them about the meaning of $2^{\frac{1}{2}}$.

$$2^{\frac{1}{2}} \cdot 2^{\frac{1}{2}} = 2 \text{ and } \sqrt{2} \cdot \sqrt{2} = 2$$

- What do these two statements tell us about the meaning of $2^{\frac{1}{2}}$?
  - *Since both statements involve multiplying a number by itself and getting 2, and we know that there is only one number that does that, we can conclude that $2^{\frac{1}{2}} = \sqrt{2}$.*

At this point, have students confirm these results by using a calculator to approximate both $2^{\frac{1}{2}}$ and $\sqrt{2}$ to several decimal places. In the Opening, $2^{\frac{1}{2}}$ was approximated graphically, and now it has been shown to be an irrational number.

MP.7

Next, ask students to think about the meaning of $2^{\frac{1}{3}}$ using a similar line of reasoning.

> *Scaffolding:*
>
> If needed, demonstrate the argument using perfect cubes. For example, use a base of 8 instead of a base of 2.
>
> Show $\left(8^{\frac{1}{3}}\right)^3 = 8$
>
> and $\left(\sqrt[3]{8}\right)^3 = 8$.

- Assume that whatever $2^{\frac{1}{3}}$ means will satisfy $b^m \cdot b^n = b^{m+n}$. What is the value of $\left(2^{\frac{1}{3}}\right)\left(2^{\frac{1}{3}}\right)\left(2^{\frac{1}{3}}\right)$?
  - *The value is 2 because $\left(2^{\frac{1}{3}}\right)\left(2^{\frac{1}{3}}\right)\left(2^{\frac{1}{3}}\right) = 2^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = 2^1 = 2$.*
- What is the value of $\sqrt[3]{2} \cdot \sqrt[3]{2} \cdot \sqrt[3]{2}$?
  - *The value is 2 because $\sqrt[3]{2} \cdot \sqrt[3]{2} \cdot \sqrt[3]{2} = \left(\sqrt[3]{2}\right)^3 = 2$.*
- What appears to be the meaning of $2^{\frac{1}{3}}$?
  - *Since both the exponent expression and the radical expression involve multiplying a number by itself three times and the result is equal to 2, we know that $2^{\frac{1}{3}} = \sqrt[3]{2}$.*

Students can also confirm using a calculator that the decimal approximations of $2^{\frac{1}{3}}$ and $\sqrt[3]{2}$ are the same. Next, they are asked to generalize their findings.

MP.8

- Can we generalize this relationship? Does $2^{\frac{1}{4}} = \sqrt[4]{2}$? Does $2^{\frac{1}{10}} = \sqrt[10]{2}$? What is $2^{\frac{1}{n}}$, for any positive integer n? Why?
  - $2^{\frac{1}{n}} = \sqrt[n]{2}$ *because*

$$\left(2^{\frac{1}{n}}\right)^n = \underbrace{\left(2^{\frac{1}{n}}\right)\left(2^{\frac{1}{n}}\right)\cdots\left(2^{\frac{1}{n}}\right)}_{n\text{ times}} = 2^{\overbrace{\frac{1}{n}+\frac{1}{n}+\cdots+\frac{1}{n}}^{n\text{ times}}} = 2^1 = 2.$$

Have students confirm these results using a calculator as well as checking to see if the decimal approximations of $2^{\frac{1}{n}}$ and $\sqrt[n]{2}$ are the same for different values of n such as $4, 5, 6, 10, \ldots$. Be sure to share the generalization shown above on the board to help students understand why it makes sense to define $2^{\frac{1}{n}}$ to be $\sqrt[n]{2}$.

However, be careful to not stop here; there is a problem with the reasoning if $\sqrt[n]{2}$ is not defined. In previous courses, only square roots and cube roots were defined.

It is first necessary to define the n^{th} root of a number; there may be more than one, as in the case where $2^2 = 4$ and $(-2)^2 = 4$. It is said that both -2 and 2 are square roots of 4. However, priority is given to the positive-valued square root, and it is said that 2 is the *principal square root* of 4. Often *the* square root of 4 is referred to when what is meant is the *principal* square root of 4. The definition of n^{th} root presented below is consistent with allowing complex n^{th} roots, which students encounter in Precalculus and in college if they pursue engineering or higher mathematics. If complex n^{th} roots are allowed, there are three cube roots of 2: $\sqrt[3]{2}$, $\sqrt[3]{2}\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)$, and $\sqrt[3]{2}\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)$, and the real number $\sqrt[3]{2}$ is referred to as the *principal cube root* of 2. There is no need to discuss this with any but the most advanced students.

The n^{th} root of 2, $\sqrt[n]{2}$, is the positive real number a such that $a^n = 2$. In general, if a is positive, then the n^{th} root of a exists for any positive integer n, and if a is negative, then the n^{th} root of a exists only for odd integers n. This even/odd condition is handled subtly in the definition below; the n^{th} root exists only if there is already an exponential relationship $b = a^n$.

Present the following definitions to students, and have them record them in their notes.

n^{TH} ROOT OF A NUMBER: Let a and b be numbers, and let n be a positive integer. If $b = a^n$, then a is a n^{th} *root of* b. If $n = 2$, then the root is a called a *square root*. If $n = 3$, then the root is called a *cube root*.

PRINCIPAL n^{TH} ROOT OF A NUMBER: Let b be a real number that has at least one real n^{th} root. The *principal* n^{th} *root of* b is the real n^{th} root that has the same sign as b and is denoted by a radical symbol: $\sqrt[n]{b}$.

Every positive number has a unique principal n^{th} root. We often refer to the principal n^{th} root of b as just the n^{th} *root of* b. For any positive integer n, the n^{th} root of 0 is 0.

Students have already learned about square and cube roots in previous grades. In Module 1 and at the beginning of this lesson, students worked with radical expressions involving cube and square roots. Explain that the n^{th} roots of a number satisfy the same properties of radicals learned previously. Have students record these properties in their notes.

> If $a \geq 0$, $b \geq 0$ ($b \neq 0$ when b is a denominator), and n is a positive integer, then
> $$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b} \quad \text{and} \quad \sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}.$$

Background information regarding n^{th} roots and their uniqueness is provided below. Consider sharing this with students working above grade level or the entire class if extending this lesson to an additional day.

The existence of the principal n^{th} root of a positive real number b is a consequence of the fundamental theorem of algebra: Consider the polynomial function $f(x) = x^n - b$. When n is odd, we know that f has at least one real zero because the graph of f must cross the x-axis. That zero is a positive number, which (after showing that it is the only real zero) is the n^{th} root. The case for when n is even follows a similar argument.

To show uniqueness of the n^{th} root, suppose there are two n^{th} roots of a number b, x, and y, such that $x > 0$, $y > 0$, $x^n = b$, and $y^n = b$. Then, $x^n - y^n = b - b = 0$, and the expression $x^n - y^n$ factors (see Lesson 7 in Module 1).

$$0 = x^n - y^n$$
$$0 = (x - y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + xy^{n-2} + y^{n-1})$$

Since both x and y are positive, the second factor is never zero. Thus, for $x^n - y^n = 0$, we must have $x - y = 0$, and it follows that $x = y$. Thus, there is only one n^{th} root of b.

A proof of the first radical property is shown below for background information. Consider sharing this proof with students working above grade level or the entire class if extending this lesson to an additional day.

Prove that $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$.

Let $x \geq 0$ be the number such that $x^n = a$, and let $y \geq 0$ be the number such that $y^n = b$, so that $x = \sqrt[n]{a}$ and $y = \sqrt[n]{b}$. Then, by a property of exponents, $(xy)^n = x^n y^n = ab$. Thus, xy must be the n^{th} root of ab. Writing this using our notation gives

$$\sqrt[n]{ab} = xy = \sqrt[n]{a} \cdot \sqrt[n]{b}.$$

Example (3 minutes)

This example familiarizes students with the wording in the definition presented above.

> **Example**
>
> a. **What is the 4^{th} root of 16?**
>
> ***$x^4 = 16$ when $x = 2$ because $2^4 = 16$. Thus, $\sqrt[4]{16} = 2$.***
>
> b. **What is the cube root of 125?**
>
> ***$x^3 = 125$ when $x = 5$ because $5^3 = 125$. Thus, $\sqrt[3]{125} = 5$.***

c. **What is the 5^{th} root of $100,000$?**

$x^5 = 100,000$ when $x = 10$ because $10^5 = 100\,000$. Thus, $\sqrt[5]{100,000} = 10$.

Exercise 1 (2 minutes)

In these brief exercises, students work with the definition of n^{th} roots and the multiplication property presented above. Have students check their work with a partner and briefly discuss any questions that arise as a whole class.

Exercise 1

1. **Evaluate each expression.**

   a. $\sqrt[4]{81}$

   3

   b. $\sqrt[5]{32}$

   2

   c. $\sqrt[3]{9} \cdot \sqrt[3]{3}$

   $\sqrt[3]{27} = 3$

   d. $\sqrt[4]{25} \cdot \sqrt[4]{100} \cdot \sqrt[4]{4}$

   $\sqrt[4]{10,000} = 10$

Scaffolding:

If needed, continue to support students who struggle with abstraction by including additional numeric examples.

Discussion (8 minutes)

Return to the question posted in the title of this lesson: What are $2^{\frac{1}{2}}$ and $2^{\frac{1}{3}}$? Now we know the answer, $2^{\frac{1}{2}} = \sqrt{2}$ and $2^{\frac{1}{3}} = \sqrt[3]{2}$. So far, we have given meaning to $2^{\frac{1}{n}}$ by equating $2^{\frac{1}{n}}$ and $\sqrt[n]{2}$. Ask students if they believe these results extend to any base $b > 0$.

- We just did some exercises where the b-value was a number different from 2. In our earlier work, was there something special about using 2 as the base of our exponential expression? Would these results generalize to expressions of the form $3^{\frac{1}{n}}$? $7^{\frac{1}{n}}$? $10^{\frac{1}{n}}$? $b^{\frac{1}{n}}$, for any positive real number b?
  - *There is nothing inherently special about the base 2 in the above discussion. These results should generalize to expressions of the form $b^{\frac{1}{n}}$ for any positive real number base b because we have defined an n^{th} root for any positive base b.*

Now that we know the meaning of the n^{th} root of a number b, $\sqrt[n]{b}$, the work done earlier with base 2 suggests that $b^{\frac{1}{n}}$ should also be defined as the n^{th} root of b. Discuss the definition below with the class. If students are unclear on the definition, do some numerical examples. For example, $(-32)^{\frac{1}{5}} = -2$ because $(-2)^5 = -32$, but $(-16)^{\frac{1}{4}}$ does not exist because there is no principal 4^{th} root of a negative number.

For a real number b and a positive integer n, define $b^{\frac{1}{n}}$ to be the principal n^{th} root of b when it exists. That is,

$$b^{\frac{1}{n}} = \sqrt[n]{b}.$$

Note that the definition of $\sqrt[n]{b}$ holds for any real number b if n is an odd integer and for positive real numbers b if n is an even integer. Consider emphasizing this with the class. Thus, when b is negative and n is an odd integer, the expression $b^{\frac{1}{n}}$ is negative. If n is an even integer, then b must be restricted to positive real numbers only, and $b^{\frac{1}{n}}$ is positive.

In the next lesson, students see that with this definition, $b^{\frac{1}{n}}$ satisfies all the usual properties of exponents, so it makes sense to define it in this way.

At this point, consider revisiting the original question with students one more time.

- What is the value of $2^{\frac{1}{2}}$? What does it mean? What is the value of $b^{\frac{1}{2}}$ for any positive number b? How are radicals related to rational exponents?
  - *We estimated numerically that $2^{\frac{1}{2}} \approx 1.4$ in part (e) of the Opening Exercise. We now know that $2^{\frac{1}{2}}$ is equal to $\sqrt{2}$. In general, $b^{\frac{1}{2}} = \sqrt{b}$ for any positive real number b. A number that contains a radical can be expressed using rational exponents in place of the radical.*

In the following Discussion, the definition of exponentiation with exponents of the form $\frac{1}{n}$ is extended to exponentiation with any positive rational exponent. Begin by considering an example: What do we mean by $2^{\frac{3}{4}}$? Give students a few minutes to respond individually in writing to this question on their student pages, and then have them discuss their reasoning with a partner. Make sure to correct any blatant misconceptions and to clarify incomplete thinking while leading the Discussion that follows.

MP.3

Discussion

If $2^{\frac{1}{2}} = \sqrt{2}$ and $2^{\frac{1}{3}} = \sqrt[3]{2}$, what does $2^{\frac{3}{4}}$ equal? Explain your reasoning.

Student solutions and explanations will vary. One possible solution would be $2^{\frac{3}{4}} = \left(2^{\frac{1}{4}}\right)^3$, so it must mean that $2^{\frac{3}{4}} = \left(\sqrt[4]{2}\right)^3$. Since the properties of exponents and the meaning of an exponent made sense with integers and now for rational numbers in the form $\frac{1}{n}$, it would make sense that they would work for all rational numbers, too.

Now that there is a definition for exponential expressions of the form $b^{\frac{1}{n}}$, use the discussion below to define $b^{\frac{m}{n}}$, where m and n are integers, $n \neq 0$, and b is a positive real number. Make sure students understand that the interpretation of $b^{\frac{m}{n}}$ must be consistent with the exponent properties (which hold for integer exponents) and the definition of $b^{\frac{1}{n}}$.

- How can we rewrite the exponent of $2^{\frac{3}{4}}$ using integers and rational numbers in the form $\frac{1}{n}$?
  - *We can write $2^{\frac{3}{4}} = \left(2^{\frac{1}{4}}\right)^3$, or we can write $2^{\frac{3}{4}} = (2^3)^{\frac{1}{4}}$.*

- Now, apply our definition of $b^{\frac{1}{n}}$.
  - $2^{\frac{3}{4}} = \left(2^{\frac{1}{4}}\right)^3 = \left(\sqrt[4]{2}\right)^3$ *or* $2^{\frac{3}{4}} = (2^3)^{\frac{1}{4}} = \sqrt[4]{2^3} = \sqrt[4]{8}$
- Does this make sense? If $2^{\frac{3}{4}} = \sqrt[4]{8}$, then if we raise $2^{\frac{3}{4}}$ to the fourth power, we should get 8. Does this happen?
  - $\left(2^{\frac{3}{4}}\right)^4 = \left(2^{\frac{3}{4}}\right)\left(2^{\frac{3}{4}}\right)\left(2^{\frac{3}{4}}\right)\left(2^{\frac{3}{4}}\right) = 2^{\left(4\cdot\frac{3}{4}\right)} = 2^3 = 8$
  - *So, 8 is the product of four equal factors, which we denote by $2^{\frac{3}{4}}$. Thus, $2^{\frac{3}{4}} = \sqrt[4]{8}$.*

Take a few minutes to allow students to think about generalizing their work above to $2^{\frac{m}{n}}$ and then to $b^{\frac{m}{n}}$. Have them write a response to the following questions and share it with a partner before proceeding as a whole class.

- Can we generalize this result? How would you define $2^{\frac{m}{n}}$, for positive integers m and n?
  - *Conjecture: $2^{\frac{m}{n}} = \sqrt[n]{2^m}$, or equivalently, $2^{\frac{m}{n}} = \left(\sqrt[n]{2}\right)^m$.*
- Can we generalize this result to any positive real number base b? What is $3^{\frac{m}{n}}$? $7^{\frac{m}{n}}$? $10^{\frac{m}{n}}$? $b^{\frac{m}{n}}$?
  - *There is nothing inherently special about the base 2 in the above Discussion. These results should generalize to expressions of the form $b^{\frac{m}{n}}$ for any positive real number base b.*
  - *Then we are ready to define $b^{\frac{m}{n}} = \sqrt[n]{b^m}$ for positive integers m and n and positive real numbers b.*

This result is summarized in the box below.

> For any positive integers m and n, and any real number b for which $b^{\frac{1}{n}}$ exists, we define
>
> $b^{\frac{m}{n}} = \sqrt[n]{b^m}$, which is equivalent to $b^{\frac{m}{n}} = \left(\sqrt[n]{b}\right)^m$.

Note that this property holds for any real number b if n is an odd integer. Consider emphasizing this with the class. When b is negative and n is an odd integer, the expression $b^{\frac{m}{n}}$ is negative. If n is an even integer, then b must be restricted to positive real numbers only.

Exercises 2–8 (4 minutes)

In these exercises, students use the definitions above to rewrite and evaluate expressions. Have students check their work with a partner and briefly discuss as a whole class any questions that arise.

> **Exercises 2–12**
>
> **Rewrite each exponential expression as a radical expression.**
>
> **2.** $3^{\frac{1}{2}}$
>
> $3^{\frac{1}{2}} = \sqrt{3}$

3. $11^{\frac{1}{5}}$

$11^{\frac{1}{5}} = \sqrt[5]{11}$

4. $\left(\frac{1}{4}\right)^{\frac{1}{5}}$

$\left(\frac{1}{4}\right)^{\frac{1}{5}} = \sqrt[5]{\frac{1}{4}}$

5. $6^{\frac{1}{10}}$

$6^{\frac{1}{10}} = \sqrt[10]{6}$

Rewrite the following exponential expressions as equivalent radical expressions. If the number is rational, write it without radicals or exponents.

6. $2^{\frac{3}{2}}$

$2^{\frac{3}{2}} = \sqrt{2^3} = 2\sqrt{2}$

7. $4^{\frac{5}{2}}$

$4^{\frac{5}{2}} = \sqrt{4^5} = \left(\sqrt{4}\right)^5 = 2^5 = 32$

8. $\left(\frac{1}{8}\right)^{\frac{5}{3}}$

$\left(\frac{1}{8}\right)^{\frac{5}{3}} = \sqrt[3]{\left(\frac{1}{8}\right)^5} = \left(\sqrt[3]{\frac{1}{8}}\right)^5 = \left(\frac{1}{2}\right)^5 = \frac{1}{32}$

Exercise 9 (3 minutes)

In this exercise, students are asked to consider a negative rational exponent. Have students work directly with a partner, and ask them to use thinking similar to that in the preceding Discussion. Correct and extend student thinking while reviewing the solution.

9. **Show why the following statement is true:**

$2^{-\frac{1}{2}} = \frac{1}{2^{\frac{1}{2}}}$

Student solutions and explanations will vary. One possible solution would be

$2^{-\frac{1}{2}} = \left(2^{\frac{1}{2}}\right)^{-1} = \left(\sqrt{2}\right)^{-1} = \frac{1}{\sqrt{2}} = \frac{1}{2^{\frac{1}{2}}}.$

Share the following property with the class, and show how the work they did in the previous exercises supports this conclusion. Consider verifying these properties using an argument similar to the ones presented earlier for the meaning of $b^{\frac{m}{n}}$.

> For any positive integers m and n, and any nonzero number b for which $b^{\frac{1}{n}}$ exists, we define
> $$b^{-\frac{m}{n}} = \frac{1}{\sqrt[n]{b^m}}$$
> or, equivalently,
> $$b^{-\frac{m}{n}} = \frac{1}{\left(\sqrt[n]{b}\right)^m}.$$

Exercises 10–12 (3 minutes)

Rewrite the following exponential expressions as equivalent radical expressions. If the number is rational, write it without radicals or exponents.

10. $4^{-\frac{3}{2}}$

$$4^{-\frac{3}{2}} = \frac{1}{\sqrt{4^3}} = \frac{1}{\left(\sqrt{4}\right)^3} = \frac{1}{8}$$

11. $27^{-\frac{2}{3}}$

$$27^{-\frac{2}{3}} = \frac{1}{27^{\frac{2}{3}}} = \frac{1}{\left(\sqrt[3]{27}\right)^2} = \frac{1}{3^2} = \frac{1}{9}$$

12. $\left(\frac{1}{4}\right)^{-\frac{1}{2}}$

We have $\left(\frac{1}{4}\right)^{-\frac{1}{2}} = \left(\sqrt{\frac{1}{4}}\right)^{-1} = \left(\frac{1}{2}\right)^{-1} = 2$. *Alternatively,* $\left(\frac{1}{4}\right)^{-\frac{1}{2}} = \left(\left(\frac{1}{4}\right)^{-1}\right)^{\frac{1}{2}} = (4)^{\frac{1}{2}} = \sqrt{4} = 2$.

Closing (3 minutes)

Have students summarize the key points of the lesson in writing. Circulate around the classroom to informally assess understanding and provide assistance. Student work should reflect the summary provided below.

Lesson Summary

n^{TH} ROOT OF A NUMBER: Let a and b be numbers, and let n be a positive integer. If $b = a^n$, then a is a n^{th} *root of* b. If $n = 2$, then the root is a called a *square root*. If $n = 3$, then the root is called a *cube root*.

PRINCIPAL n^{TH} ROOT OF A NUMBER: Let b be a real number that has at least one real n^{th} root. The *principal* n^{th} *root of* b is the real n^{th} root that has the same sign as b and is denoted by a radical symbol: $\sqrt[n]{b}$.

Every positive number has a unique principal n^{th} root. We often refer to the principal n^{th} root of b as just *the* n^{th} *root of* b. The n^{th} root of 0 is 0.

For any positive integers m and n, and any real number b for which the principal n^{th} root of b exists, we have

$$b^{\frac{1}{n}} = \sqrt[n]{b}$$
$$b^{\frac{m}{n}} = \sqrt[n]{b^m} = \left(\sqrt[n]{b}\right)^m$$
$$b^{-\frac{m}{n}} = \frac{1}{\sqrt[n]{b^m}} \text{ for } b \neq 0.$$

Exit Ticket (4 minutes)

Name ______________________________ Date ________________

Lesson 3: Rational Exponents—What are $2^{\frac{1}{2}}$ and $2^{\frac{1}{3}}$?

Exit Ticket

1. Write the following exponential expressions as equivalent radical expressions.

   a. $2^{\frac{1}{2}}$

   b. $2^{\frac{3}{4}}$

   c. $3^{-\frac{2}{3}}$

2. Rewrite the following radical expressions as equivalent exponential expressions.

   a. $\sqrt{5}$

   b. $2\sqrt[4]{3}$

   c. $\frac{1}{\sqrt[3]{16}}$

3. Provide a written explanation for each question below.

   a. Is it true that $\left(1000^{\frac{1}{3}}\right)^3 = \left(1000^3\right)^{\frac{1}{3}}$? Explain how you know.

   b. Is it true that $\left(4^{\frac{1}{2}}\right)^3 = \left(4^3\right)^{\frac{1}{2}}$? Explain how you know.

   c. Suppose that m and n are positive integers and b is a real number so that the principal n^{th} root of b exists. In general, does $\left(b^{\frac{1}{n}}\right)^m = \left(b^m\right)^{\frac{1}{n}}$? Explain how you know.

Exit Ticket Sample Solutions

1. **Rewrite the following exponential expressions as equivalent radical expressions.**

   a. $2^{\frac{1}{2}}$

   $2^{\frac{1}{2}} = \sqrt{2}$

   b. $2^{\frac{3}{4}}$

   $2^{\frac{3}{4}} = \sqrt[4]{2^3} = \sqrt[4]{8}$

   c. $3^{-\frac{2}{3}}$

   $3^{-\frac{2}{3}} = \frac{1}{\sqrt[3]{3^2}} = \frac{1}{\sqrt[3]{9}}$

2. **Rewrite the following radical expressions as equivalent exponential expressions.**

   a. $\sqrt{5}$

   $\sqrt{5} = 5^{\frac{1}{2}}$

   b. $2\sqrt[4]{3}$

   $2\sqrt[4]{3} = \sqrt[4]{2^4 \cdot 3} = \sqrt[4]{48} = 48^{\frac{1}{4}}$

   c. $\frac{1}{\sqrt[3]{16}}$

   $\frac{1}{\sqrt[3]{16}} = (2^4)^{-\frac{1}{3}} = 2^{-\frac{4}{3}}$

   $\frac{1}{\sqrt[3]{16}} = (16)^{-\frac{1}{3}}$

3. **Provide a written explanation for each question below.**

   a. **Is it true that $\left(1000^{\frac{1}{3}}\right)^3 = (1000^3)^{\frac{1}{3}}$? Explain how you know.**

   $\left(1000^{\frac{1}{3}}\right)^3 = \left(\sqrt[3]{1000}\right)^3 = 10^3 = 1000$

   $(1000^3)^{\frac{1}{3}} = (1000000000)^{\frac{1}{3}} = 1000$

   So, this statement is true.

   b. **Is it true that $\left(4^{\frac{1}{2}}\right)^3 = (4^3)^{\frac{1}{2}}$? Explain how you know.**

   $\left(4^{\frac{1}{2}}\right)^3 = \left(\sqrt{4}\right)^3 = 2^3 = 8$

   $(4^3)^{\frac{1}{2}} = 64^{\frac{1}{2}} = \sqrt{64} = 8$

   So, this statement is true.

MP.3 & MP.7

c. Suppose that m and n are positive integers and b is a real number so that the principal n^{th} root of b exists. In general, does $\left(b^{\frac{1}{n}}\right)^m = (b^m)^{\frac{1}{n}}$? Explain how you know.

From the two examples we have seen, it appears that we can extend the property $\left(b^{\frac{1}{n}}\right)^m = (b^m)^{\frac{1}{n}}$ for integers m and n to rational exponents.

We know that, in general, we have

$$\begin{aligned}\left(b^{\frac{1}{n}}\right)^m &= \left(\sqrt[n]{b}\right)^m \\ &= \sqrt[n]{b^m} \\ &= (b^m)^{\frac{1}{n}}.\end{aligned}$$

Problem Set Sample Solutions

1. Select the expression from (A), (B), and (C) that correctly completes the statement.

		(A)	(B)	(C)
a.	$x^{\frac{1}{3}}$ is equivalent to ________.	$\frac{1}{3}x$	$\sqrt[3]{x}$	$\frac{3}{x}$
b.	$x^{\frac{2}{3}}$ is equivalent to ________.	$\frac{2}{3}x$	$\sqrt[3]{x^2}$	$\left(\sqrt{x}\right)^3$
c.	$x^{-\frac{1}{4}}$ is equivalent to ________.	$-\frac{1}{4}x$	$\frac{4}{x}$	$\frac{1}{\sqrt[4]{x}}$
d.	$\left(\frac{4}{x}\right)^{\frac{1}{2}}$ is equivalent to ________.	$\frac{2}{x}$	$\frac{4}{x^2}$	$\frac{2}{\sqrt{x}}$

a. *(B)*

b. *(B)*

c. *(C)*

d. *(C)*

2. Identify which of the expressions (A), (B), and (C) are equivalent to the given expression.

		(A)	(B)	(C)
a.	$16^{\frac{1}{2}}$	$\left(\frac{1}{16}\right)^{-\frac{1}{2}}$	$8^{\frac{2}{3}}$	$64^{\frac{3}{2}}$
b.	$\left(\frac{2}{3}\right)^{-1}$	$-\frac{3}{2}$	$\left(\frac{9}{4}\right)^{\frac{1}{2}}$	$\frac{27^{\frac{1}{3}}}{6}$

a. *(A) and (B)*

b. *(B) only*

3. **Rewrite in radical form. If the number is rational, write it without using radicals.**

a. $6^{\frac{3}{2}}$

$\sqrt{216}$

b. $\left(\frac{1}{2}\right)^{\frac{1}{4}}$

$\sqrt[4]{\frac{1}{2}}$

c. $3(8)^{\frac{1}{3}}$

$3\sqrt[3]{8} = 6$

d. $\left(\frac{64}{125}\right)^{-\frac{2}{3}}$

$\left(\sqrt[3]{\frac{125}{64}}\right)^2 = \frac{25}{16}$

e. $81^{-\frac{1}{4}}$

$\frac{1}{\sqrt[4]{81}} = \frac{1}{3}$

4. **Rewrite the following expressions in exponent form.**

a. $\sqrt{5}$

$5^{\frac{1}{2}}$

b. $\sqrt[3]{5^2}$

$5^{\frac{2}{3}}$

c. $\sqrt{5^3}$

$5^{\frac{3}{2}}$

d. $\left(\sqrt[3]{5}\right)^2$

$5^{\frac{2}{3}}$

5. **Use the graph of $f(x) = 2^x$ shown to the right to estimate the following powers of 2.**

a. $2^{\frac{1}{4}}$ ≈ 1.2

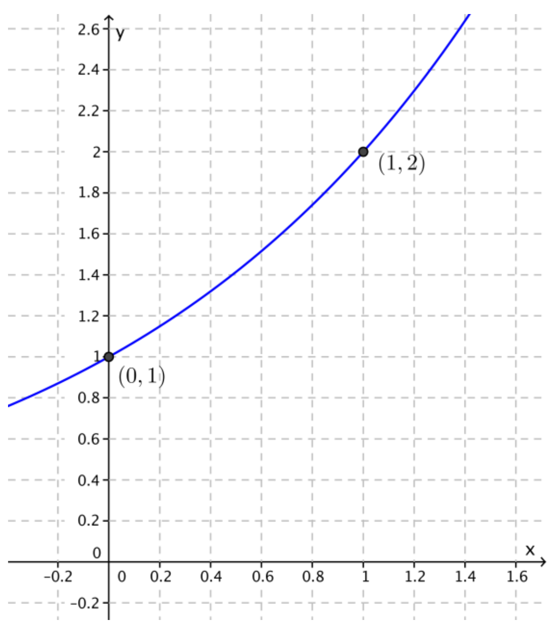
b. $2^{\frac{2}{3}}$ ≈ 1.6

c. $2^{\frac{3}{4}}$ ≈ 1.7

d. $2^{0.2}$ ≈ 1.15

e. $2^{1.2}$ ≈ 2.3

f. $2^{-\frac{1}{5}}$ ≈ 0.85



6. Rewrite each expression in the form kx^n, where k is a real number, x is a positive real number, and n is rational.

a. $\sqrt[4]{16x^3}$

$2x^{\frac{3}{4}}$

b. $\frac{5}{\sqrt{x}}$

$5x^{-\frac{1}{2}}$

c. $\sqrt[3]{\ }$

$x^{-\frac{4}{3}}$

d. $\frac{4}{\sqrt[3]{8x^3}}$

$2x^{-1}$

e. $\frac{27}{\sqrt{9x^4}}$

$9x^{-2}$

f. $\left(\frac{125}{x^2}\right)^{-\frac{1}{3}}$

$\frac{1}{5}x^{\frac{2}{3}}$

7. Find the value of x for which $2x^{\frac{1}{2}} = 32$.

256

8. Find the value of x for which $x^{\frac{4}{3}} = 81$.

27

9. If $x^{\frac{3}{2}} = 64$, find the value of $4x^{-\frac{3}{4}}$.

$x = 16$, *so* $4(16)^{-\frac{3}{4}} = 4(2)^{-3} = \frac{4}{8} = \frac{1}{2}$.

10. Evaluate the following expressions when $b = \frac{1}{9}$.

a. $b^{-\frac{1}{2}}$

$\left(\frac{1}{9}\right)^{-\frac{1}{2}} = 9^{\frac{1}{2}} = 3$

b. $b^{\frac{5}{2}}$

$\left(\frac{1}{9}\right)^{\frac{5}{2}} = \left(\frac{1}{3}\right)^5 = \frac{1}{243}$

c. $\sqrt[3]{3b^{-1}}$

$\sqrt[3]{3\left(\frac{1}{9}\right)^{-1}} = \sqrt[3]{27} = 3$

11. Show that each expression is equivalent to $2x$. Assume x is a positive real number.

a. $\sqrt[4]{16x^4}$

$\sqrt[4]{16} \cdot \sqrt[4]{x^4} = 2x$

b. $\dfrac{\left(\sqrt[3]{8x^3}\right)^2}{\sqrt{4x^2}}$

$\dfrac{(2x)^2}{(4x^2)^{\frac{1}{2}}} = \dfrac{4x^2}{2x} = 2x$

c. $\dfrac{6x^3}{\sqrt[3]{27x^6}}$

$\dfrac{6x^3}{3x^{\frac{6}{3}}} = \dfrac{6x^3}{3x^2} = 2x$

MP.3

12. Yoshiko said that $16^{\frac{1}{4}} = 4$ because 4 is one-fourth of 16. Use properties of exponents to explain why she is or is not correct.

Yoshiko's reasoning is not correct. By our exponent properties, $\left(16^{\frac{1}{4}}\right)^4 = 16^{\left(\frac{1}{4}\right)\cdot 4} = 16^1 = 16$, *but* $4^4 = 256$. *Since* $\left(16^{\frac{1}{4}}\right)^4 \neq 4^4$, *we know that* $16^{\frac{1}{4}} \neq 4$.

13. Jefferson said that $8^{\frac{4}{3}} = 16$ because $8^{\frac{1}{3}} = 2$ and $2^4 = 16$. Use properties of exponents to explain why he is or is not correct.

Jefferson's reasoning is correct. We know that $8^{\frac{4}{3}} = \left(8^{\frac{1}{3}}\right)^4$, *so* $8^{\frac{4}{3}} = 2^4$, *and thus* $8^{\frac{4}{3}} = 16$.

14. Rita said that $8^{\frac{2}{3}} = 128$ because $8^{\frac{2}{3}} = 8^2 \cdot 8^{\frac{1}{3}}$, so $8^{\frac{2}{3}} = 64 \cdot 2$, and then $8^{\frac{2}{3}} = 128$. Use properties of exponents to explain why she is or is not correct.

Rita's reasoning is not correct because she did not apply the properties of exponents correctly. She should also realize that raising 8 *to a positive power less than* 1 *would produce a number less than* 8. *The correct calculation is below.*

$$\begin{aligned} 8^{\frac{2}{3}} &= \left(8^{\frac{1}{3}}\right)^2 \\ &= 2^2 \\ &= 4 \end{aligned}$$

15. Suppose for some positive real number a that $\left(a^{\frac{1}{4}} \cdot a^{\frac{1}{2}} \cdot a^{\frac{1}{4}}\right)^2 = 3$.

   a. What is the value of a?

$$\left(a^{\frac{1}{4}} \cdot a^{\frac{1}{2}} \cdot a^{\frac{1}{4}}\right)^2 = 3$$
$$\left(a^{\frac{1}{4}+\frac{1}{2}+\frac{1}{4}}\right)^2 = 3$$
$$(a^1)^2 = 3$$
$$a^2 = 3$$
$$a = \sqrt{3}$$

   b. Which exponent properties did you use to find your answer to part (a)?

   We used the properties $b^n \cdot b^m = b^{m+n}$ and $(b^m)^n = b^{mn}$.

16. In the lesson, you made the following argument:

$$\begin{aligned}\left(2^{\frac{1}{3}}\right)^3 &= 2^{\frac{1}{3}} \cdot 2^{\frac{1}{3}} \cdot 2^{\frac{1}{3}} \\ &= 2^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} \\ &= 2^1 \\ &= 2.\end{aligned}$$

Since $\sqrt[3]{2}$ is a number so that $\left(\sqrt[3]{2}\right)^3 = 2$ and $2^{\frac{1}{3}}$ is a number so that $\left(2^{\frac{1}{3}}\right)^3 = 2$, you concluded that $2^{\frac{1}{3}} = \sqrt[3]{2}$. Which exponent property was used to make this argument?

We used the property $b^n \cdot b^m = b^{m+n}$. (Students may also mention the uniqueness of n^{th} roots.)

Lesson 4: Properties of Exponents and Radicals

Student Outcomes

- Students rewrite expressions involving radicals and rational exponents using the properties of exponents.

Lesson Notes

In Lesson 1, students reviewed the properties of exponents for integer exponents before establishing the meaning of the n^{th} root of a positive real number and how it can be expressed as a rational exponent in Lesson 3. In Lesson 4, students extend properties of exponents that applied to expressions with integer exponents to expressions with rational exponents. In each case, the notation $b^{\frac{1}{n}}$ specifically indicates the principal root (e.g., $2^{\frac{1}{2}}$ is $\sqrt{2}$, as opposed to $-\sqrt{2}$).

This lesson extends students' thinking using the properties of radicals and the definitions from Lesson 3 so that they can see why it makes sense that the properties of exponents hold for any rational exponents (**N-RN.A.1**). Examples and exercises work to establish fluency with the properties of exponents when the exponents are rational numbers and emphasize rewriting expressions and evaluating expressions using the properties of exponents and radicals (**N-RN.A.2**).

Scaffolding:

- Throughout the lesson, remind students of past properties of integer exponents and radicals either through an anchor chart posted on the wall or by recording relevant properties as they come up. Included is a short list of previous properties used in this module.
- For all real numbers $a, b > 0$, and all integers m, n for which the expressions are defined:

$$b^m \cdot b^n = b^{m+n}$$

$$(b^m)^n = b^{mn}$$

$$(ab)^m = a^m \cdot b^m$$

$$b^{-m} = \frac{1}{b^m}$$

Additionally, if $n > 0$,

$$\sqrt[n]{b} = b^{\frac{1}{n}}$$

$$\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$$

$$\sqrt[n]{b^n} = \left(\sqrt[n]{b}\right)^n = b$$

$$\sqrt[n]{b^m} = \left(\sqrt[n]{b}\right)^m = b^{\frac{m}{n}}.$$

Classwork

Opening (2 minutes)

Students revisit the properties of square roots and cube roots studied in Module 1 to remind them that those were extended to any n^{th} root in Lesson 3. So, they are now ready to verify that the properties of exponents hold for rational exponents.

Draw students' attention to a chart posted prominently on the wall or to their notebooks where the properties of exponents and radicals are displayed, including those developed in Lesson 3.

Remind students of the description of exponential expressions of the form $b^{\frac{m}{n}}$, which they make use of throughout the lesson:

Let b be any positive real number, and m, n be any integers with $n > 0$; then $b^{\frac{m}{n}} = \sqrt[n]{b^m} = \left(\sqrt[n]{b}\right)^m$.

Opening Exercise (5 minutes)

These exercises briefly review content from Module 1 and the last lesson.

Opening Exercise

Write each exponential expression as a radical expression, and then use the definition and properties of radicals to write the resulting expression as an integer.

a. $7^{\frac{1}{2}} \cdot 7^{\frac{1}{2}}$

$\sqrt{7} \cdot \sqrt{7} = \sqrt{49} = 7$

b. $3^{\frac{1}{3}} \cdot 3^{\frac{1}{3}} \cdot 3^{\frac{1}{3}}$

$\sqrt[3]{3} \cdot \sqrt[3]{3} \cdot \sqrt[3]{3} = \sqrt[3]{9} \cdot \sqrt[3]{3} = \sqrt[3]{27} = 3$

c. $12^{\frac{1}{2}} \cdot 3^{\frac{1}{2}}$

$\sqrt{12} \cdot \sqrt{3} = \sqrt{12 \cdot 3} = \sqrt{36} = 6$

d. $\left(64^{\frac{1}{3}}\right)^{\frac{1}{2}}$

$\sqrt{\sqrt[3]{64}} = \sqrt{4} = 2$

To transition from the Opening Exercise to Example 1, ask students to write parts (a) and (b) of the Opening Exercise in exponent form. Then, ask them to discuss with a partner whether or not it would be true in general that $b^{\frac{m}{n}} \cdot b^{\frac{p}{q}} = b^{\frac{m}{n}+\frac{p}{q}}$ for positive real numbers b where m, n, p, and q are integers with $n \neq 0$ and $q \neq 0$.

MP.3 & MP.8

- How could you write $\sqrt{7} \cdot \sqrt{7} = 7$ with rational exponents? How about $\sqrt[3]{3} \cdot \sqrt[3]{3} \cdot \sqrt[3]{3} = 3$?
  - $7^{\frac{1}{2}} \cdot 7^{\frac{1}{2}} = 7^1$ *and* $3^{\frac{1}{3}} \cdot 3^{\frac{1}{3}} \cdot 3^{\frac{1}{3}} = 3^1$
- Based on these examples, does the exponent property $b^m \cdot b^n = b^{m+n}$ appear to be valid when m and n are rational numbers? Explain how you know.
  - *Since the exponents on the left side of each statement add up to the exponents on the right side, it appears to be true. However, the right side exponent was always* 1*. If we work with* $8^{\frac{1}{3}} \cdot 8^{\frac{1}{3}} = 8^{\frac{2}{3}}$ *and write it in radical form, then* $\sqrt[3]{8} \cdot \sqrt[3]{8} = \sqrt[3]{8^2} = \sqrt[3]{64}$*. So, it may be true in general.*
    Note that examples alone do not prove that a mathematical statement is always true.

In the rest of this lesson, students make sense of these observations in general to extend the properties of exponents to rational numbers by applying the definition of the n^{th} root *of* b and the properties of radicals introduced in Lesson 3.

Examples 1–3 (10 minutes)

In the previous lesson, it was assumed that the exponent property $b^m b^n = b^{m+n}$ for positive real numbers b and integers m and n would also hold for rational exponents when the exponents were of the form $\frac{1}{n}$, where n was a positive integer. This example helps students see that the property below makes sense for any rational exponent.

$$b^{\frac{m}{n}} \cdot b^{\frac{p}{q}} = b^{\frac{m}{n}+\frac{p}{q}}, \text{ where } m, n, p, \text{ and } q \text{ are integers with } n > 0 \text{ and } q > 0.$$

Consider modeling Example 1 below and having students work with a partner on Example 2. Make sure students include a justification for each step in the problem. The following discussion questions can be used to guide students through Example 3.

MP.7 & MP.8

- How can we write these expressions using radicals?
  - *In Lesson 3, we learned that $b^{\frac{1}{n}} = \sqrt[n]{b}$ and $b^{\frac{m}{n}} = \sqrt[n]{b^m}$ for positive real numbers b and positive integers m and n.*
- Which properties help us to write the expression as a single radical?
  - *The property of radicals that states $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$ for positive real numbers a and b and positive integer n, and the property of exponents that states $b^m \cdot b^n = b^{m+n}$ for positive real numbers b and integers m and n*
- How do we rewrite this expression in exponent form?
  - *In Lesson 3, we related radicals and rational exponents by $b^{\frac{m}{n}} = \sqrt[n]{b^m}$.*
- What makes Example 3 different from Examples 1 and 2?
  - *The exponents have different denominators, so when we write the expression in radical form, the roots are not the same, and we cannot apply the property that $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$.*
- Can you think of a way to rewrite the problem so it looks more like Examples 1 and 2?
  - *We can write the exponents as equivalent fractions with the same denominator.*

Examples 1–3

Write each expression in the form $b^{\frac{m}{n}}$ for positive real numbers b and integers m and n with $n > 0$ by applying the properties of radicals and the definition of n^{th} root.

1. $b^{\frac{1}{4}} \cdot b^{\frac{1}{4}}$

By the definition of n^{th} root,

$b^{\frac{1}{4}} \cdot b^{\frac{1}{4}} = \sqrt[4]{b} \cdot \sqrt[4]{b}.$

$= \sqrt[4]{b \cdot b}$ ***By the properties of radicals and properties of exponents***

$= \sqrt[4]{b^2}$

$= b^{\frac{2}{4}}$ ***By the definition of $b^{\frac{m}{n}}$***

The rational number $\frac{2}{4}$ is equal to $\frac{1}{2}$. Thus,

$b^{\frac{1}{4}} \cdot b^{\frac{1}{4}} = b^{\frac{1}{2}}.$

Scaffolding:

- Throughout the lesson, create parallel problems to demonstrate that these problems work with numerical values as well.
- For example, in part (a), substitute 4 for b.
- In part (b), substitute a perfect cube such as 8 or 27 for b.

2. $b^{\frac{1}{3}} \cdot b^{\frac{4}{3}}$

$$\begin{aligned} b^{\frac{1}{3}} \cdot b^{\frac{4}{3}} &= \sqrt[3]{b} \cdot \sqrt[3]{b^4} && \textit{By the definition of } b^{\frac{1}{n}} \textit{ and } b^{\left(\frac{m}{n}\right)} \\ &= \sqrt[3]{b \cdot b^4} && \textit{By the properties of radicals and properties of exponents} \\ &= \sqrt[3]{b^5} \\ &= b^{\frac{5}{3}} && \textit{By the definition of } b^{\frac{m}{n}} \end{aligned}$$

Thus,

$b^{\frac{1}{3}} \cdot b^{\frac{4}{3}} = b^{\frac{5}{3}}$.

3. $b^{\frac{1}{5}} \cdot b^{\frac{3}{4}}$

Write the exponents as equivalent fractions with the same denominator.

$$b^{\frac{1}{5}} \cdot b^{\frac{3}{4}} = b^{\frac{4}{20}} \cdot b^{\frac{15}{20}}$$

Rewrite in radical form.

$$= \sqrt[20]{b^4} \cdot \sqrt[20]{b^{15}}$$

Rewrite as a single radical expression.

$$= \sqrt[20]{b^4 \cdot b^{15}}$$
$$= \sqrt[20]{b^{19}}$$

Rewrite in exponent form using the definition.

$$= b^{\frac{19}{20}}$$

Thus,

$$b^{\frac{1}{5}} \cdot b^{\frac{3}{4}} = b^{\frac{19}{20}}.$$

- Now, add the exponents in each example. What is $\frac{1}{4} + \frac{1}{4}$? $\frac{1}{3} + \frac{4}{3}$? $\frac{1}{5} + \frac{3}{4}$?
  - $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$, $\frac{1}{3} + \frac{4}{3} = \frac{5}{3}$, *and* $\frac{1}{5} + \frac{3}{4} = \frac{19}{20}$
- What do you notice about these sums and the value of the exponent when we rewrote each expression?
  - *The sum of the exponents was equal to the exponent of the answer.*

Based on these examples, particularly the last one, it seems reasonable to extend the properties of exponents to hold when the exponents are any rational number. Thus, the following property can be stated:

- For any integers m, n, p and q, with $n > 0$ and $q > 0$, and any real numbers b so that $b^{\frac{1}{n}}$ and $b^{\frac{1}{q}}$ are defined,
  $b^{\frac{m}{n}} \cdot b^{\frac{p}{q}} = b^{\frac{m}{n}+\frac{p}{q}}$.

Have students copy this property into their notes along with the ones listed below. Also, consider writing these properties on a piece of chart paper and displaying them in the classroom. These properties are listed in the Lesson Summary.

In a similar fashion, the other properties of exponents can be extended to hold for any rational exponents as well.

- For any integers m, n, p, and q, with $n > 0$ and $q > 0$, and any real numbers a and b so that $a^{\frac{1}{n}}$, $b^{\frac{1}{n}}$, and $b^{\frac{1}{q}}$ are defined,

$$b^{\frac{m}{n}} = \sqrt[n]{b^m}$$
$$\left(b^{\frac{1}{n}}\right)^n = b$$
$$(b^n)^{\frac{1}{n}} = b$$
$$(ab)^{\frac{m}{n}} = a^{\frac{m}{n}} \cdot b^{\frac{m}{n}}$$
$$\left(b^{\frac{m}{n}}\right)^{\frac{p}{q}} = b^{\frac{mp}{nq}}$$
$$b^{-\frac{m}{n}} = \frac{1}{b^{\frac{m}{n}}}.$$

At this point, consider having the class look at the Opening Exercise again and asking them which property could be used to simplify each problem.

For advanced learners, a derivation of the property explored in Example 1 is provided below.

Rewrite $b^{\frac{m}{n}}$ and $b^{\frac{p}{q}}$ as equivalent exponential expressions in which the exponents have the same denominator, and apply the definition of $b^{\frac{m}{n}}$ as the n^{th} root.

By the definition of $b^{\frac{m}{n}}$ and then using properties of algebra, students can rewrite the exponent to be $\frac{m}{n} + \frac{p}{q}$.

$$\begin{aligned} b^{\frac{m}{n}} \cdot b^{\frac{p}{q}} &= b^{\frac{mq}{nq}} b^{\frac{np}{nq}} \\ &= \sqrt[nq]{b^{mq}} \cdot \sqrt[nq]{b^{np}} \\ &= \sqrt[nq]{b^{mq} \cdot b^{np}} \\ &= \sqrt[nq]{b^{mq+np}} \\ &= b^{\frac{mq+np}{nq}} \\ &= b^{\frac{mq}{nq}+\frac{np}{nq}} \\ &= b^{\frac{m}{n}+\frac{p}{q}} \end{aligned}$$

Exercises 1–4 (6 minutes)

Have students work with a partner or in small groups to rewrite expressions with rational exponents using the properties presented above. As students work, emphasize that it is not necessary to write these expressions using radicals since it has just been established that the properties of exponents hold for rational numbers. In the last two exercises, students have to use their knowledge of radicals to rewrite the answers without exponents.

> *Scaffolding:*
> - When students get to these exercises, it may be necessary to remind them that it is often easier to rewrite $b^{\frac{m}{n}}$ as $\left(\sqrt[n]{b}\right)^m$ when evaluating radical expressions.
> - Provide a scientific calculator for students who struggle with arithmetic, but encourage them to use the radical and exponent properties to show the steps.

Exercises 1–4

Write each expression in the form $b^{\frac{m}{n}}$. If a numeric expression is a rational number, then write your answer without exponents.

1. $b^{\frac{2}{3}} \cdot b^{\frac{1}{2}}$

   $b^{\frac{2}{3}+\frac{1}{2}} = b^{\frac{7}{6}}$

2. $\left(b^{-\frac{1}{5}}\right)^{\frac{2}{3}}$

   $b^{-\frac{1}{5}\cdot\frac{2}{3}} = b^{-\frac{2}{15}}$

3. $64^{\frac{1}{3}} \cdot 64^{\frac{3}{2}}$

   $$\begin{aligned} 64^{\frac{1}{3}+\frac{3}{2}} &= 64^{\frac{11}{6}} \\ &= \left(\sqrt[6]{64}\right)^{11} \\ &= 2^{11} \\ &= 2048 \end{aligned}$$

4. $\left(\frac{9^3}{4^2}\right)^{\frac{3}{2}}$

   $$\begin{aligned} \left(\frac{9^3}{4^2}\right)^{\frac{3}{2}} &= \frac{9^{\frac{9}{2}}}{4^3} \\ &= \frac{\left(\sqrt[2]{9}\right)^9}{64} \\ &= \frac{3^9}{64} \\ &= \frac{19\,683}{64} \end{aligned}$$

Example 4 (5 minutes)

- We can rewrite radical expressions using properties of exponents. There are other methods for rewriting radical expressions, but this example models using the properties of exponents. Often, textbooks and exams give directions to simplify an expression, which is vague unless we specify what it means. We want students to develop fluency in applying the properties, so the directions here say to rewrite in a specific fashion.

Example 4

Rewrite the radical expression $\sqrt{48x^5y^4z^2}$ so that no perfect square factors remain inside the radical.

$$\begin{aligned}\sqrt{48\cdot x^5\cdot y^4\cdot z^2} &= (4^2\cdot 3\cdot x^5\cdot y^4\cdot z^2)^{\frac{1}{2}}\\ &= 4^{\frac{2}{2}}\cdot 3^{\frac{1}{2}}\cdot x^{\frac{5}{2}}\cdot y^{\frac{4}{2}}\cdot z^{\frac{2}{2}}\\ &= 4\cdot 3^{\frac{1}{2}}\cdot x^{2+\frac{1}{2}}\cdot y^2\cdot z\\ &= 4x^2y^2z\cdot(3x)^{\frac{1}{2}}\\ &= 4x^2y^2z\sqrt{3x}\end{aligned}$$

- Although this process may seem drawn out, once it has been practiced, most of the steps can be internalized, and expressions are quickly rewritten using this technique.

Exercise 5 (5 minutes)

For the values $x = 50$, $y = 12$, and $z = 3$, the expressions in Exercises 5(a) and (b) are difficult to evaluate. Students need to rewrite these expressions in a simpler form by minimizing fractions in the exponents before attempting to evaluate them. Do not allow calculators to be used on these exercises.

Exercise 5

5. **Use the definition of rational exponents and properties of exponents to rewrite each expression with rational exponents containing as few fractions as possible. Then, evaluate each resulting expression for $x = 50$, $y = 12$, and $z = 3$.**

   a. $\sqrt{8x^3y^2}$

   $$\begin{aligned}\sqrt{8x^3y^2} &= 2^{\frac{3}{2}}x^{\frac{3}{2}}y^{\frac{2}{2}}\\ &= 2xy\cdot(2x)^{\frac{1}{2}}\end{aligned}$$

   Evaluating, we get $2(50)(12)(2\cdot 50)^{\frac{1}{2}} = 100\cdot 12\cdot 10 = 12\,000$.

   b. $\sqrt[3]{54y^7z^2}$

   $$\begin{aligned}\sqrt[3]{54y^7z^2} &= 27^{\frac{1}{3}}\cdot 2^{\frac{1}{3}}\cdot y^{\frac{7}{3}}\cdot z^{\frac{2}{3}}\\ &= 3y^2\cdot(2yz^2)^{\frac{1}{3}}\end{aligned}$$

   Evaluating, we get $3(12)^2(2\cdot 12\cdot 3^2)^{\frac{1}{3}} = 3(144)(216)^{\frac{1}{3}} = 3\cdot 144\cdot 6 = 2592$.

Exercise 6 (5 minutes)

This exercise reminds students that rational numbers can be represented in decimal form and gives them a chance to work on their numeracy skills. Students should work on this exercise with a partner or in their groups to encourage dialogue and debate. Have a few students demonstrate their results to the entire class. There is more than one possible approach, so when debriefing, try to share different approaches that show varied reasoning. Conclude with one or two strong arguments. Students can confirm their reasoning using a calculator.

MP.3

Exercise 6

6. **Order these numbers from smallest to largest. Explain your reasoning.**

$16^{2.5}$ $9^{3.6}$ $32^{1.2}$

The number $16^{2.5}$ is between 256 and $4,096$. We can rewrite $16^{2.5} = (2^4)^{2.5}$, which is 2^{10}, so $16^{2.5} = 2^{10} = 1024$.

The number $32^{1.2}$ is between 32 and $1,024$. We can rewrite $32^{1.2} = (2^5)^{1.2}$, which is 2^6, so $32^{1.2} = 2^6 = 64$.

The number $9^{3.6}$ is larger than 9^3, so $9^{3.6}$ is larger than 729.

Thus, $32^{1.2}$ is clearly the smallest number, but we need to determine if $9^{3.6}$ is greater than or less than $1,024$. To do this, we know that $9^{3.6} = 9^{3+0.6} = 9^3 \cdot 9^{0.6}$. This means that $9^{3.6} > 9^3 \cdot 9^{0.5}$, and $9^3 \cdot 9^{0.5} = 729 \cdot 3$, which is greater than $1,024$.

Thus, the numbers in order from smallest to largest are $32^{1.2}$, $16^{2.5}$, and $9^{3.6}$.

Closing (2 minutes)

Have students summarize the definition and properties of rational exponents and any important ideas from the lesson by creating a list of what they have learned so far about the properties of exponents and radicals. Circulate around the classroom to informally assess understanding. Reinforce the properties of exponents listed below.

Lesson Summary

The properties of exponents developed in Grade 8 for integer exponents extend to rational exponents.

That is, for any integers m, n, p, and q, with $n > 0$ and $q > 0$, and any real numbers a and b so that $a^{\frac{1}{n}}$, $b^{\frac{1}{n}}$, and $b^{\frac{1}{q}}$ are defined, we have the following properties of exponents:

1. $b^{\frac{m}{n}} \cdot b^{\frac{p}{q}} = b^{\frac{m}{n}+\frac{p}{q}}$
2. $b^{\frac{m}{n}} = \sqrt[n]{b^m}$
3. $\left(b^{\frac{1}{n}}\right)^n = b$
4. $(b^n)^{\frac{1}{n}} = b$
5. $(ab)^{\frac{m}{n}} = a^{\frac{m}{n}} \cdot b^{\frac{m}{n}}$
6. $\left(b^{\frac{m}{n}}\right)^{\frac{p}{q}} = b^{\frac{mp}{nq}}$
7. $b^{-\frac{m}{n}} = \frac{1}{b^{\frac{m}{n}}}$.

Exit Ticket (5 minutes)

Name ____ Date ____________________

Lesson 4: Properties of Exponents and Radicals

Exit Ticket

1. Find the exact value of $9^{\frac{11}{10}} \cdot 9^{\frac{2}{5}}$ without using a calculator.

2. Justify that $\sqrt[3]{8} \cdot \sqrt[3]{8} = \sqrt{16}$ using the properties of exponents in at least two different ways.

Exit Ticket Sample Solutions

1. Find the exact value of $9^{\frac{11}{10}} \cdot 9^{\frac{2}{5}}$ without using a calculator.

$$\begin{aligned} 9^{\frac{11}{10}} \cdot 9^{\frac{2}{5}} &= 9^{\frac{11}{10}+\frac{2}{5}} \\ &= 9^{\frac{15}{10}} \\ &= 9^{\frac{3}{2}} \\ &= \left(\sqrt[2]{9}\right)^3 \\ &= 27 \end{aligned}$$

2. Justify that $\sqrt[3]{8} \cdot \sqrt[3]{8} = \sqrt{16}$ using the properties of exponents in at least two different ways.

$$\begin{aligned} 8^{\frac{1}{3}} \cdot 8^{\frac{1}{3}} &= 8^{\frac{2}{3}} \\ &= \left(2^3\right)^{\frac{2}{3}} \\ &= 2^2 \\ &= 2^{\frac{4}{2}} \\ &= \left(2^4\right)^{\frac{1}{2}} \\ &= \sqrt{16} \end{aligned}$$

$$\begin{aligned} 16^{\frac{1}{2}} &= (4 \cdot 4)^{\frac{1}{2}} \\ &= 4^{\frac{1}{2}} \cdot 4^{\frac{1}{2}} \\ &= 2 \cdot 2 \\ &= 8^{\frac{1}{3}} \cdot 8^{\frac{1}{3}} \\ &= \sqrt[3]{8} \cdot \sqrt[3]{8} \end{aligned}$$

Problem Set Sample Solutions

1. Evaluate each expression for $a = 27$ and $b = 64$.

   a. $\sqrt[3]{a}\sqrt{b}$

   $\sqrt[3]{27} \cdot \sqrt{64} = 3 \cdot 8 = 24$

   b. $\left(3\sqrt[3]{a}\sqrt{b}\right)^2$

   $(3 \cdot 3 \cdot 8)^2 = 5184$

   c. $\left(\sqrt[3]{a} + 2\sqrt{b}\right)^2$

   $(3 + 2 \cdot 8)^2 = 361$

   d. $a^{-\frac{2}{3}} + b^{\frac{3}{2}}$

   $\frac{1}{\left(\sqrt[3]{27}\right)^2} + \left(\sqrt{64}\right)^3 = \frac{1}{9} + 512 = 512\frac{1}{9}$

   e. $\left(a^{-\frac{2}{3}} \cdot b^{\frac{3}{2}}\right)^{-1}$

   $\left(\frac{1}{9} \cdot 512\right)^{-1} = \frac{9}{512}$

   f. $\left(a^{-\frac{2}{3}} - \frac{1}{8}b^{\frac{3}{2}}\right)^{-1}$

   $\left(\frac{1}{9} - \frac{1}{8} \cdot 512\right)^{-1} = \left(-\frac{575}{9}\right)^{-1} = -\frac{9}{575}$

2. Rewrite each expression so that each term is in the form kx^n, where k is a real number, x is a positive real number, and n is a rational number.

a. $x^{-\frac{2}{3}} \cdot x^{\frac{1}{3}}$

$x^{-\frac{1}{3}}$

b. $2x^{\frac{1}{2}} \cdot 4x^{-\frac{5}{2}}$

$8x^{-2}$

c. $\dfrac{10x^{\frac{1}{3}}}{2x^2}$

$5x^{-\frac{5}{3}}$

d. $\left(3x^{\frac{1}{4}}\right)^{-2}$

$\frac{1}{9}x^{-\frac{1}{2}}$

e. $x^{\frac{1}{2}}\left(2x^2 - \frac{4}{x}\right)$

$2x^{\frac{5}{2}} - 4x^{-\frac{1}{2}}$

f. $\sqrt[3]{\dfrac{27}{x^6}}$

$3x^{-2}$

g. $\sqrt[3]{x} \cdot \sqrt[3]{-8x^2} \cdot \sqrt[3]{27x^4}$

$-6x^{\frac{7}{3}}$

h. $\dfrac{2x^4 - x^2 - 3x}{\sqrt{x}}$

$2x^{\frac{7}{2}} - x^{\frac{3}{2}} - 3x^{\frac{1}{2}}$

i. $\dfrac{\sqrt{x} - 2x^{-3}}{4x^2}$

$\frac{1}{4}x^{-\frac{3}{2}} - \frac{1}{2}x^{-5}$

3. Show that $\left(\sqrt{x} + \sqrt{y}\right)^2$ is not equal to $x^1 + y^1$ when $x = 9$ and $y = 16$.

When $x = 9$ and $y = 16$, the two expressions are $\left(\sqrt{9} + \sqrt{16}\right)^2$ and $9 + 16$. The first expression simplifies to 49, and the second simplifies to 25. The two expressions are not equal.

4. Show that $\left(x^{\frac{1}{2}} + y^{\frac{1}{2}}\right)^{-1}$ is not equal to $\dfrac{1}{x^{\frac{1}{2}}} + \dfrac{1}{y^{\frac{1}{2}}}$ when $x = 9$ and $y = 16$.

When $x = 9$ and $y = 16$, the two expressions are $\left(\sqrt{9} + \sqrt{16}\right)^{-1}$ and $\frac{1}{\sqrt{9}} + \frac{1}{\sqrt{16}}$. The first expression is $\frac{1}{7}$, and the second one is $\frac{1}{3} + \frac{1}{4} = \frac{7}{12}$. The two expressions are not equal.

5. From these numbers, select (a) one that is negative, (b) one that is irrational, (c) one that is not a real number, and (d) one that is a perfect square:

$$3^{\frac{1}{2}} \cdot 9^{\frac{1}{2}},\ 27^{\frac{1}{3}} \cdot 144^{\frac{1}{2}},\ 64^{\frac{1}{3}} - 64^{\frac{2}{3}},\text{ and } \left(4^{-\frac{1}{2}} - 4^{\frac{1}{2}}\right)^{\frac{1}{2}}.$$

The first number, $3^{\frac{1}{2}} \cdot 9^{\frac{1}{2}}$, is irrational; the second number, $27^{\frac{1}{3}} \cdot 144^{\frac{1}{2}}$, is a perfect square; the third number, $64^{\frac{1}{3}} - 64^{\frac{2}{3}}$, is negative; and the last number, $\left(4^{-\frac{1}{2}} - 4^{\frac{1}{2}}\right)^{\frac{1}{2}}$, is not a real number.

6. Show that for any rational number n, the expression $2^n \cdot 4^{n+1} \cdot \left(\frac{1}{8}\right)^n$ is equal to 4.

$2^n \cdot 2^{2n+2} \cdot 2^{-3n} = 2^2 = 4$

7. Let n be any rational number. Express each answer as a power of 10.

a. Multiply 10^n by 10.

$10^n \cdot 10 = 10^{n+1}$

b. Multiply $\sqrt{10}$ by 10^n.

$10^{\frac{1}{2}} \cdot 10^n = 10^{\frac{1}{2}+n}$

c. Square 10^n.

$(10^n)^2 = 10^{2n}$

d. Divide $100 \cdot 10^n$ by 10^{2n}.

$\frac{100 \cdot 10^n}{10^{2n}} = 10^{2+n-2n} = 10^{2-n}$

e. Show that $10^n = 11 \cdot 10^n - 10^{n+1}$.

$$\begin{aligned} 11 \cdot 10^n - 10^{n+1} &= 11 \cdot 10^n - 10 \cdot 10^n \\ &= 10^n(11 - 10) \\ &= 10^n \cdot 1 \\ &= 10^n \end{aligned}$$

8. Rewrite each of the following radical expressions as an equivalent exponential expression in which each variable occurs no more than once.

a. $\sqrt{8x^2y}$

$$\begin{aligned} \sqrt{8x^2y} &= 2^{\frac{2}{2}}x^{\frac{2}{2}}(2y)^{\frac{1}{2}} \\ &= 2x \cdot (2y)^{\frac{1}{2}} \\ &= 2^{\frac{3}{2}}\, x\, y^{\frac{1}{2}} \end{aligned}$$

b. $\sqrt[5]{96x^3y^{15}z^6}$

$$\begin{aligned} \sqrt[5]{96x^3y^{15}z^6} &= (32 \cdot 3 \cdot x^3 \cdot y^{15} \cdot z^6)^{\frac{1}{5}} \\ &= 32^{\frac{1}{5}} \cdot 3^{\frac{1}{5}} \cdot x^{\frac{3}{5}} \cdot y^{\frac{15}{5}} \cdot z^{\frac{6}{5}} \\ &= 2 \cdot 3^{\frac{1}{5}} \cdot x^{\frac{3}{5}} \cdot y^3 \cdot z^{\frac{6}{5}} \end{aligned}$$

9. Use properties of exponents to find two integers that are upper and lower estimates of the value of $4^{1.6}$.

$$4^{1.5} < 4^{1.6} < 4^2$$

$4^{1.5} = 2^3 = 8$ *and* $4^2 = 16$, *so* $8 < 4^{1.6} < 16$.

10. Use properties of exponents to find two integers that are upper and lower estimates of the value of $8^{2.3}$.

$$8^2 < 8^{2.3} < 8^{2+\frac{1}{3}}$$

$$8^2 = 64 \text{ and } 8^{\frac{1}{3}} = 2, \text{ so } 8^{2+\frac{1}{3}} = 8^2 \cdot 8^{\frac{1}{3}} = 128. \text{ Thus, } 64 < 8^{2.3} < 128.$$

11. Kepler's third law of planetary motion relates the average distance, a, of a planet from the sun to the time, t, it takes the planet to complete one full orbit around the sun according to the equation $t^2 = a^3$. When the time, t, is measured in Earth years, the distance, a, is measured in astronomical units (AUs). (One AU is equal to the average distance from Earth to the sun.)

a. Find an equation for t in terms of a and an equation for a in terms of t.

$$t^2 = a^3$$
$$t = a^{\frac{3}{2}}$$
$$a = t^{\frac{2}{3}}$$

b. Venus takes about 0.616 Earth year to orbit the sun. What is its average distance from the sun?

Because $a = (0.616)^{\frac{2}{3}} \approx 0.724$, *the average distance from Venus to the sun is* 0.724 AU.

c. Mercury is an average distance of 0.387 AU from the sun. About how long is its orbit in Earth years?

Because $t = (0.387)^{\frac{3}{2}} \approx 0.241$, *the length of Mercury's orbit is approximately* 0.241 *Earth years.*

Lesson 5: Irrational Exponents—What are $2^{\sqrt{2}}$ and 2^{π}?

Student Outcomes

- Students approximate the value of quantities that involve positive irrational exponents.
- Students extend the domain of the function $f(x) = b^x$ for positive real numbers b to all real numbers.

Lesson Notes

The goal today is to define 2 to an irrational power. There is already a definition for 2 to a rational power $\frac{p}{q}$: $2^{\frac{p}{q}} = \sqrt[q]{2^p}$, but irrational numbers cannot be written as "an integer divided by an integer." By defining 2 to an irrational power, it is possible to state definitively that the domain of the function $f(x) = 2^x$ is all real numbers. This result can be extended to any exponential function $f(x) = b^x$ where b is a positive real number. These important results are necessary to proceed to the study of logarithms. The lesson provides a new way to reinforce standard **8.NS.A.2** when students determine a recursive process for calculation from a context (**F-BF.A.1a**) when they use rational approximations of irrational numbers to approximate first $\sqrt{2}$ and then $2^{\sqrt{2}}$. Extending rational exponents to real exponents is an application of **N-RN.A.1**. The foundational work done in this lesson with exponential expressions is extended to logarithms in later lessons so that logarithmic functions in base 2, 10, and e are well defined and can be used to solve exponential equations. Understanding the domain of exponential functions also allows students to correctly graph exponential and logarithmic functions in Topic C. The work done in Lesson 5 also helps demystify irrational numbers, which eases the introduction to Euler's number, e, in Lesson 6.

Classwork

Opening (5 minutes)

Use the Opening to recall the definitions of rational and irrational numbers and solicit examples and characteristics from the class. Randomly select students to explain what they know about rational and irrational numbers. Then, make a list including examples and characteristics of both. Alternatively, have students give rational and irrational numbers, make a class list, and then have students generalize characteristics of rational and irrational numbers in their notebooks. Rational and irrational numbers along with some characteristics and examples are described below.

RATIONAL NUMBER: A *rational number* is a number that can be represented as $\frac{p}{q}$ where p and q are integers with $q \neq 0$.

IRRATIONAL NUMBER: An *irrational number* is a real number that cannot be represented as $\frac{p}{q}$ for any integers p and q with $q \neq 0$.

- What are some characteristics of rational numbers?
  - *A rational number can be represented as a finite or repeating decimal; that is, a rational number can be written as a fraction.*

- What are some characteristics of irrational numbers?
  - *An irrational number cannot be represented as a finite or repeating decimal, so it must be represented symbolically or as an infinite, nonrepeating decimal.*
- What are some examples of irrational numbers?
  - $\sqrt{2}, \pi, \sqrt[3]{17}$
- We usually assume that the rules we develop for rational numbers hold true for irrational numbers, but what could something like $2^{\sqrt{2}}$ or 2^{π} mean?
  - *Solicit ideas from the class. Students may consider numbers like this to be between rational exponents or "filling the gaps" from rational exponents.*
- Let's find out more about exponents raised to irrational powers and how we can get a handle on their values.

Exercise 1 (8 minutes)

Have students work on the following exercises independently or in pairs. Students need to use calculators. After students finish, debrief them with the questions that follow the exercises.

Exercise 1

a. Write the following finite decimals as fractions (you do not need to reduce to lowest terms).

$$1, \quad 1.4, \quad 1.41, \quad 1.414, \quad 1.4142, \quad 1.41421$$

$1.4 = \frac{14}{10}$

$1.41 = \frac{141}{100}$

$1.414 = \frac{1414}{1000}$

$1.4142 = \frac{14\,142}{10\,000}$

$1.41421 = \frac{141\,421}{100000}$

b. Write $2^{1.4}$, $2^{1.41}$, $2^{1.414}$, and $2^{1.4142}$ in radical form ($\sqrt[n]{2^m}$).

$2^{1.4} = 2^{14/10} = \sqrt[10]{2^{14}}$

$2^{1.41} = 2^{141/100} = \sqrt[100]{2^{141}}$

$2^{1.414} = 2^{1414/1000} = \sqrt[1000]{2^{1414}}$

$2^{1.4142} = 2^{14142/10000} = \sqrt[10000]{2^{14142}}$

$2^{1.41421} = 2^{141421/100000} = \sqrt[100000]{2^{141421}}$

c. **Use a calculator to compute decimal approximations of the radical expressions you found in part (b) to 5 decimal places. For each approximation, underline the digits that are also in the previous approximation, starting with 2.00000 done for you below. What do you notice?**

$$2^1 = 2 = 2.00000$$

$2^{1.4} = \sqrt[10]{2^{14}} \approx \underline{2.}63901$

$2^{1.41} = \sqrt[100]{2^{141}} \approx \underline{2.6}5737$

$2^{1.414} = \sqrt[1000]{2^{1414}} \approx \underline{2.6}6474$

$2^{1.4142} = \sqrt[10000]{2^{14142}} \approx \underline{2.66}511$

$2^{1.41421} = \sqrt[100000]{2^{141421}} \approx \underline{2.6651}4$

More and more of the digits are the same.

Note to teacher: Students cannot find 2^{1414} on most calculators due to the number being 426 digits long. They need to calculate $\left(\sqrt[1000]{2}\right)^{1414}$ instead of $\sqrt[100]{2^{1414}}$. At this point, it may be a good time to switch to using the decimal approximation within the exponent, reminding students that the calculator is evaluating the decimal by using the radical form, that is, $b^{\frac{m}{n}} = \sqrt[n]{b^m}$. Ideally, a student suggests using the decimal exponent first. If roots are used, make sure that the root is taken before the exponent for large exponents. Examples and possible solutions throughout the lesson assume that roots are used so the true meaning of rational exponents is emphasized.

MP.8

- How are the exponents in each power of 2 changing?
  - *A new digit is included in the exponent each time:* 1.4, $1.4\underline{1}$, $1.41\underline{4}$, $1.414\underline{2}$, $1.414\,2\underline{1}$.
- If we kept including more digits, what do you conjecture will happen to the decimal approximations?
  - *A greater and greater number of digits in each approximation would remain the same.*
- Let's see!

Example (6 minutes)

Students should already be aware that rational exponents are defined using roots and exponents.

Write a decimal approximation for $2^{1.4142135}$.

$$2^{1.4142135} \text{ is the } 10{,}000{,}000^{\text{th}} \text{ root of } 2^{14142135}.$$

Remember to take the root first. We get

$$2^{1.4142135} \approx \underline{2.665\,14}.$$

- Can anyone tell the class what the exponents $1.4, 1.41, 1.414, \ldots$ approximate?

Hopefully, one student says $\sqrt{2}$, but if not, ask them to find $\sqrt{2}$ on their calculators, and ask again.

- Yes, $\sqrt{2} \approx 1.414\,213\,562$.

  The goal of this lesson is to find a meaning for $2^{\sqrt{2}}$. We now know enough to discuss both the problem and solution to defining 2 to an irrational power such as $\sqrt{2}$.

- First, the problem: Each time we took a better finite decimal approximation of the irrational number $2^{\sqrt{2}}$, we needed to take a greater n^{th} root. However, an irrational number has an infinite number of digits in its decimal expansion. We cannot take an ∞^{th} root! In particular, while we have always assumed $2^{\sqrt{2}}$ and 2^{π} existed (because when we show the graph of $f(x) = 2^x$, we drew a solid curve—not one with "holes" at $x = \sqrt{2}, \pi$, etc.), we do not as of yet have a way to define what $2^{\sqrt{2}}$ and 2^{π} really are.
- Fortunately, our beginning exercise suggests a solution using a limit process (much the way we defined the area of a circle in Geometry, Module 3 in terms of limits).
- Let a_k stand for the term of the sequence of finite decimal approximations of $\sqrt{2}$ with k digits after the decimal point:
$$\{1, 1.4, 1.41, 1.414, 1.4142, 1.414\,21, 1.414\,213, 1.414\,213\,5, \ldots\},$$
and label these as $a_0 = 1,\ a_1 = 1.4,\ a_2 = 1.41,\ a_3 = 1.414$. Then, define $2^{\sqrt{2}}$ to be the limit of the values of 2^{a_k}. Thus,
$$2^{a_k} \to 2^{\sqrt{2}} \text{ as } k \to \infty.$$

The important point to make to students is that each 2^{a_k} can be computed since each a_k is a rational number and therefore has a well-defined value in terms of n^{th} roots.

This is how calculators and computers are programmed to compute approximations of $2^{\sqrt{2}}$. Try it: The calculator says that $2^{\sqrt{2}} \approx 2.665\,144\,14$.

Exercise 2 (5 minutes)

Students should attempt the following exercise independently or in pairs. After the exercise, use the Discussion to debrief and informally assess understanding.

Exercise 2

a. **Write six terms of a sequence that a calculator can use to approximate 2^{π}. (Hint: $\pi = 3.141\,59\ldots$)**

$\{2^3, 2^{3.1}, 2^{3.14}, 2^{3.141}, 2^{3.1415}, 2^{3.14159}, \ldots\}$

b. **Compute $2^{3.14}$ and 2^{π} on your calculator. In which digit do they start to differ?**

$2^{3.14} = \sqrt[100]{2^{314}} \approx 8.81524$
$2^{\pi} \approx 8.82497$
They start to differ in the hundredths place.

c. **How could you improve the accuracy of your estimate of 2^{π}?**

Include more digits of the decimal approximation of π in the exponent.

Scaffolding:
- Have students working above grade level give the most accurate estimate they can for part (b). Most calculators can provide an additional three to four decimal places of π. For reference, $\pi \approx 3.141\,592\,653\,589\,793\,238\,46$.
- Another option for students working above grade level is to discuss the sequence of upper bounds of π $\{4, 3.2, 3.15, 3.142, 3.1416, \ldots\}$ and whether this sequence leads to an accurate estimate of 2^{π}.

Discussion (10 minutes)

- Why does the sequence $2^3, 2^{3.1}, 2^{3.14}, 2^{3.141}, 2^{3.1415}, \ldots$ get closer and closer to 2^π?

Allow students to make some conjectures, but be sure to go through the reasoning below.

- We can trap 2^π in smaller and smaller intervals, each one contained in the previous interval.

Write the following incomplete inequalities on the board, and ask students to help complete them before continuing. Mention that in this process, they are *squeezing* π between two rational numbers that are each getting closer and closer to the value of π.

$3 < \pi < 4$	
$3.1 < \pi < ?$	3.2
$3.14 < \pi < ?$	3.15
$3.141 < \pi < ?$	3.142
$3.1415 < \pi < ?$	3.1416
$\vdots$	

- Since $3 < \pi < 4$, and the function $f(x) = 2^x$ increases, we know that $2^3 < \pi < 2^4$. Likewise, we can use the smaller intervals that contain π to find smaller intervals that contain 2^π. In this way, we can squeeze 2^π between rational powers of 2.

Now, have students use calculators to estimate the endpoints of each interval created by the upper and lower estimates of the values of 2^π, and write the numerical approximations of each interval on the board so students can see the endpoints of the intervals getting closer together, squeezing the value of 2^π between them. Record values to four decimal places.

	Decimal Form
$2^3 < 2^\pi < 2^4$	$8.0000 < 2^\pi < 16.0000$
$2^{3.1} < 2^\pi < 2^{3.2}$	$8.5742 < 2^\pi < 9.1896$
$2^{3.14} < 2^\pi < 2^{3.15}$	$8.8152 < 2^\pi < 8.8766$
$2^{3.141} < 2^\pi < 2^{3.142}$	$8.8214 < 2^\pi < 8.8275$
$2^{3.1415} < 2^\pi < 2^{3.1416}$	$8.8244 < 2^\pi < 8.8250$
$\vdots$	$\vdots$

- What is the approximate value of 2^π? How many digits of this number do we know?
  - *Because our upper and lower estimates agree to two decimal places, our best approximation is* $2^\pi \approx 8.82$.
- How could we get a more accurate estimate of 2^π?
  - *Use more and more digits of* π *as exponents.*

- As the exponents get closer to the value of π, what happens to the size of the interval?
  - *The intervals get smaller; the endpoints of the interval get closer together.*
- What does every interval share in common?
  - *Every interval contains* 2^{π}.
- The only number that is guaranteed to be contained in every interval is 2^{π}. (Emphasize this fact.)
- There was nothing special about our choice of 2 in this discussion, or $\sqrt{2}$ or π. In fact, with a little more work, we could define $\pi^{\sqrt{2}}$ using the same ideas.

Closing (6 minutes)

Ask students to respond to the following questions either in writing or with a partner. Use this as an opportunity to informally assess understanding. The summative point of the lesson is that the domain of an exponential function $f(x) = b^x$ is *all real numbers*, so emphasize the final question below.

- For any positive real number $b > 0$ and any rational number r, how do we define b^r?
  - *If r is rational, then $r = \frac{p}{q}$ for some integers p and q. Then $b^r = \sqrt[q]{b^p}$. For example, $5^{\frac{2}{3}} = \sqrt[3]{5^2}$.*
- For any positive real number $b > 0$ and any irrational number r, how do we define b^r?
  - *If r is irrational, b^r is the limit of the values a^{b_n} where b_n is the finite decimal approximation of b to n decimal places.*
- If b is any positive real number, then consider the function $f(x) = b^x$. How is $f(x)$ defined if x is a rational number?
  - *If x is a rational number, then there are integers p and q so that $x = \frac{p}{q}$. Then $f(x) = b^{\frac{p}{q}} = \sqrt[q]{b^p}$.*
- How is $f(x)$ defined if x is an irrational number?
  - *If x is an irrational number, we find a sequence of rational numbers $\{a_0, a_1, a_2, \dots\}$ that gets closer and closer to x. Then the sequence $\{b^{a_0}, b^{a_1}, b^{a_2}, \dots\}$ approaches $f(x)$.*
- What is the domain of the exponential function $f(x) = b^x$?
  - *The domain of the function $f(x) = b^x$ is all real numbers.*

Exit Ticket (5 minutes)

Name ______________________________ Date ______________

Lesson 5: Irrational Exponents—What are $2^{\sqrt{2}}$ and 2^{π}?

Exit Ticket

Use the process outlined in the lesson to approximate the number $2^{\sqrt{3}}$. Use the approximation $\sqrt{3} \approx 1.732\,050\,8$.

a. Find a sequence of five intervals that contain $\sqrt{3}$ whose endpoints get successively closer to $\sqrt{3}$.

b. Find a sequence of five intervals that contain $2^{\sqrt{3}}$ whose endpoints get successively closer to $2^{\sqrt{3}}$. Write your intervals in the form $2^r < 2^{\sqrt{3}} < 2^s$ for rational numbers r and s.

c. Use your calculator to find approximations to four decimal places of the endpoints of the intervals in part (b).

d. Based on your work in part (c), what is your best estimate of the value of $2^{\sqrt{3}}$?

Exit Ticket Sample Solutions

Use the process outlined in the lesson to approximate the number $2^{\sqrt{3}}$. Use the approximation $\sqrt{3} \approx 1.732\,050\,8$.

a. **Find a sequence of five intervals that contain $\sqrt{3}$ whose endpoints get successively closer to $\sqrt{3}$.**

$1 < \sqrt{3} < 2$
$1.7 < \sqrt{3} < 1.8$
$1.73 < \sqrt{3} < 1.74$
$1.732 < \sqrt{3} < 1.733$
$1.7320 < \sqrt{3} < 1.7321$

b. **Find a sequence of five intervals that contain $2^{\sqrt{3}}$ whose endpoints get successively closer to $2^{\sqrt{3}}$. Write your intervals in the form $2^r < 2^{\sqrt{3}} < 2^s$ for rational numbers r and s.**

$2^1 < 2^{\sqrt{3}} < 2^2$
$2^{1.7} < 2^{\sqrt{3}} < 2^{1.8}$
$2^{1.73} < 2^{\sqrt{3}} < 2^{1.74}$
$2^{1.732} < 2^{\sqrt{3}} < 2^{1.733}$
$2^{1.7320} < 2^{\sqrt{3}} < 2^{1.7321}$

c. **Use your calculator to find approximations to four decimal places of the endpoints of the intervals in part (b).**

$2.0000 < 2^{\sqrt{3}} < 4.0000$
$3.2490 < 2^{\sqrt{3}} < 3.4822$
$3.3173 < 2^{\sqrt{3}} < 3.3404$
$3.3219 < 2^{\sqrt{3}} < 3.3242$
$3.3219 < 2^{\sqrt{3}} < 3.3221$

d. **Based on your work in part (c), what is your best estimate of the value of $2^{\sqrt{3}}$?**

$2^{\sqrt{3}} \approx 3.322$

Problem Set Sample Solutions

1. **Is it possible for a number to be both rational and irrational?**

   No. Either the number can be written as $\frac{p}{q}$ for integers p and q or it cannot. If it can, the number is rational. If it cannot, the number is irrational.

2. **Use properties of exponents to rewrite the following expressions as a number or an exponential expression with only one exponent.**

   a. $\left(2^{\sqrt{3}}\right)^{\sqrt{3}}$ 8

b. $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}}$ **2**

c. $\left(3^{1+\sqrt{5}}\right)^{1-\sqrt{5}}$ $\frac{1}{81}$

d. $3^{\frac{1+\sqrt{5}}{2}} \cdot 3^{\frac{1-\sqrt{5}}{2}}$ **3**

e. $3^{\frac{1+\sqrt{5}}{2}} \div 3^{\frac{1-\sqrt{5}}{2}}$ $3^{\sqrt{5}}$

f. $3^{2\cos^2(x)} \cdot 3^{2\sin^2(x)}$ **9**

3.

a. **Between what two integer powers of 2 does $2^{\sqrt{5}}$ lie?**

$2^2 < 2^{\sqrt{5}} < 2^3$

b. **Between what two integer powers of 3 does $3^{\sqrt{10}}$ lie?**

$3^3 < 3^{\sqrt{10}} < 3^4$

c. **Between what two integer powers of 5 does $5^{\sqrt{3}}$ lie?**

$5^1 < 5^{\sqrt{3}} < 5^2$

4. Use the process outlined in the lesson to approximate the number $2^{\sqrt{5}}$. Use the approximation $\sqrt{5} \approx 2.236\,067\,98$.

a. **Find a sequence of five intervals that contain $\sqrt{5}$ whose endpoints get successively closer to $\sqrt{5}$.**

$2 < \sqrt{5} < 3$
$2.2 < \sqrt{5} < 2.3$
$2.23 < \sqrt{5} < 2.24$
$2.236 < \sqrt{5} < 2.237$
$2.2360 < \sqrt{5} < 2.2361$

b. **Find a sequence of five intervals that contain $2^{\sqrt{5}}$ whose endpoints get successively closer to $2^{\sqrt{5}}$. Write your intervals in the form $2^r < 2^{\sqrt{5}} < 2^s$ for rational numbers r and s.**

$2^2 < 2^{\sqrt{5}} < 2^3$
$2^{2.2} < 2^{\sqrt{5}} < 2^{2.3}$
$2^{2.23} < 2^{\sqrt{5}} < 2^{2.24}$
$2^{2.236} < 2^{\sqrt{5}} < 2^{2.237}$
$2^{2.2360} < 2^{\sqrt{5}} < 2^{2.2361}$

c. **Use your calculator to find approximations to four decimal places of the endpoints of the intervals in part (b).**

$4.0000 < 2^{\sqrt{5}} < 8.0000$
$4.5948 < 2^{\sqrt{5}} < 4.9246$
$4.6913 < 2^{\sqrt{5}} < 4.7240$
$4.7109 < 2^{\sqrt{5}} < 4.7142$
$4.7109 < 2^{\sqrt{5}} < 4.7112$

d. Based on your work in part (c), what is your best estimate of the value of $2^{\sqrt{5}}$?

$2^{\sqrt{5}} \approx 4.711$

e. Can we tell if $2^{\sqrt{5}}$ is rational or irrational? Why or why not?

No. We do not have enough information to determine whether $2^{\sqrt{5}}$ has a repeated pattern in its decimal representation or not.

5. Use the process outlined in the lesson to approximate the number $3^{\sqrt{10}}$. Use the approximation $\sqrt{10} \approx 3.162\,277\,7$.

a. Find a sequence of five intervals that contain $3^{\sqrt{10}}$ whose endpoints get successively closer to $3^{\sqrt{10}}$. Write your intervals in the form $3^r < 3^{\sqrt{10}} < 3^s$ for rational numbers r and s.

$3^3 < 3^{\sqrt{10}} < 3^4$

$3^{3.1} < 3^{\sqrt{10}} < 3^{3.2}$

$3^{3.16} < 3^{\sqrt{10}} < 3^{3.17}$

$3^{3.162} < 3^{\sqrt{10}} < 3^{3.163}$

$3^{3.1622} < 3^{\sqrt{10}} < 3^{3.1623}$

b. Use your calculator to find approximations to four decimal places of the endpoints of the intervals in part (a).

$9.0000 < 3^{\sqrt{10}} < 81.0000$

$30.1353 < 3^{\sqrt{10}} < 33.6347$

$32.1887 < 3^{\sqrt{10}} < 32.5443$

$32.2595 < 3^{\sqrt{10}} < 32.2949$

$32.2666 < 3^{\sqrt{10}} < 32.2701$

c. Based on your work in part (b), what is your best estimate of the value of $3^{\sqrt{10}}$?

$3^{\sqrt{10}} \approx 32.27$

6. Use the process outlined in the lesson to approximate the number $5^{\sqrt{7}}$. Use the approximation $\sqrt{7} \approx 2.645\,751\,31$.

a. Find a sequence of seven intervals that contain $5^{\sqrt{7}}$ whose endpoints get successively closer to $5^{\sqrt{7}}$. Write your intervals in the form $5^r < 5^{\sqrt{7}} < 5^s$ for rational numbers r and s.

$5^2 < 5^{\sqrt{7}} < 5^3$

$5^{2.6} < 5^{\sqrt{7}} < 5^{2.7}$

$5^{2.64} < 5^{\sqrt{7}} < 5^{2.65}$

$5^{2.645} < 5^{\sqrt{7}} < 5^{2.646}$

$5^{2.6457} < 5^{\sqrt{7}} < 5^{2.6458}$

$5^{2.64575} < 5^{\sqrt{7}} < 5^{2.64576}$

$5^{2.645751} < 5^{\sqrt{7}} < 5^{2.645752}$

b. **Use your calculator to find approximations to four decimal places of the endpoints of the intervals in part (a).**

$25.0000 < 5^{\sqrt{7}} < 125.0000$

$65.6632 < 5^{\sqrt{7}} < 77.1292$

$70.0295 < 5^{\sqrt{7}} < 71.1657$

$70.5953 < 5^{\sqrt{7}} < 70.7090$

$70.6749 < 5^{\sqrt{7}} < 70.6862$

$70.6805 < 5^{\sqrt{7}} < 70.6817$

$70.6807 < 5^{\sqrt{7}} < 70.6808$

c. **Based on your work in part (b), what is your best estimate of the value of $5^{\sqrt{7}}$?**

$5^{\sqrt{7}} \approx 70.681$

7. **A rational number raised to a rational power can either be rational or irrational. For example, $4^{\frac{1}{2}}$ is rational because $4^{\frac{1}{2}} = 2$, and $2^{\frac{1}{4}}$ is irrational because $2^{\frac{1}{4}} = \sqrt[4]{2}$. In this problem, you will investigate the possibilities for an irrational number raised to an irrational power.**

a. **Evaluate $\left(\sqrt{2}\right)^{\left(\sqrt{2}\right)^{\sqrt{2}}}$.**

$\left(\sqrt{2}\right)^{\left(\sqrt{2}\right)^{\sqrt{2}}} = \left(\sqrt{2}\right)^{\sqrt{2}\cdot\sqrt{2}} = \left(\sqrt{2}\right)^{2} = 2$

b. **Can the value of an irrational number raised to an irrational power ever be rational?**

Yes. For instance, in part (a) above, $\sqrt{2}$ is irrational, and the number $\sqrt{2}^{\sqrt{2}}$ is either irrational or rational. If $\sqrt{2}^{\sqrt{2}}$ is rational, then this is an example of an irrational number raised to an irrational power that is rational. Otherwise, $\sqrt{2}^{\sqrt{2}}$ is irrational, and part (a) is an example of an irrational number raised to an irrational power that is rational.

Lesson 6: Euler's Number, e

Student Outcomes

- Students write an exponential function that represents the amount of water in a tank after t seconds if the height of the water doubles every 10 seconds.
- Students discover Euler's number e by numerically approaching the constant for which the height of water in a tank equals the rate of change of the height of the water in the tank.
- Students calculate the average rate of change of a function.

Lesson Notes

Leonhard Euler (pronounced "Oiler"), 1707–1783, was a prolific Swiss mathematician and physicist who made many important contributions to mathematics such as much of the modern terminology and notation, including function notation, and popularizing the use of π to represent the circumference of a circle divided by its diameter. Euler also discovered many properties of the irrational number e, which is now known as *Euler's number*. Euler's number naturally occurs in various applications, and a comparison can be made to π, which also occurs naturally. During the lesson, students determine an explicit expression for the height of water in a water tank from its context (**F-BF.A.1a**) and calculate the average rate of change over smaller and smaller intervals to create a sequence that converges to e (**F-IF.B.6**). It is important to stress that the water tank exploration is a way to *define* e. Yes, it is remarkable, but when students discover it, the teacher's reaction should not be "Ta-da! It's magic!" Instead, the teacher should stress that students have defined this special constant (similar to how π is defined as the ratio of any circle's circumference to its diameter) that will be used extensively in the near future and occurs in many different applications.

Classwork

Exercises 1–3 (8 minutes)

In these exercises, students find exponential equations that model the increasing height of water in a cylindrical tank as it doubles over a fixed time interval. These preliminary exercises lead to the discovery of Euler's number, e, at the end of the lesson. As a demonstration, show students the 47-second video in which the height of water in a tank doubles repeatedly until it fills the tank completely; note how long it takes for the height to appear to change at all. Although this situation is contrived, it provides a good visual representation of the power of exponential growth. This is a good time to discuss constraints and how quantities cannot realistically increase exponentially without bound due to physical constraints to the growth. In this case, the water tank has a finite volume, and there is only a finite amount of water on the planet. Likewise, the main constraint to population growth is the availability of such resources as food and land.

After watching the video, students may work individually or in pairs. Point out to students that the growth shown in the video happened much more quickly than it happens in the problems below, but the underlying concept is the same. Students should be prepared to share their solutions with the class.

MP.4

Exercises 1–3

1. **Assume that there is initially 1 cm of water in the tank, and the height of the water doubles every 10 seconds. Write an equation that could be used to calculate the height $H(t)$ of the water in the tank at any time t.**

   The height of the water at time t seconds can be modeled by $H(t) = 2^{t/10}$.

2. **How would the equation in Exercise 1 change if…**
   a. **the initial depth of water in the tank was 2 cm?**

      $H(t) = 2 \cdot 2^{t/10}$

   b. **the initial depth of water in the tank was $\frac{1}{2}$ cm?**

      $H(t) = \frac{1}{2} \cdot 2^{t/10}$

   c. **the initial depth of water in the tank was 10 cm?**

      $H(t) = 10 \cdot 2^{t/10}$

   d. **the initial depth of water in the tank was A cm, for some positive real number A?**

      $H(t) = A \cdot 2^{t/10}$

MP.4

3. **How would the equation in Exercise 2, part (d), change if…**
   a. **the height tripled every ten seconds?**

      $H(t) = A \cdot 3^{t/10}$

   b. **the height doubled every five seconds?**

      $H(t) = A \cdot 2^{t/5}$

   c. **the height quadrupled every second?**

      $H(t) = A \cdot 4^{t}$

   d. **the height halved every ten seconds?**

      $H(t) = A \cdot (0.5^{t/10})$

Scaffolding:

Students working below grade level can create a table of water depths to visualize the accumulation of water. Since the doubling happens every 10 seconds, have them deduce the exponent by asking, "How many times would doubling occur in 30 seconds? How many times would doubling occur in one minute?"

Time (s)	Depth (cm)
0	1
10	2
20	4
30	8

Discussion (2 minutes)

Students have worked informally with the average rate of change of a function before in Algebra I, Modules 3 and 4. For the next examples, the following definition is needed. Go through this definition, and post it on the board or in another prominent place before beginning the next example. Students continue to work with the average rate of change of a function in the Problem Set.

AVERAGE RATE OF CHANGE: Given a function f whose domain contains the interval of real numbers $[a, b]$ and whose range is a subset of the real numbers, the *average rate of change on the interval* $[a, b]$ is defined by the number:

$$\frac{f(b) - f(a)}{b - a}.$$

Example (4 minutes)

Use this example to model the process of finding the average rate of change of the height of the water that is increasing according to one of the exponential functions in the hypothetical scenario. Students repeat this calculation in the exercises that follow. The student materials contain the images below of the three water tanks but not the accompanying formulas.

Example

Consider two identical water tanks, each of which begins with a height of water 1 cm and fills with water at a different rate. Which equations can be used to calculate the height of water in each tank at time t? Use H_1 for tank 1 and H_2 for tank 2.

$H_1(t) = 2^t$ $\qquad$ $H_2(t) = 3^t$

a. **If both tanks start filling at the same time, which one fills first?**

Tank 2 fills first because the level is rising more quickly.

b. **We want to know the average rate of change of the height of the water in these tanks over an interval that starts at a fixed time T as they are filling up. What is the formula for the average rate of change of a function f on an interval $[a, b]$?**

$$\frac{f(b) - f(a)}{b - a}$$

c. **What is the formula for the average rate of change of the function H_1 on an interval $[a, b]$?**

$$\frac{H_1(b) - H_1(a)}{b - a}$$

d. **Let's calculate the average rate of change of the function H_1 on the interval $[T, T + 0.1]$, which is an interval one-tenth of a second long starting at an unknown time T.**

$$\begin{aligned}\frac{H_1(T+0.1) - H_1(T)}{T + 0.1 - T} &= \frac{(2^{T+0.1}) - (2^T)}{0.1} \\ &= \frac{2^T \cdot 2^{0.1} - 2^T}{0.1} \\ &= \frac{2^T(2^{0.1} - 1)}{0.1} \\ &\approx 2^T(0.717735) \\ &\approx 0.717735\, H_1(T)\end{aligned}$$

- So, the average rate of change of the height function is a multiple of the value of the function. This means that the speed at which the height is changing at time T depends on the depth of water at that time. On average, over the interval $[T, T+0.1]$, the water in tank 1 rises at a rate of approximately $0.717\,735\ H_1(T)$ centimeters per second.
- Let's say that at time T there is a height of 5 cm of water in the tank. Then, after one-tenth of a second, the height of the water would increase by $\frac{1}{10}\left(0.717\,735(5)\right) \approx 0.3589$ cm. But if there is a height of 20 cm of water in the tank, after one-tenth of a second, the height of the water would increase by $\frac{1}{10}\left(0.717\,735(20)\right) \approx 1.4355$ cm.

Exercises 4–5 (10 minutes)

Students need to use calculators to compute the numerical constants in the exercises below.

Exercises 4–8

4. **For the second tank, calculate the average change in the height, H_2, from time T seconds to $T+0.1$ second. Express the answer as a number times the value of the original function at time T. Explain the meaning of these findings.**

$$\begin{aligned}\frac{H_2(T+0.1)-H_2(T)}{0.1} &= \frac{3^{T+0.1}-3^T}{0.1}\\ &= \frac{3^T\cdot 3^{0.1}-3^T}{0.1}\\ &= \frac{3^T(3^{0.1}-1)}{0.1}\\ &\approx \frac{3^T(0.116123)}{0.1}\\ &\approx 1.16123\cdot 3^T\\ &\approx 1.16123\cdot H_2(T)\end{aligned}$$

On average, over the time interval $[T, T+0.1]$, the water in tank 2 rises at a rate of approximately $1.16123H_2(T)$ centimeters per second.

5. **For each tank, calculate the average change in height from time T seconds to $T+0.001$ second. Express the answer as a number times the value of the original function at time T. Explain the meaning of these findings.**

Tank 1:

$$\begin{aligned}\frac{H_1(T+0.001)-H_1(T)}{0.001} &= \frac{2^{T+0.001}-2^T}{0.001}\\ &= \frac{2^T\cdot 2^{0.001}-2^T}{0.001}\\ &= \frac{2^T(2^{0.001}-1)}{0.001}\\ &\approx \frac{2^T(0.000693)}{0.001}\\ &\approx 0.69339\cdot 2^T\\ &\approx 0.69339\cdot H_1(T)\end{aligned}$$

On average, over the time interval $[T, T+0.001]$, the water in tank 1 rises at a rate of approximately $0.693387H_1(T)$ centimeter per second.

Tank 2:

$$\begin{aligned}\frac{H_2(T+0.001)-H_2(T)}{0.001} &= \frac{3^{T+0.001}-3^T}{0.001}\\ &= \frac{3^T\cdot 3^{0.001}-3^T}{0.001}\\ &= \frac{3^T(3^{0.001}-1)}{0.001}\\ &\approx \frac{3^T(0.001\,10)}{0.001}\\ &\approx 1.09922\cdot 3^T\\ &\approx 1.09922\cdot H_2(T)\end{aligned}$$

Over the time interval $[T, T+0.001]$*, the water in tank 2 rises at an average rate of approximately* $1.09922H_2(T)$ *centimeters per second.*

Exercises 6–8 (12 minutes)

The following exercises lead to discovery of the constant e that occurs naturally in many situations that can be modeled mathematically. Looking at the results of the previous three exercises, if the height of the water doubles, then the expression for the average rate of change contains a factor less than one. If the height of the water triples, then the expression for the average rate of change contains a factor greater than one. Under what conditions does the expression for the average rate of change contain a factor of exactly one? Answering this question leads to e.

6. **In Exercise 5, the average rate of change of the height of the water in tank 1 on the interval $[T, T+0.001]$ can be described by the expression $c_1\cdot 2^T$, and the average rate of change of the height of the water in tank 2 on the interval $[T, T+0.001]$ can be described by the expression $c_2\cdot 3^T$. What are approximate values of c_1 and c_2?**

   $c_1 \approx 0.69339$ *and* $c_2 \approx 1.09922$

7. **As an experiment, let's look for a value of b so that if the height of the water can be described by $H(t)=b^t$, then the expression for the average rate of change on the interval $[T, T+0.001]$ is $1\cdot H(T)$.**

   a. **Write out the expression for the average rate of change of $H(t)=b^t$ on the interval $[T, T+0.001]$.**

   $$\frac{H_b(T+0.001)-H_b(T)}{0.001}$$

   b. **Set your expression in part (a) equal to $1\cdot H(T)$, and reduce to an expression involving a single b.**

   $$\begin{aligned}\frac{H_b(T+0.001)-H_b(T)}{0.001} &= 1\cdot H_b(T)\\ \frac{b^{T+0.001}-b^T}{0.001} &= b^t\\ \frac{b^T(b^{0.001}-1)}{0.001} &= b^T\\ b^{0.001}-1 &= 0.001\\ b^{0.001} &= 1.001\end{aligned}$$

c. Now we want to find the value of b that satisfies the equation you found in part (b), but we do not have a way to explicitly solve this equation. Look back at Exercise 6; which two consecutive integers have b between them?

We are looking for the base of the exponent that produces a rate of change on a small interval near t that is $1 \cdot H(t)$. When that base is 2, the value of the rate is roughly $0.69H(t)$. When the base is 3, the value of the rate is roughly $1.1H$. Since $0.69 < 1 < 1.1$, the base we are looking for is somewhere between 2 and 3.

d. Use your calculator and a guess-and-check method to find an approximate value of b to 2 decimal places.

Students may choose to use a table such as the table shown below. Make sure that students are maintaining enough decimal places of $b^{0.001}$ to determine which value is closest to 0.001.

b	$b^{0.001}$
2.0	1.00069
2.1	1.00074
2.2	1.00079
2.3	1.00083
2.4	1.00088
2.5	1.00092
2.6	1.00096
2.7	$\underline{1.00099}$
2.8	1.00103
2.9	1.00107
3.0	1.00110

b	$b^{0.001}$
2.70	1.000994
2.71	1.000997
2.72	$\underline{1.001001}$
2.73	1.001005
2.74	
2.75	
2.76	
2.77	
2.78	
2.79	
2.80	

Then $b \approx 2.72$.

8. Verify that for the value of b found in Exercise 7, $\frac{H_b(T+0.001)-H_b(T)}{0.001} \approx H_b(T)$, where $H_b(T) = b^T$.

$$\begin{aligned}\frac{H_b(T+0.001)-H_b(T)}{0.001} &= \frac{2.72^{T+0.001} - 2.72^{0.001}}{0.001} \\ &= \frac{2.72^T(2.72^{0.001} - 1)}{0.001} \\ &\approx \frac{2.72^T(0.001000)}{0.001} \\ &\approx 1.00 \cdot 2.72^T \\ &\approx 1.00 \cdot H_b(T)\end{aligned}$$

When the height of the water increases by a factor of 2.72 units per second, the height at any time is equal to the rate of change of height at that time.

Discussion (2 minutes)

MP.8

If there is time, perform the calculation of b several more times, over smaller and smaller time intervals and finding more and more digits of b. If not, then just present students with the fact below.

- What happens to the value of b?
  - *If we were to keep finding the average rate of change of the function H_b on smaller and smaller time intervals and solving the equation $H_b(t) = A \cdot b^t$, we would find that the height of the water increases by a factor that gets closer and closer to the number $2.718281828\,4$ ….*
- The number that this process leads to is called *Euler's number* and is denoted by e. Like π, e is an irrational number, so it cannot be accurately represented by a decimal expansion. The approximation of e to 13 decimal places is $e \approx 2.7182818284590$.

- Like π, e is important enough to merit inclusion on scientific calculators. Depending on the calculator, e may appear alone, as the base of an exponential expression e^x, or both. Find the e button on your calculator, and experiment with its use. Make sure you can use your calculator to provide an approximation of e, and use the button to calculate e^2 and $2e$.

Closing (4 minutes)

Summarize the lesson with students, and ensure the first two points below are addressed. Have students highlight what they think is important about the lesson in writing or with a partner. Use this as an opportunity to informally assess learning.

- We just discovered the number e, which is important in the world of mathematics. It naturally occurred in our water tank exploration. It also occurs naturally in many other applications, such as finance and population growth.
- Just as we can create and use an exponential function $f(x) = 2^x$ or $f(x) = 10^x$, we can also create and use an exponential function $f(x) = e^x$. The interesting thing about the exponential function base e is that the rate of change of this function at a value a is the same as the value of this function at a.
- Euler's number will surface in a variety of different places in your future exposure to mathematics, and you will see that it is one of the numbers on which much of the mathematics we practice is based.

Lesson Summary

- **Euler's number, e, is an irrational number that is approximately equal to $2.718\,281\,828\,459\,0$.**
- **AVERAGE RATE OF CHANGE: Given a function f whose domain contains the interval of real numbers $[a, b]$ and whose range is a subset of the real numbers, the *average rate of change on the interval* $[a, b]$ is defined by the number**

$$\frac{f(b) - f(a)}{b - a}.$$

Exit Ticket (3 minutes)

Name ____ Date ____________________

Lesson 6: Euler's Number, e

Exit Ticket

1. Suppose that water is entering a cylindrical water tank so that the initial height of the water is 3 cm and the height of the water doubles every 30 seconds. Write an equation of the height of the water at time t seconds.

2. Explain how the number e arose in our exploration of the average rate of change of the height of the water in the water tank.

Exit Ticket Sample Solutions

1. **Suppose that water is entering a cylindrical water tank so that the initial height of the water is 3 cm and the height of the water doubles every 30 seconds. Write an equation of the height of the water at time t seconds.**

   $H(t) = 3\left(2^{\frac{t}{30}}\right)$

2. **Explain how the number e arose in our exploration of the average rate of change of the height of the water in the water tank.**

   We first noticed that if the water level in the tank was doubling every second, then the average rate of change of the height of the water was roughly 0.69 times the height of the water at that time. And if the water level in the tank was tripling every second, then the average rate of change of the height of the water was roughly 1.1 times the height of the water at that time. When we went looking for a base b so that the average rate of change of the height of the water was 1.0 times the height of the water at that time, we found that the base was roughly e. Calculating the average rate of change over shorter intervals gave a better approximation of e.

Problem Set Sample Solutions

Problems 1–5 address other occurrences of e and some fluency practice with the number e, and the remaining problems focus on the average rate of change of a function. The last two problems are extension problems that introduce some ideas of calculus with the familiar formulas for the area and circumference of a circle and the volume and surface area of a sphere.

1. **The product $4 \cdot 3 \cdot 2 \cdot 1$ is called 4 *factorial* and is denoted by $4!$. Then $10! = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$, and for any positive integer n, $n! = n(n-1)(n-2) \cdot \cdots \cdot 3 \cdot 2 \cdot 1$.**

   a. **Complete the following table of factorial values:**

n	1	2	3	4	5	6	7	8
$n!$	1	2	6	24	120	720	5040	40320

   b. **Evaluate the sum $1 + \frac{1}{1!}$.**

   2

   c. **Evaluate the sum $1 + \frac{1}{1!} + \frac{1}{2!}$.**

   2.5

   d. **Use a calculator to approximate the sum $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!}$ to 7 decimal places. Do not round the fractions before evaluating the sum.**

   $\frac{8}{3} \approx 2.6666667$

   e. **Use a calculator to approximate the sum $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!}$ to 7 decimal places. Do not round the fractions before evaluating the sum.**

   $\frac{65}{24} \approx 2.7083333$

f. Use a calculator to approximate sums of the form $1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{k!}$ to 7 decimal places for $k=5,6,7,8,9,10$. Do not round the fractions before evaluating the sums with a calculator.

If $k=5$, the sum is $\frac{163}{60}\approx 2.1766667$.

If $k=6$, the sum is $\frac{1957}{720}\approx 2.7180556$.

If $k=7$, the sum is $\frac{685}{252}\approx 2.7182540$.

If $k=8$, the sum is $\frac{109\,601}{40\,320}\approx 2.7182788$.

If $k=9$, the sum is $\frac{98\,461}{36\,288}\approx 2.7182815$.

If $k=10$, the sum is $\frac{9\,864\,101}{3\,628\,800}\approx 2.7182818$.

g. Make a conjecture about the sums $1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{k!}$ for positive integers k as k increases in size.

It seems that as k gets larger, the sums $1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{k!}$ get closer to e.

h. Would calculating terms of this sequence ever yield an exact value of e? Why or why not?

No. The number e is irrational, so it cannot be written as a quotient of integers. Any finite sum $1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{k!}$ can be expressed as a single rational number with denominator $k!$, so the sums are all rational numbers. However, the more terms that are calculated, the closer to e the sum becomes, so these sums provide better and better rational number approximations of e.

2. Consider the sequence given by $a_n=\left(1+\frac{1}{n}\right)^n$, where $n\geq 1$ is an integer.

a. Use your calculator to approximate the first 5 terms of this sequence to 7 decimal places.

$$a_1=\left(1+\frac{1}{1}\right)^1=2$$
$$a_2=\left(1+\frac{1}{2}\right)^2=2.25$$
$$a_3=\left(1+\frac{1}{3}\right)^3\approx 2.3703704$$
$$a_4=\left(1+\frac{1}{4}\right)^4\approx 2.4414063$$
$$a_5=\left(1+\frac{1}{5}\right)^5=2.4883200$$

b. Does it appear that this sequence settles near a particular value?

No, the numbers get bigger, but we cannot tell if it keeps getting bigger or settles on or near a particular value.

c. **Use a calculator to approximate the following terms of this sequence to 7 decimal places.**

i. a_{100} $= 2.7081383$

ii. a_{1000} $= 2.7169239$

iii. $a_{10,000}$ $= 2.7181459$

iv. $a_{100,000}$ $= 2.7182682$

v. $a_{1,000,000}$ $= 2.7182805$

vi. $a_{10,000,000}$ $= 2.7182816$

vii. $a_{100,000,000}$ $= 2.7182818$

d. **Does it appear that this sequence settles near a particular value?**

Yes, it appears that as n gets really large (at least $100,000,000$), the terms a_n of the sequence settle near the value of e.

e. **Compare the results of this exercise with the results of Problem 1. What do you observe?**

It took about 10 terms of the sum in Problem 1 to see that the sum settled at the value e, but it takes $100,000,000$ terms of the sequence in this problem to see that the sum settles at the value e.

3. **If $x = 5a^4$ and $a = 2e^3$, express x in terms of e, and approximate to the nearest whole number.**

If $x = 5a^4$ and $a = 2e^3$, then $x = 5(2e^3)^4$. Rewriting the right side in an equivalent form gives $x = 80e^{12} \approx 13020383$.

4. **If $a = 2b^3$ and $b = -\frac{1}{2}e^{-2}$, express a in terms of e, and approximate to four decimal places.**

If $a = 2b^3$ and $b = -\frac{1}{2}e^{-2}$, then $a = 2\left(-\frac{1}{2}e^{-2}\right)^3$. Rewriting the right side in an equivalent form gives $a = -\frac{1}{4}e^{-6} \approx -0.0006$.

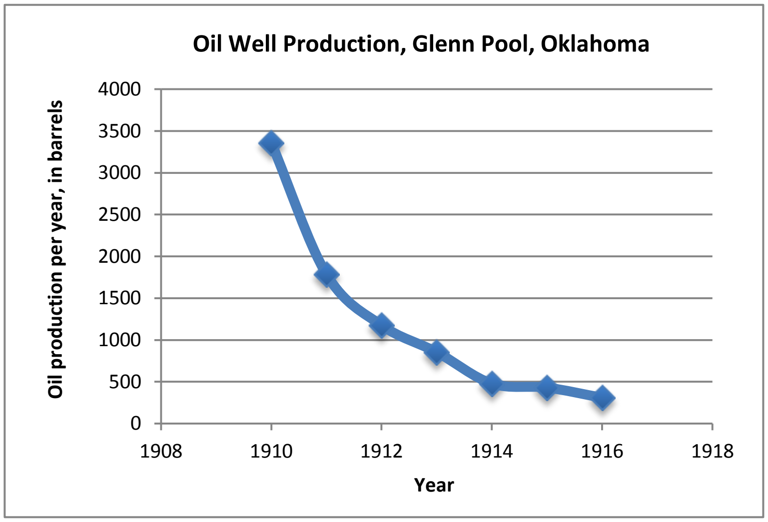
5. **If $x = 3e^4$ and $e = \frac{s}{2x^3}$, show that $s = 54e^{13}$, and approximate s to the nearest whole number.**

Rewrite the equation $e = \frac{s}{2x^3}$ to isolate the variable s.

$$e = \frac{s}{2x^3}$$
$$2x^3e = s$$

By the substitution property, if $s = 2x^3e$ and $x = 3e^4$, then $s = 2(3e^4)^3 \cdot e$. Rewriting the right side in an equivalent form gives $s = 2 \cdot 27e^{12} \cdot e = 54e^{13} \approx 23890323$.

6. The following graph shows the number of barrels of oil produced by the Glenn Pool well in Oklahoma from 1910 to 1916.



Source: Cutler, Willard W., Jr. Estimation of Underground Oil Reserves by Oil-Well Production Curves, U.S. Department of the Interior, 1924.

a. Estimate the average rate of change of the amount of oil produced by the well on the interval $[1910, 1916]$, and explain what that number represents.

Student responses will vary based on how they read the points on the graph. Over the interval $[1910, 1916]$*, the average rate of change is roughly*

$$\frac{300 - 3200}{1916 - 1910} = -\frac{2900}{6} \approx -483.33.$$

This says that the production of the well decreased by an average of about 483 *barrels of oil each year between* 1910 *and* 1916.

b. Estimate the average rate of change of the amount of oil produced by the well on the interval $[1910, 1913]$, and explain what that number represents.

Student responses will vary based on how they read the points on the graph. Over the interval $[1910, 1913]$*, the average rate of change is roughly*

$$\frac{800 - 3200}{1913 - 1910} = -\frac{2400}{3} = -800.$$

This says that the production of the well decreased by an average of about 800 *barrels of oil per year between* 1910 *and* 1913*.*

c. Estimate the average rate of change of the amount of oil produced by the well on the interval $[1913, 1916]$, and explain what that number represents.

Student responses will vary based on how they read the points on the graph. Over the interval $[1913, 1916]$*, the average rate of change is roughly*

$$\frac{300 - 800}{1916 - 1913} = -\frac{500}{3} \approx -166.67.$$

This says that the production of the well decreased by an average of about 166.67 *barrels of oil per year between* 1913 *and* 1916*.*

d. **Compare your results for the rates of change in oil production in the first half and the second half of the time period in question in parts (b) and (c). What do those numbers say about the production of oil from the well?**

The production dropped much more rapidly in the first three years than it did in the second three years. Looking at the graph, it looks like the oil in the well might be running out, so less and less can be extracted each year.

e. **Notice that the average rate of change of the amount of oil produced by the well on any interval starting and ending in two consecutive years is always negative. Explain what that means in the context of oil production.**

Because the average rate of change of oil production over a one-year period is always negative, the well is producing less oil each year than it did the year before.

7. **The following table lists the number of hybrid electric vehicles (HEVs) sold in the United States between 1999 and 2013.**

Year	Number of HEVs Sold in U.S.
1999	17
2000	9350
2001	20,282
2002	36,035
2003	47,600
2004	84,199
2005	209,711
2006	252,636

Year	Number of HEVs Sold in U.S.
2007	352,274
2008	312,386
2009	290,271
2010	274,210
2011	268,752
2012	434,498
2013	495,685

Source: U.S. Department of Energy, Alternative Fuels and Advanced Vehicle Data Center, 2013.

a. **During which one-year interval is the average rate of change of the number of HEVs sold the largest? Explain how you know.**

The average rate of change of the number of HEVs sold is largest during $[2011, 2012]$ because the number of HEVs sold increases by the largest amount between those two years.

b. **Calculate the average rate of change of the number of HEVs sold on the interval $[2003, 2004]$, and explain what that number represents.**

On the interval $[2003, 2004]$, the average rate of change in sales of HEVs is $\frac{84\,199-47\,600}{2004-2003}$, which is $36,599$. This means that during this one-year period, HEVs were selling at a rate of $36,599$ vehicles per year.

c. **Calculate the average rate of change of the number of HEVs sold on the interval $[2003, 2008]$, and explain what that number represents.**

On the interval $[2003, 2008]$, the average rate of change in sales of HEVs is $\frac{312{,}386-47{,}600}{2008-2003}$, which is $52,957.2$. This means that during this five-year period, HEVs were selling at an average rate of $52,957$ vehicles per year.

d. **What does it mean if the average rate of change of the number of HEVs sold is negative?**

If the average rate of change of the vehicles sold is negative, then the sales are declining. This means that fewer cards were sold than in the previous year.

Extension:

8. The formula for the area of a circle of radius r can be expressed as a function $A(r) = \pi r^2$.

   a. Find the average rate of change of the area of a circle on the interval $[4, 5]$.

   $$\frac{A(5) - A(4)}{5 - 4} = \frac{25\pi - 16\pi}{1} = 9\pi$$

   b. Find the average rate of change of the area of a circle on the interval $[4, 4.1]$.

   $$\frac{A(4.1) - A(4)}{4.1 - 4} = \frac{16.81\pi - 16\pi}{0.1} = 8.1\pi$$

   c. Find the average rate of change of the area of a circle on the interval $[4, 4.01]$.

   $$\frac{A(4.01) - A(4)}{4.01 - 4} = \frac{16.0801\pi - 16\pi}{0.01} = 8.01\pi$$

   d. Find the average rate of change of the area of a circle on the interval $[4, 4.001]$.

   $$\frac{A(4.001) - A(4)}{4.001 - 4} = \frac{16.008001\pi - 16\pi}{0.001} = 8.001\pi$$

   e. What is happening to the average rate of change of the area of the circle as the interval gets smaller and smaller?

   The average rate of change of the area of the circle appears to be getting close to 8π.

   f. Find the average rate of change of the area of a circle on the interval $[4, 4 + h]$ for some small positive number h.

   $$\begin{aligned}\frac{A(4+h) - A(4)}{(4+h) - 4} &= \frac{(4+h)^2\pi - 16\pi}{h} \\ &= \frac{(16 + 8h + h^2)\pi - 16\pi}{h} \\ &= \frac{1}{h}(8h + h^2)\pi \\ &= \frac{1}{h} \cdot h(8 + h)\pi \\ &= (8 + h)\pi\end{aligned}$$

   g. What happens to the average rate of change of the area of the circle on the interval $[4, 4 + h]$ as $h \to 0$? Does this agree with your answer to part (d)? Should it agree with your answer to part (e)?

   As $h \to 0$, $8 + h \to 8$, so as h gets smaller, the average rate of change approaches 8. This agrees with my response to part (e), and it should because as $h \to 0$, the interval $[4, 4 + h]$ gets smaller.

h. Find the average rate of change of the area of a circle on the interval $[r_0, r_0 + h]$ for some positive number r_0 and some small positive number h.

$$\begin{aligned}\frac{A(r_0+h)-A(r_0)}{(r_0+h)-r_0} &= \frac{(r_0+h)^2\pi - r_0^2\pi}{h}\\ &= \frac{(r_0^2+2r_0h+h^2)\pi - r_0^2\pi}{h}\\ &= \frac{1}{h}(2r_0h+h^2)\pi\\ &= \frac{1}{h}\cdot h\,(2r_0+h)\pi\\ &= (2r_0+h)\pi\end{aligned}$$

i. What happens to the average rate of change of the area of the circle on the interval $[r_0, r_0 + h]$ as $h \to 0$? Do you recognize the resulting formula?

As $h \to 0$, the expression for the average rate of change becomes $2\pi r_0$, which is the circumference of the circle with radius r_0.

9. The formula for the volume of a sphere of radius r can be expressed as a function $V(r) = \frac{4}{3}\pi r^3$. As you work through these questions, you will see the pattern develop more clearly if you leave your answers in the form of a coefficient times π. Approximate the coefficient to five decimal places.

a. Find the average rate of change of the volume of a sphere on the interval $[2, 3]$.

$$\frac{V(3)-V(2)}{3-2} = \frac{\frac{4}{3}\cdot 27\pi - \frac{4}{3}\cdot 8\pi}{1} = \frac{4}{3}\cdot 19\pi \approx 25.33333\pi$$

b. Find the average rate of change of the volume of a sphere on the interval $[2, 2.1]$.

$$\frac{V(2.1)-V(2)}{2.1-2} = \frac{\frac{4}{3}\pi(2.1^3 - 8)}{0.1} \approx 16.81333\pi$$

c. Find the average rate of change of the volume of a sphere on the interval $[2, 2.01]$.

$$\frac{V(2.01)-V(2)}{2.01-2} = \frac{\frac{4}{3}\pi(2.01^3 - 8)}{0.01} \approx 16.08013\pi$$

d. Find the average rate of change of the volume of a sphere on the interval $[2, 2.001]$.

$$\frac{V(2.001)-V(2)}{2.001-2} = \frac{\frac{4}{3}\pi(2.001^3 - 8)}{0.001} \approx 16.00800\pi$$

e. What is happening to the average rate of change of the volume of a sphere as the interval gets smaller and smaller?

The average rate of change of the volume of the sphere appears to be getting close to 16π.

f. **Find the average rate of change of the volume of a sphere on the interval $[2, 2+h]$ for some small positive number h.**

$$\begin{aligned}\frac{V(2+h)-V(2)}{(2+h)-2} &= \frac{\frac{4}{3}\pi((2+h)^3-8)}{h}\\ &= \frac{4}{3}\pi\cdot\frac{1}{h}(8+12h+6h^2+h^3-8)\\ &= \frac{4\pi}{3h}(12h+6h^2+h^3)\\ &= \frac{4\pi}{3h}\cdot h\,(12+6h+h^2)\\ &= \frac{4\pi}{3}(12+6h+h^2)\end{aligned}$$

g. **What happens to the average rate of change of the volume of a sphere on the interval $[2, 2+h]$ as $h \to 0$? Does this agree with your answer to part (e)? Should it agree with your answer to part (e)?**

As $h \to 0$, the value of the polynomial $12 + 6h + h^2$ approaches 12. Then the average rate of change approaches $\frac{4\pi}{3} \cdot 12 = 16$. This agrees with my response to part (e), and it should because as $h \to 0$, the interval $[2, 2+h]$ gets smaller.

h. **Find the average rate of change of the volume of a sphere on the interval $[r_0, r_0+h]$ for some positive number r_0 and some small positive number h.**

$$\begin{aligned}\frac{V(r_0+h)-V(r_0)}{(r_0+h)-r_0} &= \frac{\frac{4}{3}\pi\left((r_0+h)^3-r_0^3\right)}{h}\\ &= \frac{4}{3}\pi\cdot\frac{1}{h}\left(r_0^3+3r_0^2h+3r_0h^2+h^3-r_0^3\right)\\ &= \frac{4\pi}{3h}(3r_0^2h+3r_0h^2+h^3)\\ &= \frac{4\pi}{3h}\cdot h(3r_0^2+3r_0h+h^2)\\ &= \frac{4\pi}{3}(3r_0^2+3r_0h+h^2)\end{aligned}$$

i. **What happens to the average rate of change of the volume of a sphere on the interval $[r_0, r_0+h]$ as $h \to 0$? Do you recognize the resulting formula?**

As $h \to 0$, the expression for the average rate of change becomes $4\pi r_0^2$, which is the surface area of the sphere with radius r_0.

Lesson 7: Bacteria and Exponential Growth

Student Outcomes

- Students solve simple exponential equations numerically.

Lesson Notes

The lessons in Topic A familiarized students with the laws and properties of real-valued exponents. Topic B introduces the logarithm and develops logarithmic properties through exploration of logarithmic tables, primarily in base 10. This lesson introduces simple exponential equations whose solutions do not follow from equating exponential terms of equal bases. Because students have no sophisticated tools for solving exponential equations until logarithms are introduced in later lessons, numerical methods must be used to approximate solutions to exponential equations, a process that asks students to determine a recursive process from a context to solve $2^x = 10$ (**F-BF.A.1a**, **F-BF.B.4a**, **A-CED.A.1**). Students have many opportunities to solve such equations algebraically throughout the module, using both the technique of equating exponents of exponential expressions with the same base and using properties of logarithms. The goals of this lesson are to help students understand (1) why logarithms are useful by introducing a situation (i.e., solving $2^x = 10$) offering students no option other than numerical methods to solve it, (2) that it is often possible to solve equations numerically by trapping the solution through better and better approximation, and (3) that the better and better approximations are converging on a (possibly) irrational number.

Exponential equations are used frequently to model bacteria and population growth, and both of those scenarios occur in this lesson.

Classwork

Opening Exercise (6 minutes)

In this exercise, students work in groups to solve simple exponential equations that can be solved by rewriting the expressions on each side of the equation as a power of the same base and equating exponents. It is also possible for students to use a table of values to solve these problems numerically; either method is valid, and both should be discussed at the end of the exercise. Asking students to solve equations of this type demands that they think deeply about the meaning of exponential expressions. Because students have not solved exponential equations previously, the exercises are scaffolded to begin very simply and progress in difficulty; the early exercises may be merely solved by inspection. When students are finished, ask for volunteers to share their solutions on the board and discuss different solution methods.

Opening Exercise

Work with your partner or group to solve each of the following equations for x.

a. $2^x = 2^3$

$x = 3$

b. $2^x = 2$

$x = 1$

Scaffolding:

Encourage struggling students to make a table of values of the powers of 2 to use as a reference for these exercises.

c. $2^x = 16$

$2^x = 2^4$
$x = 4$

d. $2^x - 64 = 0$

$2^x = 64$
$2^x = 2^6$
$x = 6$

e. $2^x - 1 = 0$

$2^x = 1$
$2^x = 2^0$
$x = 0$

f. $2^{3x} = 64$

$2^{3x} = 2^6$
$3x = 6$
$x = 2$

g. $2^{x+1} = 32$

$2^{x+1} = 2^5$
$x + 1 = 5$
$x = 4$

Scaffolding:
Give early finishers a more challenging equation where both bases need to be changed, such as $4^{2x} = 8^{x+3}$.

Discussion (3 minutes)

This Discussion should emphasize that the equations in the Opening Exercise have straightforward solutions because both sides can be written as an exponential expression with base 2.

MP.7

- How did the structure of the expressions in these equations allow you to solve them easily?
  - *Both sides of the equations could be written as exponential expressions with base* 2.
- Suppose the Opening Exercise had asked us to solve the equation $2^x = 10$ instead of the equation $2^x = 8$. Why is it far more difficult to solve the equation $2^x = 10$?
  - *We do not know how to express* 10 *as a power of* 2. *In the Opening Exercise, it is straightforward that* 8 *can be expressed as* 2^3.
- Can we find two integers that are under and over estimates of the solution to $2^x = 10$? That is, can we find a and b so that $a < x < b$?
  - *Yes; the unknown* x *is between* 3 *and* 4 *because* $2^3 < 10 < 2^4$.
- In the next example, we will use a calculator (or other technology) to find a more accurate estimate of the solutions to $2^x = 10$.

Example (12 minutes)

The purpose of this exercise is to numerically pinpoint the solution d to the equation $2^d = 10$ by squeezing the solution between numbers that get closer and closer together. Start with $3 < d < 4$, and then find that $2^{3.3} < 10$ and $10 < 2^{3.4}$, so $3.3 < d < 3.4$. Continuing with this logic, squeeze $3.32 < d < 3.33$ and then $3.321 < d < 3.322$. The point of this exercise is that it is possible to continue squeezing d between numbers with more and more digits, meaning that there is an approximation of d to greater and greater accuracy.

In the student materials, the tables for the Discussion on the following pages are presented next to each other, but they are spread out here to show how they fit into the Discussion.

Example

The *Escherichia coli* bacteria (commonly known as *E. coli*) reproduces once every 30 minutes, meaning that a colony of *E. coli* can double every half hour. *Mycobacterium tuberculosis* has a generation time in the range of 12 to 16 hours. Researchers have found evidence that suggests certain bacteria populations living deep below the surface of the earth may grow at extremely slow rates, reproducing once every several thousand years. With this variation in bacterial growth rates, it is reasonable that we assume a 24-hour reproduction time for a hypothetical bacteria colony in this example.

Suppose we have a bacteria colony that starts with 1 bacterium, and the population of bacteria doubles every day.

What function P can we use to model the bacteria population on day t?

$P(t) = 2^t$, for real numbers $t \geq 0$.

Have students volunteer values of $P(t)$ to help complete the following table.

t	$P(t)$
1	2
2	4
3	8
4	16
5	32

How many days will it take for the bacteria population to reach 8?

It will take 3 days because $P(3) = 2^3 = 8$.

How many days will it take for the bacteria population to reach 16?

It will take 4 days because $P(4) = 2^4 = 16$.

Roughly how long will it take for the population to reach 10?

Between 3 and 4 days; the number d so that $2^d = 10$

We already know from our previous discussion that if $2^d = 10$, then $3 < d < 4$, and the table confirms that. At this point, we have an underestimate of 3 and an overestimate of 4 for d. How can we find better under and over estimates for d?

(Note to teacher: Once students respond, have them volunteer values to complete the table.)

Calculate the values of $2^{3.1}$, $2^{3.2}$, $2^{3.3}$, etc., until we find two consecutive values that have 10 between them.

t	$P(t)$
3.1	8.574
3.2	9.190
3.3	9.849
3.4	10.556

MP.8

From our table, we now know another set of under and over estimates for the number d that we seek. What are they?

We know d is between 3.3 and 3.4. That is, $3.3 < d < 3.4$.

Continue this process of "squeezing" the number d between two numbers until you are confident you know the value of d to two decimal places.

t	$P(t)$
3.31	9.918
3.32	9.987
3.33	10.056

t	$P(t)$
3.321	9.994
3.322	10.001

Since $3.321 < d < 3.322$, and both numbers round to 3.32, we can say that $d \approx 3.32$. We see that the population reaches 10 after 3.32 days (i.e., $2^{3.32} \approx 10$).

What if we had wanted to find d to 5 decimal places?

Keep squeezing d between under and over estimates until they agree to the first 5 decimal places. (Note to teacher: To 5 decimal places, $3.321928 < d < 3.321929$, so $d \approx 3.32193$.)

To the nearest minute, when does the population of bacteria become 10?

It takes 3.322 days, which is roughly 3 days, 7 hours, and 43 minutes.

Discussion (2 minutes)

- Could we repeat the same process to find the time required for the bacteria population to reach 20 (or 100 or 500)?
  - *Yes. We could start by determining between which two integers the solution to the equation $2^t = 20$ must lie and then continue the same process to find the solution.*
- Could we achieve the same level of accuracy as we did in the example? Could we make our solution more accurate?
  - *Yes. We can continue to repeat the process and eventually "trap" the solution to as many decimal places as we would like.*

Lead students to the idea that for any positive number x, they can repeat the process above to approximate the exponent L so that $2^L = x$ to as many decimal places of accuracy as they would like. Likewise, they can approximate an exponent so that they can write the number x as a power of 10 or a power of 3 or a power of e or a power of any positive number other than 1.

Note that there is a little bit of a theoretical hole here that is filled in later when the logarithm function is introduced. In this lesson, students are only finding a rational approximation to the value of the exponent, which is the logarithm and is generally an irrational number. That is, in this example, they are not truly writing 10 as a power of 2, but they are only finding a close approximation. If students question this subtle point, let them know that later in the module they have definitive ways to write any positive number exactly as a power of the base.

Exercise (8 minutes)

Divide students into groups of two or three, and assign each group a different equation to solve from the list below. Students should repeat the process of the Example to solve these equations by squeezing the solution between more and more precise under and over estimates. Record the solutions in a way that students can see the entire list either written on poster board, written on the white board, or projected through the document camera.

Exercise

Use the method from the Example to approximate the solution to the equations below to two decimal places.

a. $2^x = 1000$ $x \approx 9.97$

b. $3^x = 1000$ $x \approx 6.29$

c. $4^x = 1000$ $x \approx 4.98$

d. $5^x = 1000$ $x \approx 4.29$

e. $6^x = 1000$ $x \approx 3.86$

f. $7^x = 1000$ $x \approx 3.55$

g. $8^x = 1000$ $x \approx 3.32$

h. $9^x = 1000$ $x \approx 3.14$

i. $11^x = 1000$ $x \approx 2.88$

j. $12^x = 1000$ $x \approx 2.78$

k. $13^x = 1000$ $x \approx 2.69$

l. $14^x = 1000$ $x \approx 2.62$

m. $15^x = 1000$ $x \approx 2.55$

n. $16^x = 1000$ $x \approx 2.49$

Discussion (2 minutes)

MP.3

- Do you observe a pattern in the solutions to the equations in the above Exercise?
  - *Yes. The larger the base, the smaller the solution.*
- Why would that be?
  - *The larger the base, the smaller the exponent needs to be in order to reach* 1,000.

Closing (4 minutes)

Have students respond to the following questions individually in writing or orally with a partner.

- Explain when a simple exponential equation, such as those we have seen today, can be solved exactly using our current methods.
  - *If both sides of the equation can be written as exponential expressions with the same base, then the equation can be solved exactly.*
- When a simple exponential equation cannot be solved by hand, what can we do?
  - *We can give crude under and over estimates for the solution using integers.*
  - *We can use a calculator to find increasingly accurate under and over estimates to the solution until we are satisfied.*

Exit Ticket (8 minutes)

Name ____ Date ____________________

Lesson 7: Bacteria and Exponential Growth

Exit Ticket

Loggerhead turtles reproduce every 2 to 4 years, laying approximately 120 eggs in a clutch. Using this information, we can derive an approximate equation to model the turtle population. As is often the case in biological studies, we will count only the female turtles. If we start with a population of one female turtle in a protected area and assume that all turtles survive, we can roughly approximate the population of female turtles by $T(t) = 5^t$. Use the methods of the Example to find the number of years, Y, it will take for this model to predict that there will be 300 female turtles. Give your answer to two decimal places.

Exit Ticket Sample Solutions

Loggerhead turtles reproduce every 2 to 4 years, laying approximately 120 eggs in a clutch. Using this information, we can derive an approximate equation to model the turtle population. As is often the case in biological studies, we will count only the female turtles. If we start with a population of one female turtle in a protected area and assume that all turtles survive, we can roughly approximate the population of female turtles by $T(t) = 5^t$. Use the methods of the Example to find the number of years, Y, it will take for this model to predict that there will be 300 female turtles. Give your answer to two decimal places.

Since $5^3 = 125$ and $5^4 = 625$, we know that $3 < Y < 4$.

Since $5^{3.5} \approx 279.5084$ and $5^{3.6} \approx 328.3160$, we know that $3.5 < Y < 3.6$.

Since $5^{3.54} \approx 298.0944$ and $5^{3.55} \approx 302.9308$, we know that $3.54 < Y < 3.55$.

Since $5^{3.543} \approx 299.5372$ and $5^{3.544} \approx 300.0196$, we know that $3.543 < Y < 3.544$.

Thus, to two decimal places, we have $Y \approx 3.54$. So, it will take roughly $3\frac{1}{2}$ years for the population to grow to 300 female turtles.

Problem Set Sample Solutions

The Problem Set gives students an opportunity to practice using the numerical methods established in the lesson for approximating solutions to exponential equations.

1. Solve each of the following equations for x using the same technique as was used in the Opening Exercise.

a. $2^x = 32$

$x = 5$

b. $2^{x-3} = 2^{2x+5}$

$x = -8$

c. $2^{x^2-3x} = 2^{-2}$

$x = 1$ *or* $x = 2$

d. $2^x - 2^{4x-3} = 0$

$x = 1$

e. $2^{3x} \cdot 2^5 = 2^7$

$x = \frac{2}{3}$

f. $2^{x^2-16} = 1$

$x = 4$ *or* $x = -4$

g. $3^{2x} = 27$

$x = \frac{3}{2}$

h. $3^{\frac{2}{x}} = 81$

$x = \frac{1}{2}$

i. $\frac{3^{x^2}}{3^{5x}} = 3^6$

$x = 6$ *or* $x = -1$

2. Solve the equation $\frac{2^{2x}}{2^{x+5}} = 1$ algebraically using two different initial steps as directed below.

a. **Write each side as a power of 2.**

$$2^{2x-(x+5)} = 2^0$$
$$x - 5 = 0$$
$$x = 5$$

b. **Multiply both sides by 2^{x+5}.**

$$2^{2x} = 2^{x+5}$$
$$2x = x + 5$$
$$x = 5$$

3. Find consecutive integers that are under and over estimates of the solutions to the following exponential equations.

a. $2^x = 20$

$2^4 = 16$ *and* $2^5 = 25$, *so* $4 < x < 5$.

b. $2^x = 100$

$2^6 = 64$ *and* $2^7 = 128$, *so* $6 < x < 7$.

c. $3^x = 50$

$3^3 = 27$ *and* $3^4 = 81$, *so* $3 < x < 4$.

d. $10^x = 432,901$

$10^5 = 100,000$ *and* $10^6 = 1,000,000$, *so* $5 < x < 6$.

e. $2^{x-2} = 750$

$2^9 = 512$ *and* $2^{10} = 1,024$, *so* $9 < x - 2 < 10$; *thus*, $11 < x < 12$.

f. $2^x = 1.35$

$2^0 = 1$ *and* $2^1 = 2$, *so* $0 < x < 1$.

4. Complete the following table to approximate the solution to $10^x = 34,198$ to three decimal places.

x	10^x
1	10
2	100
3	1,000
4	10,000
5	100,000

x	10^x
4.1	12,589.254
4.2	15,848.932
4.3	19,952.623
4.4	25,118.864
4.5	31,622.777
4.6	39,810.717

x	10^x
4.51	32,359.366
4.52	33,113.112
4.53	33,884.416
4.54	34,673.685

x	10^x
4.531	33,962.527
4.532	34,040.819
4.533	34,119.291
4.534	34,197.944
4.535	34,276.779

$10^x = 34,198$

$10^{4.534} \approx 34,198$, *so* $x \approx 4.534$.

5. Complete the following table to approximate the solution to $2^x = 19$ to three decimal places.

x	2^x
1	2
2	4
3	8
4	16
5	32

x	2^x
4.1	17.1484
4.2	18.3792
4.3	19.6983

x	2^x
4.21	18.5070
4.22	18.6357
4.23	18.7654
4.24	18.8959
4.25	19.0273

x	2^x
4.241	18.9090
4.242	18.9221
4.243	18.9352
4.244	18.9483
4.245	18.9615
4.246	18.9746
4.247	18.9878
4.248	19.0010

$2^x = 19$

$2^{4.248} \approx 19$, *so* $x \approx 4.248$.

6. Approximate the solution to $5^x = 5555$ to four decimal places.

Since $5^5 = 3125$ *and* $5^6 = 15\,625$, *we know that* $5 < x < 6$.

Since $5^{5.3} \approx 5064.5519$ *and* $5^{5.4} \approx 5948.9186$, *we know that* $5.3 < x < 5.4$.

Since $5^{5.35} \approx 5488.9531$ *and* $5^{5.36} \approx 5578.0092$, *we know that* $5.35 < x < 5.36$.

Since $5^{5.357} \approx 5551.1417$ *and* $5^{5.358} \approx 5560.0831$, *we know that* $5.357 < x < 5.358$.

Since $5^{5.3574} \approx 5554.7165$ *and* $5^{5.3575} \approx 5555.6106$, *we know that* $5.3574 < x < 5.3575$.

Since $5^{5.35743} \approx 5554.9847$ *and* $5^{5.35744} \approx 5555.0741$, *we know that* $5.35743 < x < 5.35744$.

Thus, the approximate solution to this equation to four decimal places is 5.3574.

7. A dangerous bacterial compound forms in a closed environment but is immediately detected. An initial detection reading suggests the concentration of bacteria in the closed environment is one percent of the fatal exposure level. This bacteria is known to double in concentration in a closed environment every hour and can be modeled by the function $P(t) = 100 \cdot 2^t$, where t is measured in hours.

a. In the function $P(t) = 100 \cdot 2^t$, what does the 100 mean? What does the 2 mean?

The 100 *represents the initial population of bacteria, which is* 1% *of the fatal level. This means that the fatal level occurs when* $P(t) = 10,000$. *The base* 2 *represents the growth rate of the bacteria; it doubles every hour.*

b. Doctors and toxicology professionals estimate that exposure to two-thirds of the bacteria's fatal concentration level will begin to cause sickness. Without consulting a calculator or other technology, offer a rough time limit for the inhabitants of the infected environment to evacuate in order to avoid sickness.

The bacteria level is dangerous when $P(t) = 100 \cdot 2^t = \frac{2}{3}(10\,000) \approx 6666.67$.

Since $2^6 = 64$, $P(6) \approx 6400$, *inhabitants of the infected area should evacuate within* 6 *hours.*

c. **A more conservative approach is to evacuate the infected environment before bacteria concentration levels reach one-third of fatal levels. Without consulting a calculator or other technology, offer a rough time limit for evacuation in this circumstance.**

Under these guidelines, the bacteria level is dangerous when $P(t) = 100 \cdot 2^t = \frac{1}{3}(10\,000) \approx 3333.33$. Since $2^5 = 32$, $P(5) \approx 3200$, so the conservative approach is to recommend evacuation within 5 hours.

d. **Use the method of the Example to approximate when the bacteria concentration will reach 100% of the fatal exposure level, to the nearest minute.**

We need to approximate the solution to $100 \cdot 2^t = 10,000$, which is equivalent to solving $2^t = 100$.

t	2^t
1	2
2	4
3	8
4	16
5	32
6	64
7	128

t	2^t
6.1	68.5935
6.2	73.5167
6.3	78.7932
6.4	84.4485
6.5	90.5097
6.6	97.0059
6.7	103.9683

t	2^t
6.61	97.6806
6.62	98.3600
6.63	99.0442
6.64	99.7331
6.65	100.4268

t	2^t
6.641	99.8022
6.642	99.8714
6.643	99.9407
6.644	100.0010

t	2^t
6.6436	99.9822
6.6437	99.9892
6.6438	99.9961
6.6439	100.0030

Inhabitants need to evacuate within 6.644 hours, which is approximately 6 hours and 39 minutes.

Lesson 8: The "WhatPower" Function

Student Outcomes

- Students calculate a simple logarithm using the definition.

Lesson Notes

The term *logarithm* is foreign and can be intimidating, so the lesson begins with a simple renaming of the logarithm function to the more intuitive "WhatPower" function. Do not explain this function to students directly, but let them figure out what the function does. The first two exercises have already been solved to provide a hint of how the "WhatPower" function works.

This lesson is the first introduction to logarithms, and the work done here prepares students to solve exponential equations of the form $ab^{ct} = d$ (**F-LE.A.4**) and use logarithms to model relationships between two quantities (**F-BF.B.4a**) in later lessons. In the next lessons, students create logarithm tables to discover some of the basic properties of logarithms before continuing on to look at the graphs of logarithmic functions and then to finally modeling logarithmic data. In this lesson, the ideas and notation of logarithmic expressions are developed, leaving many ideas to be explored later in the module.

Classwork

Opening Exercise (12 minutes)

Allow students to work in pairs or small groups to complete these exercises. Do not explain this function to students directly, but allow them to struggle to figure out what this new "WhatPower" function means and how to evaluate these expressions. When there are about two minutes left, instruct groups that have not finished part (a) to skip to part (b) so that all groups have time to think about and state the definition of this function. Consider collecting the groups' definitions on paper and sharing some or all of them with the class using the document camera. This definition is refined through the lessons; in particular, we are interested in the allowable values of the base b.

Opening Exercise

a. **Evaluate each expression. The first two have been completed for you.**

i. $\text{WhatPower}_2(8) = 3$

3, because $2^3 = 8$

ii. $\text{WhatPower}_3(9) = 2$

2, because $3^2 = 9$

iii. $\text{WhatPower}_6(36) =$ ______

2, because $6^2 = 36$

iv. $\text{WhatPower}_2(32) =$ ______

5, *because* $2^5 = 32$

v. $\text{WhatPower}_{10}(1000) =$ ______

3, *because* $10^3 = 1000$

vi. $\text{WhatPower}_{10}(1000000) =$ ______

6, *because* $10^6 = 1,000,000$

vii. $\text{WhatPower}_{100}(1000000) =$ ______

3, *because* $100^3 = 1,000,000$

viii. $\text{WhatPower}_4(64) =$ ______

3, *because* $4^3 = 64$

ix. $\text{WhatPower}_2(64) =$ ______

6, *because* $2^6 = 64$

x. $\text{WhatPower}_9(3) =$ ______

$\frac{1}{2}$, *because* $9^{\frac{1}{2}} = 3$

xi. $\text{WhatPower}_5(\sqrt{5}) =$ ______

$\frac{1}{2}$, *because* $5^{\frac{1}{2}} = \sqrt{5}$

xii. $\text{WhatPower}_{\frac{1}{2}}\left(\frac{1}{8}\right) =$ ______

3, *because* $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$

xiii. $\text{WhatPower}_{42}(1) =$ ______

0, *because* $42^0 = 1$

xiv. $\text{WhatPower}_{100}(0.01) =$ ______

-1, *because* $100^{-1} = 0.01$

xv. $\text{WhatPower}_2\left(\frac{1}{4}\right) =$ ______

-2, *because* $2^{-2} = \frac{1}{4}$

xvi. $\mathrm{WhatPower}_{\frac{1}{4}}(2) =$ ______

$-\frac{1}{2}$, *because* $\left(\frac{1}{4}\right)^{-\frac{1}{2}} = 4^{\frac{1}{2}} = 2$

MP.8

b. **With your group members, write a definition for the function $\mathrm{WhatPower}_b$, where b is a number.**

The value of $\mathrm{WhatPower}_b$ is the number you need to raise b to in order to get x. That is, if $b^L = x$, then $L = \mathrm{WhatPower}_b(x)$.

Discussion (3 minutes)

Discuss the definitions students created, but do not settle on an official definition just yet. To reinforce the idea of how this function works, ask students a series of "WhatPower" questions, writing the expressions on the board or the document camera and reading $\mathrm{WhatPower}_b(x)$ as "What power of b is x?" Be sure that students are visually seeing the odd structure of this notation and hearing the question "What power of b is x?" to reinforce the meaning of this function that depends on both the parameter b and the variable x.

- $\mathrm{WhatPower}_2(16)$
  - 4
- $\mathrm{WhatPower}_2(4)$
  - 2
- $\mathrm{WhatPower}_2(\sqrt{2})$
  - $\frac{1}{2}$
- $\mathrm{WhatPower}_2(1)$
  - 0
- $\mathrm{WhatPower}_2\left(\frac{1}{8}\right)$
  - -3

Exercises 1–9 (8 minutes)

The point of this set of exercises is for students to determine which real numbers b make sense as a base for the $\mathrm{WhatPower}_b$ function. Have students complete this exercise in pairs or small groups, and allow time for students to debate.

Exercises 1–9

Evaluate the following expressions, and justify your answers.

1. $\mathrm{WhatPower}_7(49)$

   $\mathrm{WhatPower}_7(49) = 2$ *because* $7^2 = 49$.

2. $\mathrm{WhatPower}_0(7)$

   $\mathrm{WhatPower}_0(7)$ *does not make sense because there is no power of* 0 *that will produce* 7.

3. $\text{WhatPower}_5(1)$

$\text{WhatPower}_5(1) = 0$ because $5^0 = 1$.

4. $\text{WhatPower}_1(5)$

$\text{WhatPower}_1(5)$ does not exist because for any exponent L, $1^L = 1$, so there is no power of 1 that will produce 5.

5. $\text{WhatPower}_{-2}(16)$

$\text{WhatPower}_{-2}(16) = 4$ because $(-2)^4 = 16$.

6. $\text{WhatPower}_{-2}(32)$

$\text{WhatPower}_{-2}(32)$ does not make sense because there is no power of -2 that will produce 32.

7. $\text{WhatPower}_{\frac{1}{3}}(9)$

$\text{WhatPower}_{\frac{1}{3}}(9) = -2$ because $\left(\frac{1}{3}\right)^{-2} = 9$.

8. $\text{WhatPower}_{-\frac{1}{3}}(27)$

$\text{WhatPower}_{-\frac{1}{3}}(27)$ does not make sense because there is no power of $-\frac{1}{3}$ that will produce 27.

9. **Describe the allowable values of b in the expression $\text{WhatPower}_b(x)$. When can we define a function $f(x) = \text{WhatPower}_b(x)$? Explain how you know.**

If $b = 0$ or $b = 1$, then the expression $\text{WhatPower}_b(x)$ does not make sense. If $b < 0$, then the expression $\text{WhatPower}_b(x)$ makes sense for some values of x but not for others, so we cannot define a function $f(x) = \text{WhatPower}_b(x)$ if $b < 0$. Thus, we can define the function $f(x) = \text{WhatPower}_b(x)$ if $b > 0$ and $b \neq 1$.

Discussion (5 minutes)

Ask student groups to share their responses to Exercise 9 in which they determined which values of b are allowable in the WhatPower_b function. By the end of this Discussion, be sure that all groups understand that it is necessary to restrict b so that either $0 < b < 1$ or $b > 1$. Then, continue on to rename the WhatPower function to its true name, the logarithm base b.

- What we are calling the "WhatPower" function is known by the mathematical term *logarithm,* built from the Greek word *logos* (pronounced lo-gohs), meaning ratio, and *arithmos* (pronounced uh-rith-mohs), meaning number. The number b is the base of the logarithm, and we denote the logarithm base b of x (which means the power to which we raise b to get x) by $\log_b(x)$. That is, whenever you see $\log_b(x)$, think of $\text{WhatPower}_b(x)$.

MP.8

Discuss the definition shown in the Frayer diagram below. Ask students to articulate the definition in their own words to a partner and then share some responses. Have students work with a partner to fill in the remaining parts of the diagram and then share responses as a class. Provide some sample examples and non-examples as needed to illustrate some of the characteristics of logarithms.

Definition of the Logarithm Base b	Characteristics
If three numbers L, b, and x with $0 < b < 1$ or $b > 1$ are related by $b^L = x$, then L is the logarithm base b of x, and we write $\log_b(x) = L$.	*If $b > 1$ and $x > 1$, $\log_b(x) > 0$.* *If $0 < b < 1$ and $x > 1$, then $\log_b(x) < 0$.*
Examples	**Non-Examples**
$\log_2(8) = 3$ $\log_5(1) = 0$ $\log_{10}(0.01) = -2$	$\log_0(x)$ $\log_{-1}(x)$ $\log_{10}(-2)$

Logarithm

- What are some examples of logarithms?
  - $\log_2(4) = 2$
  - $\log_3(27) = 3$
  - $\log_{10}(0.10) = -1$
- What are some non-examples?
  - $\log_0(4)$
  - $\log_1(4)$
- Why can't $b = 0$? Why can't $b = 1$?
  - *If there is a number L so that $\log_0(4) = L$, then $0^L = 4$. But there is no number L such that 0^L is 4, so this does not make sense. Similar reasoning can be applied to $\log_1(4)$.*
- Is $\log_5(25)$ a valid example?
  - *Yes. $\log_5(25) = 2$ because $5^2 = 25$.*
- Is $\log_5(-25)$ a valid example?
  - *No. There is no number L such that $5^L = -25$. It is impossible to raise a positive base to an exponent and get a negative value.*
- Is $\log_5(0)$ a valid example?
  - *No. There is no number L such that $5^L = 0$. It is impossible to raise a positive base to an exponent and get an answer of 0.*
- So, what are some characteristics of logarithms?
  - *The base b must be a positive number not equal to 1. The input must also be a positive number. The output may be any real number (positive, negative, or 0).*

Examples (4 minutes)

Lead the class through the computation of the following logarithms. These have all been computed in the Opening Exercise using the "WhatPower" terminology.

Examples

1. $\log_2(8) = 3$

   3, because $2^3 = 8$

2. $\log_3(9) = 2$

   2, because $3^2 = 9$

3. $\log_6(36) =$ ______

   2, because $6^2 = 36$

4. $\log_2(32) =$ ______

   5, because $2^5 = 32$

5. $\log_{10}(1000) =$ ______

   3, because $10^3 = 1000$

6. $\log_{42}(1) =$ ______

   0, because $42^0 = 1$

7. $\log_{100}(0.01) =$ ______

   -1, because $100^{-1} = 0.01$

8. $\log_2\left(\frac{1}{4}\right) =$ ______

   -2, because $2^{-2} = \frac{1}{4}$

Exercise 10 (6 minutes)

Have students complete this exercise alone or in pairs.

Exercise 10

10. **Compute the value of each logarithm. Verify your answers using an exponential statement.**

   a. $\log_2(32)$

   $\log_2(32) = 5$ because $2^5 = 32$.

   b. $\log_3(81)$

   $\log_3(81) = 4$ because $3^4 = 81$.

   c. $\log_9(81)$

   $\log_9(81) = 2$ because $9^2 = 81$.

   d. $\log_5(625)$

   $\log_5(625) = 4$ because $5^4 = 625$.

   e. $\log_{10}(1,000,000,000)$

   $\log_{10}(1,000,000,000) = 9$, because $10^9 = 1,000,000,000$.

   f. $\log_{1000}(1000000000)$

   $\log_{1000}(1000000000) = 3$, because $1000^3 = 1,000,000,000$.

   g. $\log_{13}(13)$

   $\log_{13}(13) = 1$ because $13^1 = 13$.

   h. $\log_{13}(1)$

   $\log_{13}(1) = 0$ because $13^0 = 1$.

Scaffolding:

- If students are struggling with notation, give them examples where they convert between logarithmic and exponential form.
- Use this chart as a visual support.

Logarithmic Form	Exponential Form
$\log_8(64) = 2$	
	$8^{-2} = \frac{1}{64}$
$\log_{64}(4) = \frac{1}{3}$	

i. $\log_7(\sqrt{7})$

$\log_7(\sqrt{7}) = \frac{1}{2}$ *because* $7^{\frac{1}{2}} = \sqrt{7}$.

j. $\log_9(27)$

$\log_9(27) = \frac{3}{2}$ *because* $9^{\frac{3}{2}} = 3^3 = 27$

k. $\log_{\sqrt{7}}(7)$

$\log_{\sqrt{7}}(7) = 2$ *because* $\left(\sqrt{7}\right)^2 = 7$.

l. $\log_{\sqrt{7}}\left(\frac{1}{49}\right)$

$\log_{\sqrt{7}}\left(\frac{1}{49}\right) = -4$ *because* $\left(\sqrt{7}\right)^{-4} = \frac{1}{\left(\sqrt{7}\right)^4} = \frac{1}{49}$.

m. $\log_x(x^2)$

$\log_x(x^2) = 2$ *because* $(x)^2 = x^2$.

Closing (2 minutes)

Ask students to summarize the important parts of the lesson, either in writing, to a partner, or as a class. Use this as an opportunity to informally assess understanding of the lesson. The following are some important summary elements.

Lesson Summary

- If three numbers L, b, and x are related by $x = b^L$, then L is the *logarithm base* b of x, and we write $\log_b(x) = L$. That is, the value of the expression $\log_b(x)$ is the power of b needed to obtain x.
- Valid values of b as a base for a logarithm are $0 < b < 1$ and $b > 1$.

Exit Ticket (5 minutes)

Name ____ Date ____________________

Lesson 8: The "WhatPower" Function

Exit Ticket

1. Explain why we need to specify $0 < b < 1$ and $b > 1$ as valid values for the base b in the expression $\log_b(x)$.

2. Calculate the following logarithms.
   a. $\log_5(25)$

   b. $\log_{10}\left(\frac{1}{100}\right)$

   c. $\log_9(3)$

Exit Ticket Sample Solutions

1. Explain why we need to specify $0 < b < 1$ and $b > 1$ as valid values for the base b in the expression $\log_b(x)$.

   If $b = 0$, then $\log_0(x) = L$ means that $0^L = x$, which cannot be true if $x \neq 0$.

   If $b = 1$, then $\log_1(x) = L$ means that $1^L = x$, which cannot be true if $x \neq 1$.

   If $b < 0$, then $\log_b(x) = L$ makes sense for some but not all values of $x > 0$; for example, if $b = -2$ and $x = 32$, there is no power of -2 that would produce 32, so $\log_{-2}(32)$ does not makes sense.

   Thus, if $b \leq 0$ or $b = 1$, then for many values of x, the expression $\log_b(x)$ does not make sense.

2. Calculate the following logarithms.

   a. $\log_5(25)$

   $\log_5(25) = 2$

   b. $\log_{10}\left(\frac{1}{100}\right)$

   $\log_{10}\left(\frac{1}{100}\right) = -2$

   c. $\log_9(3)$

   $\log_9(3) = \frac{1}{2}$

Problem Set Sample Solutions

In this introduction to logarithms, students are only asked to find simple logarithms base b in which the logarithm is an integer or a simple fraction and the expression can be calculated by inspection.

1. Rewrite each of the following in the form $\text{WhatPower}_b(x) = L$.

   a. $3^5 = 243$

   $\text{WhatPower}_3(243) = 5$

   b. $6^{-3} = \frac{1}{216}$

   $\text{WhatPower}_6\left(\frac{1}{216}\right) = -3$

   c. $9^0 = 1$

   $\text{WhatPower}_9(1) = 0$

2. Rewrite each of the following in the form $\log_b(x) = L$.

   a. $16^{\frac{1}{4}} = 2$

   $\log_{16}(2) = \frac{1}{4}$

   b. $10^3 = 1{,}000$

   $\log_{10}(1{,}000) = 3$

   c. $b^k = r$

   $\log_b(r) = k$

3. Rewrite each of the following in the form $b^L = x$.

   a. $\log_5(625) = 4$

   $5^4 = 625$

   b. $\log_{10}(0.1) = -1$

   $10^{-1} = 0.1$

   c. $\log_{27}9 = \frac{2}{3}$

   $27^{\frac{2}{3}} = 9$

4. **Consider the logarithms base 2. For each logarithmic expression below, either calculate the value of the expression or explain why the expression does not make sense.**

   a. $\log_2(1024)$

      10

   b. $\log_2(128)$

      7

   c. $\log_2(\sqrt{8})$

      $\frac{3}{2}$

   d. $\log_2\left(\frac{1}{16}\right)$

      -4

   e. $\log_2(0)$

      This does not make sense. There is no value of L so that $2^L = 0$.

   f. $\log_2\left(-\frac{1}{32}\right)$

      This does not make sense. There is no value of L so that 2^L is negative.

5. **Consider the logarithms base 3. For each logarithmic expression below, either calculate the value of the expression or explain why the expression does not make sense.**

   a. $\log_3(243)$

      5

   b. $\log_3(27)$

      3

   c. $\log_3(1)$

      0

   d. $\log_3\left(\frac{1}{3}\right)$

      -1

   e. $\log_3(0)$

      This does not make sense. There is no value of L so that $3^L = 0$.

   f. $\log_3\left(-\frac{1}{3}\right)$

      This does not make sense. There is no value of L so that $3^L < 0$.

6. Consider the logarithms base 5. For each logarithmic expression below, either calculate the value of the expression or explain why the expression does not make sense.

a. $\log_5(3125)$

5

b. $\log_5(25)$

2

c. $\log_5(1)$

0

d. $\log_5\left(\frac{1}{25}\right)$

-2

e. $\log_5(0)$

This does not make sense. There is no value of L so that $5^L = 0$.

f. $\log_5\left(-\frac{1}{25}\right)$

This does not make sense. There is no value of L so that 5^L is negative.

7. Is there any positive number b so that the expression $\log_b(0)$ makes sense? Explain how you know.

No, there is no value of L so that $b^L = 0$. I know b has to be a positive number. A positive number raised to an exponent never equals 0.

8. Is there any positive number b so that the expression $\log_b(-1)$ makes sense? Explain how you know.

No. Since b is positive, there is no value of L so that b^L is negative. A positive number raised to an exponent never has a negative value.

9. Verify each of the following by evaluating the logarithms.

a. $\log_2(8) + \log_2(4) = \log_2(32)$

$3 + 2 = 5$

b. $\log_3(9) + \log_3(9) = \log_3(81)$

$2 + 2 = 4$

c. $\log_4(4) + \log_4(16) = \log_4(64)$

$1 + 2 = 3$

d. $\log_{10}(10^3) + \log_{10}(10^4) = \log_{10}(10^7)$

$3 + 4 = 7$

10. Looking at the results from Problem 9, do you notice a trend or pattern? Can you make a general statement about the value of $\log_b(x) + \log_b(y)$?

The sum of two logarithms of the same base is found by multiplying the input values, $\log_b(x) + \log_b(y) = \log_b(xy)$. (Note to teacher: Do not evaluate this answer harshly. This is just a preview of a property that students learn later in the module.)

11. To evaluate $\log_2(3)$, Autumn reasoned that since $\log_2(2) = 1$ and $\log_2(4) = 2$, $\log_2(3)$ must be the average of 1 and 2 and therefore $\log_2(3) = 1.5$. Use the definition of logarithm to show that $\log_2(3)$ cannot be 1.5. Why is her thinking not valid?

According to the definition of logarithm, $\log_2(3) = 1.5$ only if $2^{1.5} = 3$. According to the calculator, $2^{1.5} \approx 2.828$, so $\log_2(3)$ cannot be 1.5. Autumn was assuming that the outputs would follow a linear pattern, but since the outputs are exponents, the relationship is not linear.

12. Find the value of each of the following.

a. If $x = \log_2(8)$ and $y = 2^x$, find the value of y.

$y = 8$

b. If $\log_2(x) = 6$, find the value of x.

$x = 64$

c. If $r = 2^6$ and $s = \log_2(r)$, find the value of s.

$s = 6$

Lesson 9: Logarithms—How Many Digits Do You Need?

Student Outcomes

- Students use logarithms to determine how many characters are needed to generate unique identification numbers in different scenarios.
- Students understand that logarithms are useful when relating the number of digits in a number to the magnitude of the number and that base 10 logarithms are useful when measuring quantities that have a wide range of values such as the magnitude of earthquakes, volume of sound, and pH levels in chemistry.

Lesson Notes

In this lesson, students learn that logarithms are useful in a wide variety of situations but have extensive application when they want to generate a list of unique identifiers for a population of a given size (**N-Q.A.2**). This application of logarithms is used in computer programming, when determining how many digits are needed in a phone number to have enough unique numbers for a population, and more generally, when assigning a scale to any quantity that has a wide range of values.

In this lesson, students make sense of a simple scenario and then see how it can be applied to other real-world situations (MP.1 and MP.2). They observe and extend patterns to formulate a model (MP.7 and MP.4). They reason about and make sense of situations in context and use logarithms to draw conclusions regarding different real-world scenarios (MP.3 and MP.4).

Classwork

Opening Exercise (2 minutes)

Remind students that the WhatPower expressions are called logarithms, and announce that they will use logarithms to help them make sense of and solve some real-world problems.

Students briefly convert two WhatPower expressions into a logarithmic expression and evaluate the result.

Opening Exercise

a. **Evaluate $\text{WhatPower}_2(8)$. State your answer as a logarithm, and evaluate it.**

$\log_2(8) = 3$

b. **Evaluate $\text{WhatPower}_5(625)$. State your answer as a logarithm, and evaluate it.**

$\log_5(625) = 4$

If students struggle with these exercises, consider planning for some additional practice on problems like those found in Lesson 7.

Exploratory Challenge (15 minutes)

Divide students up into small groups, and give them about 10 minutes to work through the questions that follow. While circulating around the room, encourage students to be systematic when assigning IDs to the club members. If a group is stuck, ask questions to help move the group along.

- Remember, we only want to use A's and B's, but two-character IDs only provide enough for four people. Can you provide an example of a three-character ID using only the letters A and B?
  - *A three-character ID might be ABA or AAA.*
- How many different IDs would three characters make? How could you best organize your results?
  - *Three characters would make 8 IDs because I could take the four I already have and add an A or B onto the end. The best way to organize it is to take the existing two-character IDs and add an A and then take those existing two-character IDs again and add a B.*

Exploratory Challenge

Autumn is starting a new club with eight members including herself. She wants everyone to have a secret identification code made up of only A's and B's. For example, using two characters, her ID code could be AA.

a. Using A's and B's, can Autumn assign each club member a unique two-character ID code using only A's and B's? Justify your answer. Here's what Autumn has so far.

Club Member Name	Secret ID Code
Autumn	AA
Kris	
Tia	
Jimmy	

Club Member Name	Secret ID Code
Robert	
Jillian	
Benjamin	
Scott	

No, she cannot assign a unique 2-character ID code to each member. The only codes available are AA, BA, AB, and BB, and there are eight people.

b. Using A's and B's, how many characters would be needed to assign each club member a unique ID code? Justify your answer by showing the ID codes you would assign to each club member by completing the table above (adjust Autumn's ID if needed).

You would need three characters in each ID code. A completed table is shown below. Students could assign one of the unique codes to any club member, so this is not the only possible solution.

Club Member Name	Secret ID Code
Autumn	*AAA*
Kris	*BAA*
Tia	*ABA*
Jimmy	*BBA*

Club Member Name	Secret ID Code
Robert	*AAB*
Jillian	*BAB*
Benjamin	*ABB*
Scott	*BBB*

MP.7 & MP.4

When the club grew to 16 members, Autumn started noticing a pattern.

Using A's and B's:

i. **Two people could be given a secret ID code with 1 letter: A and B.**
ii. **Four people could be given a secret ID code with 2 letters: AA, BA, AB, BB.**
iii. **Eight people could be given a secret ID code with 3 letters: AAA, BAA, ABA, BBA, AAB, BAB, ABB, BBB.**

c. **Complete the following statement, and list the secret ID codes for the 16 people.**

16 people could be given a secret ID code with _________ letters using A's and B's.

16 people could be given a secret ID code with 4 characters. Notice the original members have their original three-character code with an A added to the end. Then, the newer members have the original three-character codes with a B added to the end.

Club Member Name	Secret ID Code
Autumn	*AAAA*
Kris	*BAAA*
Tia	*ABAA*
Jimmy	*BBAA*
Robert	*AABA*
Jillian	*BABA*
Benjamin	*ABBA*
Scott	*BBBA*

Club Member Name	Secret ID Code
Gwen	*AAAB*
Jerrod	*BAAB*
Mykel	*ABAB*
Janette	*BBAB*
Nellie	*AABB*
Serena	*BABB*
Ricky	*ABBB*
Mia	*BBBB*

d. **Describe the pattern in words. What type of function could be used to model this pattern?**

The number of people in the club is a power of 2. The number of characters needed to generate a unique ID code using only two characters is the exponent of the power of 2.

$$\log_2(2) = 1$$

$$\log_2(4) = 2$$

$$\log_2(8) = 3$$

$$\log_2(16) = 4$$

A logarithm function could be used to model this pattern. For 16 people, you will need a four-character ID code because $\log_2(16) = 4$.

To debrief this Exploratory Challenge, have different groups explain how they arrived at their solutions. If a group does not demonstrate an efficient way to organize its answers when the club membership increases, be sure to show it to the class. For example, the 16 group member ID codes were generated by adding an A onto the end of the original 8 ID codes and then adding a B onto the end of the original 8 ID codes, as shown in the solutions above.

Exercises 1–2 (3 minutes)

Give students a few minutes to answer these questions individually or in groups. Check to see if students are using logarithms when they explain their solutions. If they are not, be sure to review the answers with the entire class using logarithm notation.

MP.2 & MP.3

Exercises 1–2

In the previous problems, the letters A and B were like the digits in a number. A four-digit ID code for Autumn's club could be any four-letter arrangement of A's and B's because in her ID system, the only digits are the letters A and B.

1. **When Autumn's club grows to include more than 16 people, she will need five digits to assign a unique ID code to each club member. What is the maximum number of people that could be in the club before she needs to switch to a six-digit ID code? Explain your reasoning.**

   Since $\log_2(32) = 5$ and $\log_2(64) = 6$, she will need to switch to a six-digit ID code when the club grows to more than 32 members.

2. **If Autumn has 256 members in her club, how many digits would she need to assign each club member a unique ID code using only A's and B's? Show how you got your answers.**

   She will need 8 digits because $\log_2(256) = 8$.

Discussion (10 minutes)

Computers store keyboard characters, such as 1, 5, X, x, Q, @, /, and &, using an identification system much like Autumn's system called ASCII, which stands for American Standard Code for Information Interchange. The acronym ASCII is pronounced as "as-kee." Each character in a font list on the computer is given an ID code that a computer can recognize. A computer is essentially a lot of electrical switches, which can be in one of two states, ON or OFF, just like Autumn's A's and B's.

There are usually 256 characters in a font list, so using the solution to Exercise 2, a computer needs eight positions or digits to encode each character in a font list.

- For example, the standard ASCII code for uppercase P is 01010000. If A is zero and B is 1, how would uppercase P be encoded using Autumn's system?
  - *Using A's and B's in Autumn's code, this would be ABABAAAA.*
- How would the computer read this code?
  - *The computer reads the code as "on, off, on, off, on, on, on, on."*

If time permits, a quick Internet search for the term *ASCII* returns Web pages where the standard code for different keyboard symbols can be viewed. Each character in an ASCII code is called a *bit*, and an entire 8-character code is called a *byte*. Each byte is made up of eight bits, and each byte describes a unique character in the font list such as a P, p, %, _, 4, etc.

- When a computer saves a basic text document and reports that it contains 3,242 bytes, what do you think that means?
  - *It means that there are $3{,}242$ letters, symbols, spaces, punctuation marks, etc., in the document.*

Students have seen how to create unique ID codes using two letters for both Autumn's secret club and to encode text characters in a way that is readable by a computer using ASCII code. Next, they examine why a logarithm really is the right operation to describe the number of characters or digits needed to create unique identifiers for people by exploring some real-world situations where people are assigned a number.

Scaffolding:

Use a word wall or a chart to provide a visual reference for English language learners using the academic terms related to computers.

ASCII (American Standard Code for Information Exchange, acronym pronounced "as-kee")

Bit (one of eight positions in a byte)

Byte (a unique eight-character identifier for each ASCII symbol in font list)

Scaffolding:

Ask advanced learners to quickly estimate how many digits would be needed to generate ID numbers for the number of students enrolled in the school or in the school district. For example, a school district with more than 10,000 students would need at least a five-digit ID number.

Example (5 minutes)

This is a simplified example that uses base-10 logarithms because students are going to be assigning ID numbers using the digits 0–9. Give students a few minutes to think about the answer to the prompt in the Example, and have them discuss their ideas with a partner. Most students will likely say to assign 0 to the first person, assign 1 to the second person, and so on up to the 1000^{th} person. Be sure to tie the solution to logarithms. Since there are 10 symbols (digits), $\log_{10}(1000) = 3$ can be used to find the answer, which just counts the number of digits needed to count to 999.

- How can a logarithm help you determine the solution quickly?
  - *The logarithm counts the number of digits needed because each time we add another digit to our numbers, we are increasing by a factor of* 10*. For example,* $1 = 10^0$, $10 = 10^1$, $100 = 10^2$*, etc.*

> **Example**
>
> **A thousand people are given unique identifiers made up of the digits $0, 1, 2, \dots, 9$. How many digits would be needed for each ID number?**
>
> ***You would just need three digits:*** $\{000, 001, 002, \dots, 099, 100, 101, \dots, 998, 999\}$.

Using logarithms, it is necessary to determine the value of $\log(1000)$, which is 3. This quickly tells the number of digits needed to uniquely identify any range of numbers. Follow up by asking students to extend their thinking.

- When would you need to switch from four to five digits to assign unique numbers to a population?
  - *You could assign up to* 10^4 *people a four-digit ID number, which would be* $10{,}000$ *people. Once you exceeded that number, you would need five digits to assign each person a unique number.*

Exercises 3–4 (5 minutes)

Students should return to their small groups to work these exercises. Have different groups present their solutions to the whole class after a few minutes. Discuss different approaches, and make sure that students see the power of using a logarithm to help them quickly solve or justify a solution to the problem.

MP.1 & MP.3

> **Exercises 3–4**
>
> **3. There are approximately 317 million people in the United States. Compute and use $\log(100000000)$ and $\log(1000000000)$ to explain why Social Security numbers are 9 digits long.**
>
> ***We know that*** $\log(100000000) = 8$***, which is the number of digits needed to assign an ID number to*** 100 ***million people. We know that*** $\log(1000000000) = 9$***, which is the number of digits needed to assign an ID number to*** 1 ***billion people. The United States government will not need to increase the number of digits in a Social Security number until the United States population reaches one billion.***
>
> **4. There are many more telephones than the number of people in the United States because of people having home phones, cell phones, business phones, fax numbers, etc. Assuming we need at most 10 billion phone numbers in the United States, how many digits would be needed so that each phone number is unique? Is this reasonable? Explain.**
>
> ***Since*** $\log(10000000000) = 10$***, you would need a ten-digit phone number in order to have ten billion unique numbers. Phone numbers in the United States are*** 10 ***digits long. If you divide*** 10 ***billion by*** 317 ***million (the number of people in the United States), that would allow for approximately*** 31 ***phone numbers per person. That is plenty of numbers for individuals to have more than one number, leaving many additional numbers for businesses and the government.***

Closing (2 minutes)

Ask students to respond to the following statements in writing or with a partner. Share a few answers to close the lesson before students begin the Exit Ticket. Preview other situations where logarithms are useful, such as the Richter scale for measuring the magnitude of an earthquake.

- To increase the value of $\log_2(x)$ by 1, you would multiply x by 2. To increase the value of $\log_{10}(x)$ by 1, you would multiply x by 10. How does this idea apply to the situations in today's lesson?
  - *We saw that each time the population of Autumn's club doubled, we needed to increase the total number of digits needed for the ID numbers by* 1. *We saw that since the population of the United States was between* 100 *million and* 1 *billion, we only needed* 9 *digits (*$\log(1000000000)$*) to generate a Social Security number.*
- Situations like the ones in today's lesson can be modeled with logarithms. Can you think of a situation besides the ones we discussed today where it would make sense to use logarithms?
  - *Any time a measurement can take on a wide range of values, such as the magnitude of an earthquake or the volume level of sound (measured in decibels), a logarithm could be used to model the situation.*

Exit Ticket (3 minutes)

Name ____ Date ____________________

Lesson 9: Logarithms—How Many Digits Do You Need?

Exit Ticket

A brand-new school district needs to generate ID numbers for its student body. The district anticipates a total enrollment of $75{,}000$ students within the next ten years. Will a five-digit ID number using the symbols $0, 1, \ldots, 9$ be enough? Explain your reasoning.

Exit Ticket Sample Solutions

A brand-new school district needs to generate ID numbers for its student body. The district anticipates a total enrollment of $75,000$ students within the next ten years. Will a five-digit ID number using the symbols $0, 1, \dots, 9$ be enough? Explain your reasoning.

$\log(10000) = 4$ and $\log(100000) = 5$, so 5 digits should be enough. However, students who enter school at the kindergarten level in the tenth year of this numbering scheme would need to keep their IDs for 13 years. Dividing $75,000$ by 13 shows there would be roughly $6,000$ students per grade. Adding that many students per year would take the number of needed IDs at any one time over $100,000$ in just a few more years. The district should probably use a six-digit ID number.

Problem Set Sample Solutions

1. The student body president needs to assign each officially sanctioned club on campus a unique ID code for purposes of tracking expenses and activities. She decides to use the letters A, B, and C to create a unique three-character code for each club.

   a. How many clubs can be assigned a unique ID code according to this proposal?

      Since $\log_3(27) = 3$, the president could assign codes to 27 clubs according to this proposal.

   b. There are actually over 500 clubs on campus. Assuming the student body president still wants to use the letters A, B, and C, how many characters would be needed to generate a unique ID code for each club?

      We need to estimate $\log_3(500)$. Since $3^5 = 243$ and $3^6 = 729$, she could use a six-character combination of letters and have enough unique IDs for up to 729 clubs.

2. Can you use the numbers 1, 2, 3, and 4 in a combination of four digits to assign a unique ID code to each of 500 people? Explain your reasoning.

$$\log_4(4) = 1$$
$$\log_4(16) = 2$$
$$\log_4(64) = 3$$
$$\log_4(256) = 4$$
$$\log_4(1024) = 5$$

   No. You would need to use a five-digit ID code using combinations of 1's, 2's, 3's, and 4's such as 11111 or 12341, or you could use the numbers 1 to 5 in four characters such as 1231, 1232, 1233, 1234, 1235, etc., because $\log_5(625) = 4$.

MP.1

3. Automobile license plates typically have a combination of letters (26) and numbers (10). Over time, the state of New York has used different criteria to assign vehicle license plate numbers.

   a. From 1973 to 1986, the state used a 3-letter and 4-number code where the three letters indicated the county where the vehicle was registered. Essex County had 13 different 3-letter codes in use. How many cars could be registered to this county?

      Since $\log(10000) = 4$, the 4-digit code could be used to register up to $10,000$ vehicles. Multiply that by 13 different county codes, and up to $130,000$ vehicles could be registered in Essex County.

MP.1

b. **Since 2001, the state has used a 3-letter and 4-number code but no longer assigns letters by county. Is this coding scheme enough to register 10 million vehicles?**

Since $\log_{26}(x) = 3$ when $x = 26^3 = 17,576$, there are $17,576$ three-letter codes. Since $\log(10000) = 4$, there are $10,000$ four-digit codes. Multiply $17576 \cdot 10,000 = 175,760,000$, and we see that this scheme generates over 100 million license plate numbers.

4. **The Richter scale uses base 10 logarithms to assign a magnitude to an earthquake based on the amount of force released at the earthquake's source as measured by seismographs in various locations.**

a. **Explain the difference between an earthquake that is assigned a magnitude of 5 versus one assigned a magnitude of 7.**

The difference between 5 and 7 is 2, so a magnitude of 7 would be 10^2, or 100, times greater force.

b. **An earthquake with magnitude 2 can usually only be felt by people located near the earthquake's origin, called its *epicenter*. The strongest earthquake on record occurred in Chile in 1960 with magnitude 9.5. How many times stronger is the force of an earthquake with magnitude 9.5 than the force of an earthquake with magnitude 2?**

The difference between 2 and 9.5 is 7.5, so it would be about $10^{7.5}$ times the force. This is approximately 31 million times greater force.

c. **What is the magnitude of an earthquake whose force is $1,000$ times greater than a magnitude 4.3 quake?**

Since $10^3 = 1000$, the magnitude would be the sum of 4.3 and 3, which is 7.3.

MP.2 & MP.3

5. **Sound pressure level is measured in decibels (dB) according to the formula $L = 10\log\left(\frac{I}{I_0}\right)$, where I is the intensity of the sound and I_0 is a reference intensity that corresponds to a barely perceptible sound.**

a. **Explain why this formula would assign 0 decibels to a barely perceptible sound.**

If we let $I = I_0$, then

$$L = 10\log\left(\frac{I}{I_0}\right)$$
$$L = 10\log(1)$$
$$L = 10 \cdot 0$$
$$L = 0.$$

Therefore, the reference intensity is always 0 dB.

b. **Decibel levels above 120 dB can be painful to humans. What would be the intensity that corresponds to this level?**

$$120 = 10\log\left(\frac{I}{I_0}\right)$$
$$1.2 = \log\left(\frac{I}{I_0}\right)$$
$$\frac{I}{I_0} = 10^{1.2}$$
$$I \approx 15.8I_0$$

From this equation, we can see that the intensity is about 16 times greater than barely perceptible sound.

Lesson 10: Building Logarithmic Tables

Student Outcomes

- Students construct a table of logarithms base 10 and observe patterns that indicate properties of logarithms.

Lesson Notes

In the previous lesson, students were introduced to the concept of the logarithm by finding the power to which it is necessary to raise a base b in order to produce a given number, which was originally called the WhatPower_b function. In this lesson and the next, students build their own base-10 logarithm tables using their calculators. By taking the time to construct the table themselves (as opposed to being handed a pre-prepared table), students have a better opportunity to observe patterns in the table and practice MP.7. These observed patterns lead to formal statements of the properties of logarithms in upcoming lessons. Using logarithmic properties to rewrite logarithmic expressions satisfies the foundational standard **A-SSE.A.2**.

To answer some of the questions in this and subsequent lessons, students need an intuitive understanding that logarithmic functions with base $b > 1$ always increase; this idea is made explicit in Lesson 17 when key features of the graphs of logarithmic functions are explored. The increasing nature of a logarithmic function with base $b > 1$ is a direct consequence of the inverse relationship between a logarithmic function and the corresponding exponential function. At this point in the module, students need to understand that since $\log_b(x)$ is the power to which the base b is raised to get x, then for values of b greater than 1, if the value of x is increased, then the value of $\log_b(x)$ also increases. In Exercises 1–4 of this lesson, students need to work with this property of logarithmic functions when they squeeze the value of $\log(30)$ first between consecutive integers and then between consecutive numbers to the tenths and then the hundredths place. Ensure that students understand this property: Because $10^1 < 30 < 10^2$, it is known that $\log(10^1) < \log(30) < \log(10^2)$, which means that $1 < \log(30) < 2$.

Materials Needed

Students need access to a calculator or other technological tool able to compute exponents and logarithms base 10.

Classwork

Opening Exercise (3 minutes)

In this quick Opening Exercise, students are asked to recall the WhatPower_b function from the previous lesson and the fact that the logarithm base b is the formal name of the WhatPower_b function is reinforced. Only base-10 logarithms are considered in this lesson as the table is constructed, so this Opening Exercise is constrained to base-10 calculations.

At the end of this exercise, announce to students that the notation $\log(x)$ without the little b in the subscript means $\log_{10}(x)$. This is called the *common logarithm.*

> *Scaffolding:*
>
> Prompt struggling students to restate the logarithmic equation $\log_{10}(10^3) = x$ first as the equation $\text{WhatPower}_{10}(10^3) = x$ and then as the exponential equation $10^x = 10^3$.

Opening Exercise

Find the value of the following expressions without using a calculator.

$\text{WhatPower}_{10}(1000)$	$= 3$	$\log_{10}(1000)$	$= 3$
$\text{WhatPower}_{10}(100)$	$= 2$	$\log_{10}(100)$	$= 2$
$\text{WhatPower}_{10}(10)$	$= 1$	$\log_{10}(10)$	$= 1$
$\text{WhatPower}_{10}(1)$	$= 0$	$\log_{10}(1)$	$= 0$
$\text{WhatPower}_{10}\left(\frac{1}{10}\right)$	$= -1$	$\log_{10}\left(\frac{1}{10}\right)$	$= -1$
$\text{WhatPower}_{10}\left(\frac{1}{100}\right)$	$= -2$	$\log_{10}\left(\frac{1}{100}\right)$	$= -2$

Formulate a rule based on your results above: If k is an integer, then $\log_{10}(10^k) =$ ______.

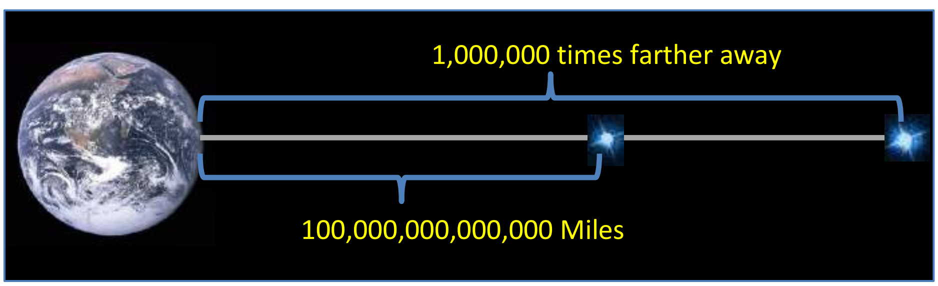
$\log_{10}(10^k) = k$

Example 1 (6 minutes)

In this example, students get their first glimpse of the property $\log_b(xy) = \log_b(x) + \log_b(y)$. Be careful not to give this formula away; by the end of the next lesson, students should have discovered it for themselves.

- Suppose that you are an astronomer, and you measure the distance to a star as $100{,}000{,}000{,}000{,}000$ miles. A second star is collinear with the first star and Earth and is 1,000,000 times farther away from Earth than the first star is. How many miles is the second star from Earth? Note: The figure is not to scale.

Example 1



- $(100{,}000{,}000{,}000{,}000)(1{,}000{,}000) = 100{,}000{,}000{,}000{,}000{,}000{,}000$*, so the second star is* 100 *quintillion miles away from Earth.*
- How did you arrive at that figure?
  - *I counted the zeros; there are* 14 *zeros in* $100{,}000{,}000{,}000{,}000$ *and* 6 *zeros in* $1{,}000{,}000$*, so there must be* 20 *zeros in the product.*
- Can we restate that in terms of exponents?
  - $(10^{14})(10^{6}) = 10^{20}$
- How are the exponents related?
  - $14 + 6 = 20$

- What are $\log(10^{14})$, $\log(10^{6})$, and $\log(10^{20})$?
  - *14, 6, and 20*
- In this case, can we state an equivalent expression for $\log(10^{14} \cdot 10^{6})$?
  - $\log(10^{14} \cdot 10^{6}) = \log(10^{14}) + \log(10^{6})$
- Why is this equation true?
  - $\log(10^{14} \cdot 10^{6}) = \log(10^{20}) = 20 = 14 + 6 = \log(10^{14}) + \log(10^{6})$

MP.8

- Generalize to find an equivalent expression for $\log(10^{m} \cdot 10^{n})$ for integers m and n. Why is this equation true?
  - $\log(10^{m} \cdot 10^{n}) = \log(10^{m+n}) = m + n = \log(10^{m}) + \log(10^{n})$
  - *This equation is true because when we multiply powers of* 10 *together, the resulting product is a power of* 10 *whose exponent is the sum of the exponents of the factors.*
- Keep this result in mind as we progress through the lesson.

Exercises 1–6 (8 minutes)

Historically, logarithms were calculated using tables because there were no calculators or computers to do the work. Every scientist and mathematician kept a book of logarithmic tables on hand to use for calculation. It is very easy to find the value of a base-10 logarithm for a number that is a power of 10, but what about for the other numbers? In this exercise, students find an approximate value of $\log(30)$ using exponentiation, the same way $\log_2(10)$ was approximated in Lesson 6. After this exercise, students rely on the logarithm button on the calculator to compute base-10 logarithms for the remainder of this lesson. Emphasize to students that logarithms are generally irrational numbers, so the results produced by the calculator are only decimal approximations. As such, care should be taken to use the approximation symbol, $\approx$, when writing out a decimal expansion of a logarithm.

Exercises

1. **Find two consecutive powers of 10 so that 30 is between them. That is, find an integer exponent k so that $10^{k} < 30 < 10^{k+1}$.**

   Since $10 < 30 < 100$, we have $k = 1$.

2. **From your result in Exercise 1, $\log(30)$ is between which two integers?**

   Since 30 is some power of 10 between 1 and 2, $1 < \log(30) < 2$.

3. **Find a number k to one decimal place so that $10^{k} < 30 < 10^{k+0.1}$, and use that to find under and over estimates for $\log(30)$.**

   Since $10^{1.4} \approx 25.1189$ and $10^{1.5} \approx 31.6228$, we have $10^{1.4} < 30 < 10^{1.5}$. Then $1.4 < \log(30) < 1.5$, and $k \approx 1.4$.

4. **Find a number k to two decimal places so that $10^{k} < 30 < 10^{k+0.01}$, and use that to find under and over estimates for $\log(30)$.**

   Since $10^{1.47} \approx 29.5121$, and $10^{1.48} \approx 30.1995$, we have $10^{1.47} < 30 < 10^{1.48}$ so that $1.47 < \log(30) < 1.48$. So, $k \approx 1.47$.

5. **Repeat this process to approximate the value of $\log(30)$ to 4 decimal places.**

Since $10^{1.477} \approx 29.9916$, and $10^{1.478} \approx 30.0608$, we have $10^{1.477} < 30 < 10^{1.478}$ so that $1.477 < \log(30) < 1.478$.

Since $10^{1.4771} \approx 29.9985$, and $10^{1.4772} \approx 30.0054$, we have $10^{1.4771} < 30 < 10^{1.4772}$ so that $1.4771 < \log(30) < 1.4772$.

Since $10^{1.47712} \approx 29.9999$, and $10^{1.47713} \approx 30.0006$, we have $10^{1.47712} < 30 < 10^{1.47713}$ so that $1.477\,12 < \log(30) < 1.477\,13$.

Thus, to four decimal places, $\log(30) \approx 1.4771$.

6. **Verify your result on your calculator, using the LOG button.**

The calculator gives $\log(30) \approx 1.477121255$.

Discussion (1 minute)

In the next exercises, students use their calculators to create a table of logarithms that they analyze to look for patterns that lead to the discovery of the logarithmic properties. The process of identifying and generalizing the observed patterns provides students with an opportunity to practice MP.7.

- Historically, since there were no calculators or computers, logarithms were calculated using a complicated algorithm involving multiple square roots. Thankfully, we have calculators and computers to do this work for us now.
- We will use our calculators to create a table of values of base-10 logarithms. Once the table is made, see what patterns you can observe.

Exercises 7–10 (6 minutes)

Put students in pairs or small groups, but have students work individually to complete the table in Exercise 7. Before working on Exercises 8–10 in groups, have students check their tables against each other. It may be necessary to remind students that $\log(x)$ means $\log_{10}(x)$.

7. **Use your calculator to complete the following table. Round the logarithms to 4 decimal places.**

x	$\log(x)$
1	0
2	0.3010
3	0.4771
4	0.6021
5	0.6990
6	0.7782
7	0.8451
8	0.9031
9	0.9542

x	$\log(x)$
10	1
20	1.3010
30	1.4771
40	1.6021
50	1.6990
60	1.7782
70	1.8451
80	1.9031
90	1.9542

x	$\log(x)$
100	2
200	2.3010
300	2.4771
400	2.6021
500	2.6990
600	2.7782
700	2.8451
800	2.9031
900	2.9542

MP.7

8. **What pattern(s) can you see in the table from Exercise 7 as x is multiplied by 10? Write the pattern(s) using logarithmic notation.**

   I found the patterns $\log(10x) = 1 + \log(x)$ and $\log(100x) = 2 + \log(x)$. I also noticed that $\log(100x) = 1 + \log(10x)$.

9. **What pattern would you expect to find for $\log(1000x)$? Make a conjecture, and test it to see whether or not it appears to be valid.**

   I would guess that the values of $\log(1000x)$ will all start with 3. That is, $\log(1000x) = 3 + \log(x)$. This appears to be the case since $\log(2000) \approx 3.3010$, $\log(5000) \approx 3.6990$, and $\log(8000) \approx 3.9031$.

10. **Use your results from Exercises 8 and 9 to make a conjecture about the value of $\log(10^k \cdot x)$ for any positive integer k.**

    It appears that $\log(10^k \cdot x) = k + \log(x)$, for any positive integer k.

Discussion (3 minutes)

Ask groups to share the patterns they observed in Exercise 8 and the conjectures they made in Exercises 9 and 10. Ensure that all students have the correct conjectures recorded in their notebooks or journals before continuing to the next set of exercises, which extend the result from Exercise 10 to all integers k (and not just positive values of k).

Exercises 11–14 (8 minutes)

In this set of exercises, students discover a rule for calculating logarithms of the form $\log(10^k \cdot x)$, where k is any integer. Have students again work individually to complete the table in Exercise 11 and to check their tables against each other before they proceed to discuss and answer Exercises 12–14 in groups.

Scaffolding:

If students are having difficulty seeing the pattern in the table for Exercise 12, nudge them to add together $\log(x)$ and $\log\left(\frac{x}{10}\right)$ for some values of x in the table.

11. **Use your calculator to complete the following table. Round the logarithms to 4 decimal places.**

x	$\log(x)$	x	$\log(x)$	x	$\log(x)$
1	0	0.1	-1	0.01	-2
2	0.3010	0.2	-0.6990	0.02	-1.6990
3	0.4771	0.3	-0.5229	0.03	-1.5229
4	0.6021	0.4	-0.3979	0.04	-1.3979
5	0.6990	0.5	-0.3010	0.05	-1.3010
6	0.7782	0.6	-0.2218	0.06	-1.2218
7	0.8451	0.7	-0.1549	0.07	-1.1549
8	0.9031	0.8	-0.0969	0.08	-1.0969
9	0.9542	0.9	-0.0458	0.09	-1.0458

MP.7

12. **What pattern(s) can you see in the table from Exercise 11? Write them using logarithmic notation.**

    I found the patterns $\log(x) - \log\left(\frac{x}{10}\right) = 1$, which can be written as $\log\left(\frac{x}{10}\right) = -1 + \log(x)$, and $\log\left(\frac{x}{100}\right) = -2 + \log(x)$.

MP.7

13. What pattern would you expect to find for $\log\left(\frac{x}{1000}\right)$? Make a conjecture, and test it to see whether or not it appears to be valid.

I would guess that the values of $\log\left(\frac{x}{1000}\right)$ *will all start with* -2 *and that* $\log\left(\frac{x}{1000}\right) = -3 + \log(x)$. *This appears to be the case since* $\log(0.002) \approx -2.6990$, *and* $-2.6990 = -3 + 0.3010$; $\log(0.005) \approx -2.3010$, *and* $-2.3010 = -3 + 0.6990$; $\log(0.008) \approx -2.0969$, *and* $-2.0969 = -3 + 0.9031$.

14. Combine your results from Exercises 10 and 12 to make a conjecture about the value of the logarithm for a multiple of a power of 10; that is, find a formula for $\log(10^k \cdot x)$ for any integer k.

It appears that $\log(10^k \cdot x) = k + \log(x)$, *for any integer* k.

Discussion (2 minutes)

Ask groups to share the patterns they observed in Exercise 12 and the conjectures they made in Exercises 13 and 14 with the class. Ensure that all students have the correct conjectures recorded in their notebooks or journals before continuing to the next example.

Examples 2–3 (2 minutes)

Lead the class through these calculations. Consider letting them work on Example 3 either alone or in groups after leading them through Example 2.

Example 2

Use the logarithm tables and the rules that have been discovered to calculate $\log(40000)$ to 4 decimal places.

$$\begin{aligned}\log(40000) &= \log(10^4 \cdot 4)\\ &= 4 + \log(4)\\ &\approx 4.6021\end{aligned}$$

Example 3

Use the logarithm tables and the rules that have been discovered to calculate $\log(0.000004)$ to 4 decimal places.

$$\begin{aligned}\log(0.000004) &= \log(10^{-6} \cdot 4)\\ &= -6 + \log(4)\\ &\approx -5.3979\end{aligned}$$

Closing (2 minutes)

Ask students to summarize the important parts of the lesson, either in writing, to a partner, or as a class. Use this as an opportunity to informally assess understanding of the lesson. The following are some important summary elements:

Lesson Summary

- **The notation $\log(x)$ is used to represent $\log_{10}(x)$.**
- **For integers k, $\log(10^k) = k$.**
- **For integers m and n, $\log(10^m \cdot 10^n) = \log(10^m) + \log(10^n)$.**
- **For integers k and positive real numbers x, $\log(10^k \cdot x) = k + \log(x)$.**

Exit Ticket (4 minutes)

Name ____ Date ____________________

Lesson 10: Building Logarithmic Tables

Exit Ticket

1. Use the logarithm table below to approximate the specified logarithms to four decimal places. Do not use a calculator.

x	$\log(x)$
1	0.0000
2	0.3010
3	0.4771
4	0.6021
5	0.6990

x	$\log(x)$
6	0.7782
7	0.8451
8	0.9031
9	0.9542
10	1.0000

   a. $\log(500)$

   b. $\log(0.0005)$

2. Suppose that A is a number with $\log(A) = 1.352$.
   a. What is the value of $\log(1000A)$?

   b. Which one of the following statements is true? Explain how you know.
      i. $A < 0$
      ii. $0 < A < 10$
      iii. $10 < A < 100$
      iv. $100 < A < 1000$
      v. $A > 1000$

Exit Ticket Sample Solutions

1. **Use the logarithm table below to approximate the specified logarithms to four decimal places. Do not use a calculator.**

x	$\log(x)$
1	0.0000
2	0.3010
3	0.4771
4	0.6021
5	0.6990

x	$\log(x)$
6	0.7782
7	0.8451
8	0.9031
9	0.9542
10	1.0000

   a. $\log(500)$

$$\begin{aligned}\log(500) &= \log(10^2 \cdot 5)\\ &= 2 + \log(5)\\ &\approx 2.6990\end{aligned}$$

   b. $\log(0.0005)$

$$\begin{aligned}\log(0.0005) &= \log(10^{-4} \cdot 5)\\ &= -4 + \log(5)\\ &\approx -3.3010\end{aligned}$$

2. **Suppose that A is a number with $\log(A) = 1.352$.**

   a. **What is the value of $\log(1000A)$?**

$\log(1000A) = \log(10^3 A) = 3 + \log(A) = 4.352$

   b. **Which one of the following statements is true? Explain how you know.**

   i. $A < 0$

   ii. $0 < A < 10$

   iii. $10 < A < 100$

   iv. $100 < A < 1000$

   v. $A > 1000$

Because $\log(A) = 1.352 = 1 + 0.352$, A is greater than 10 and less than 100. Thus, (iii) is true. In fact, from the table above, we can see that A is between 20 and 30 because $\log(20) \approx 1.3010$, and $\log(30) \approx 1.4771$.

Problem Set Sample Solutions

These problems should be solved without a calculator.

1. **Complete the following table of logarithms without using a calculator; then, answer the questions that follow.**

x	$\log(x)$
$1,000,000$	6
$100,000$	5
$10,000$	4
1000	3
100	2
10	1

x	$\log(x)$
0.1	-1
0.01	-2
0.001	-3
0.0001	-4
0.00001	-5
0.000001	-6

   a. **What is $\log(1)$? How does that follow from the definition of a base-10 logarithm?**

   Since $10^0 = 1$, we know that $\log(1) = 0$.

   b. **What is $\log(10^k)$ for an integer k? How does that follow from the definition of a base-10 logarithm?**

   By the definition of the logarithm, we know that $\log(10^k) = k$.

   c. **What happens to the value of $\log(x)$ as x gets really large?**

   For any $x > 1$, there exists $k > 0$ so that $10^k \leq x < 10^{k+1}$. As x gets really large, k gets large. Since $k \leq \log(x) < k + 1$, as k gets large, $\log(x)$ gets large.

   d. **For $x > 0$, what happens to the value of $\log(x)$ as x gets really close to zero?**

   For any $0 < x < 1$, there exists $k > 0$ so that $10^{-k} \leq x < 10^{-k+1}$. Then $-k \leq x < -k + 1$. As x gets closer to zero, k gets larger. Thus, $\log(x)$ is negative, and $|\log(x)|$ gets large as the positive number x gets close to zero.

2. **Use the table of logarithms below to estimate the values of the logarithms in parts (a)–(h).**

x	$\log(x)$
2	0.3010
3	0.4771
5	0.6990
7	0.8451
11	1.0414
13	1.1139

   a. $\log(70000)$

   4.8451

   b. $\log(0.0011)$

   -2.9586

c. $\log(20)$

1.3010

d. $\log(0.00005)$

-4.3010

e. $\log(130000)$

5.1139

f. $\log(3000)$

3.4771

g. $\log(0.07)$

-1.1549

h. $\log(11000000)$

7.0414

3. If $\log(n) = 0.6$, find the value of $\log(10n)$.

$\log(10n) = 1.6$

4. If m is a positive integer and $\log(m) \approx 3.8$, how many digits are there in m? Explain how you know.

Since $3 < \log(m) < 4$, we know $1,000 < m < 10,000$; therefore, m has 4 digits.

5. If m is a positive integer and $\log(m) \approx 9.6$, how many digits are there in m? Explain how you know.

Since $9 < \log(m) < 10$, we know $10^9 < m < 10^{10}$; therefore, m has 10 digits.

6. Vivian says $\log(452000) = 5 + \log(4.52)$, while her sister Lillian says that $\log(452000) = 6 + \log(0.452)$. Which sister is correct? Explain how you know.

Both sisters are correct. Since $452,000 = 4.52 \cdot 10^5$, we can write $\log(452000) = 5 + \log(4.52)$. However, we could also write $452,000 = 0.452 \cdot 10^6$, so $\log(452000) = 6 + \log(0.452)$. Both calculations give $\log(452,000) \approx 5.65514$.

7. Write the base-10 logarithm of each number in the form $k + \log(x)$, where k is the exponent from the scientific notation, and x is a positive real number.

a. 2.4902×10^4

$4 + \log(2.4902)$

b. 2.58×10^{13}

$13 + \log(2.58)$

c. 9.109×10^{-31}

$-31 + \log(9.109)$

8. For each of the following statements, write the number in scientific notation, and then write the logarithm base 10 of that number in the form $k + \log(x)$, where k is the exponent from the scientific notation, and x is a positive real number.

a. The speed of sound is 1116 ft/s.

$1116 = 1.116 \times 10^3$, *so* $\log(1116) = 3 + \log(1.116)$.

b. The distance from Earth to the sun is 93 million miles.

$93,000,000 = 9.3 \times 10^7$, *so* $\log(93000000) = 7 + \log(9.3)$.

c. The speed of light is $29,980,000,000\text{ cm/s}$.

$29,980,000,000 = 2.998 \times 10^{10}$, *so* $\log(29,980,000,000) = 10 + \log(2.998)$.

d. The weight of Earth is $5,972,000,000,000,000,000,000,000\text{ kg}$.

$5,972,000,000,000,000,000,000,000 = 5.972 \times 10^{24}$, *so* $\log(5,972,000,000,000,000,000,000,000) = 24 + \log(5.972)$.

e. The diameter of the nucleus of a hydrogen atom is $0.00000000000000175\text{ m}$.

$0.00000000000000175 = 1.75 \times 10^{-15}$, *so* $\log(0.00000000000000175) = -15 + \log(1.75)$.

f. For each part (a)–(e), you have written each logarithm in the form $k + \log(x)$, for integers k and positive real numbers x. Use a calculator to find the values of the expressions $\log(x)$. Why are all of these values between 0 and 1?

$\log(1.116) \approx 0.047664$

$\log(9.3) \approx 0.968483$

$\log(2.998) \approx 0.476832$

$\log(5.972) \approx 0.77612$

$\log(1.75) \approx 0.243038$

These values are all between 0 *and* 1 *because* x *is between* 1 *and* 10. *We can rewrite* $1 < x < 10$ *as* $10^0 < x < 10^1$. *If we write* $x = 10^L$ *for some exponent* L, *then* $10^0 < 10^L < 10^1$, *so* $0 < L < 1$. *This exponent* L *is the base* 10 *logarithm of* x.

Lesson 11: The Most Important Property of Logarithms

Student Outcomes

- Students construct a table of logarithms base 10 and observe patterns that indicate properties of logarithms.

Lesson Notes

In the previous lesson, students discovered that for logarithms base 10, $\log(10^k \cdot x) = k + \log(x)$. In this lesson, this result is extended to develop the most important property of logarithms: $\log(xy) = \log(x) + \log(y)$. Additionally, students discover the reciprocal property of logarithms: $\log\left(\frac{1}{x}\right) = -\log(x)$. Students continue to hone their skills at observing and generalizing patterns in this lesson as they create tables of logarithms and observe patterns, practicing MP.8. In the next lesson, these logarithmic properties are formalized and generalized for any base, but for this lesson the focus is solely on logarithms base 10. Understanding deeply the properties of logarithms helps prepare students to rewrite expressions based on their structure (**A-SSE.A.2**), solve exponential equations (**F-LE.A.4**), and interpret transformations of graphs of logarithmic functions (**F-BF.B.3**).

Materials Needed

Students need access to a calculator or other technological tool able to compute exponents and logarithms base 10.

Classwork

Opening (1 minute)

In the previous lesson, students discovered the logarithmic property $\log(10^k \cdot x) = k + \log(x)$, which is a special case of the additive property $\log(xy) = \log(x) + \log(y)$ that they discover today. The Opening Exercise reminds students of how they can use this property to compute logarithms of numbers not in the table. By the end of today's lesson, students are able to calculate any logarithm base 10 using just a table of values of $\log(x)$ for prime integers x. The only times in this lesson that calculators should be used is to create the tables in Exercises 1 and 6. Remind students that logarithm tables contain only approximations of the precise values of logarithms, which are generally irrational numbers.

Opening Exercise (4 minutes)

Students should complete this exercise without the use of a calculator.

Opening Exercise

Use the logarithm table below to calculate the specified logarithms.

x	$\log(x)$
1	0
2	0.3010
3	0.4771
4	0.6021
5	0.6990
6	0.7782
7	0.8451
8	0.9031
9	0.9542

a. $\log(80)$

$\log(80) = \log(10^1 \cdot 8) = 1 + \log(8) \approx 1.9031$

b. $\log(7000)$

$\log(7000) = \log(10^3 \cdot 7) = 3 + \log(7) \approx 3.8451$

c. $\log(0.00006)$

$\log(0.00006) = \log(10^{-5} \cdot 6) = -5 + \log(6) \approx -4.2218$

d. $\log(3.0 \times 10^{27})$

$\log(3.0 \times 10^{27}) = \log(10^{27} \cdot 3) = 27 + \log(3) \approx 27.4771$

e. $\log(9.0 \times 10^k)$ **for an integer** k

$\log(9.0 \times 10^k) = \log(10^k \cdot 9) = k + \log(9) \approx k + 0.9542$

Scaffolding:

- Consider modeling the decomposition of 80 with the whole class before students start the Opening Exercise.

$$\begin{aligned}\log(80) &= \log(10 \cdot 8)\\ &= \log(10) + \log(8)\\ &= 1 + \log(8)\end{aligned}$$

- Ask advanced students to write an expression for $\log(10^k \cdot x)$ independently.

Discussion (3 minutes)

Use this Discussion to review the formulas discovered in the previous lesson. In the next set of exercises, students use tables of logarithms to discover some other interesting properties of logarithms.

In the next set of exercises, students discover the additive property of logarithms, which is not as readily apparent as the patterns observed yesterday. Plant the seed of the idea by restating the previous property in the additive format.

- What was the formula we developed in the last class?
  - $\log(10^k \cdot x) = k + \log(x)$

- What is the value of $\log(10^k)$?
  - $\log(10^k) = k$
- So, what is another way we can write the formula $\log(10^k \cdot x) = k + \log(x)$?
  - $\log(10^k \cdot x) = \log(10^k) + \log(x)$
- Keep this statement of the formula in mind as you progress through the next set of exercises.

Exercises 1–5 (6 minutes)

Students may be confused by the fact that the formulas do not appear to be exact—for example, the table shows that $\log(4) = 0.6021$, and $2\log(2) = 0.6020$. If this question arises, remind students that since they have made approximations to irrational numbers, there is some error in rounding off the decimal expansions to four decimal places. Students should question this discrepancy in part (f) of Exercise 2.

Exercises 1–5

1. **Use your calculator to complete the following table. Round the logarithms to four decimal places.**

x	$\log(x)$	x	$\log(x)$
1	0	10	1.0000
2	0.3010	12	1.0792
3	0.4771	16	1.2041
4	0.6021	18	1.2553
5	0.6990	20	1.3010
6	0.7782	25	1.3979
7	0.8451	30	1.4771
8	0.9031	36	1.5563
9	0.9542	100	2.0000

2. **Calculate the following values. Do they appear anywhere else in the table?**

   a. $\log(2) + \log(4)$

   We see that $\log(2) + \log(4) \approx 0.9031$*, which is approximately* $\log(8)$*.*

   b. $\log(2) + \log(6)$

   We see that $\log(2) + \log(6) \approx 1.0792$*, which is approximately* $\log(12)$*.*

   c. $\log(3) + \log(4)$

   We see that $\log(3) + \log(4) \approx 1.0792$*, which is approximately* $\log(12)$*.*

   d. $\log(6) + \log(6)$

   We see that $\log(6) + \log(6) \approx 1.5663$*, which is approximately* $\log(36)$*.*

   e. $\log(2) + \log(18)$

   We see that $\log(2) + \log(18) \approx 1.5663$*, which is approximately* $\log(36)$*.*

f. $\log(3) + \log(12)$

We see that $\log(3) + \log(12) \approx 1.5664$*, which is approximately* $\log(36)$.

MP.8

3. What pattern(s) can you see in Exercise 2 and the table from Exercise 1? Write them using logarithmic notation.

I found the pattern $\log(xy) = \log(x) + \log(y)$.

4. What pattern would you expect to find for $\log(x^2)$? Make a conjecture, and test it to see whether or not it appears to be valid.

I would expect that $\log(x^2) = \log(x) + \log(x) = 2\log(x)$. *This is verified by the fact that* $\log(4) \approx 0.6021 \approx 2\log(2)$, $\log(9) \approx 0.9542 \approx 2\log(3)$, $\log(16) \approx 1.2041 \approx 2\log(4)$, *and* $\log(25) \approx 1.3980 \approx 2\log(5)$.

5. Make a conjecture for a logarithm of the form $\log(xyz)$, where x, y, and z are positive real numbers. Provide evidence that your conjecture is valid.

It appears that $\log(xyz) = \log(x) + \log(y) + \log(z)$. *This is due to applying the property from Exercise 3 twice.*

$$\begin{aligned}\log(xyz) &= \log(xy \cdot z)\\ &= \log(xy) + \log(z)\\ &= \log(x) + \log(y) + \log(z)\end{aligned}$$

OR

It appears that $\log(xyz) = \log(x) + \log(y) + \log(z)$. *We can see that*

$\log(18) \approx 1.2553 \approx 0.3010 + 0.3010 + 0.4771 \approx \log(2) + \log(2) + \log(3)$,
$\log(20) \approx 1.3010 \approx 0.3010 + 0.3010 + 0.6990 \approx \log(2) + \log(2) + \log(5)$, *and*
$\log(36) \approx 1.5563 \approx 0.3010 + 0.4771 + 0.7782 \approx \log(2) + \log(3) + \log(6)$.

Discussion (2 minutes)

Ask groups to share the patterns and conjectures they formed in Exercises 3–5 with the class; emphasize that the pattern discovered in Exercise 3 is the *most important property* of base-10 logarithms. Ensure that all students have the correct statements recorded in their notebooks or journals before continuing to the next example.

Example 1 (5 minutes)

Lead the class through these four logarithmic calculations, relying only on the values in the table from Exercise 1. Notice that since there is not a value for $\log(11)$ in the table, there is not enough information to calculate $\log(121)$. Allow students to figure this out for themselves.

Example 1

Use the logarithm table from Exercise 1 to approximate the following logarithms.

a. $\log(14)$

$\log(14) = \log(2) + \log(7) \approx 0.3010 + 0.8451$, *so* $\log(14) \approx 1.1461$.

b. $\log(35)$

$\log(35) = \log(5) + \log(7) \approx 0.6990 + 0.8451$, *so* $\log(35) \approx 1.5441$.

c. $\log(72)$
$\log(72) = \log(8) + \log(9) \approx 0.9031 + 0.9542$, *so* $\log(72) \approx 1.8573$.

d. $\log(121)$
$\log(121) = \log(11) + \log(11)$, *but we do not have a value for* $\log(11)$ *in the table, so we cannot evaluate* $\log(121)$.

Discussion (3 minutes)

- Suppose we are building a logarithm table, and we have already approximated the values of $\log(2)$ and $\log(3)$. What other values of $\log(x)$ for $4 \leq x \leq 20$ can we approximate by applying the additive property developed in Exercise 5?
  - *Since the table contains* $\log(2)$ *and* $\log(3)$, *we can figure out approximations of* $\log(4)$, $\log(6)$, $\log(8)$, $\log(9)$, $\log(12)$, *and* $\log(18)$ *since the only factors of* 4, 6, 8, 9, 12, 16, *and* 18 *are* 2 *and* 3.
- In order to develop the entire logarithm table for all integers between 1 and 20, what is the smallest set of logarithmic values that we need to know?
  - *We need to know the values of the logarithms for the prime numbers:* 2, 3, 5, 7, 11, 13, 17, *and* 19.
- Why does the additive property make sense based on what we know about exponents?
  - *We know that when you multiply two powers of the same base together, the exponents are added. For example,* $10^4 \cdot 10^5 = 10^{4+5} = 10^9$, *so* $\log(10^4 \cdot 10^5) = \log(10^9) = 9$.

Exercises 6–8 (7 minutes)

Have students again work individually to complete the table in Exercise 6 and to check their tables against each other before they proceed to discuss and answer Exercise 7 in groups. Ensure that there is enough time for a volunteer to present justification for the conjecture in Exercise 8.

Scaffolding:
Remind students to convert the fractions in the second table to decimal values, or present the table with the values as both fractions and decimals.

Exercises 6–8

6. Use your calculator to complete the following table. Round the logarithms to four decimal places.

x	$\log(x)$
2	0.3010
4	0.6021
5	0.6990
8	0.9031
10	1.0000
16	1.2041
20	1.3010
50	1.6990
100	2.0000

x	$\log(x)$
$\frac{1}{2}$	-0.3010
$\frac{1}{4}$	-0.6021
$\frac{1}{5}$	-0.6990
$\frac{1}{8}$	-0.9031
$\frac{1}{10}$	-1.0000
$\frac{1}{16}$	-1.2041
$\frac{1}{20}$	-1.3010
$\frac{1}{50}$	-1.6990
$\frac{1}{100}$	-2.0000

MP.8

7. **What pattern(s) you can see in the table from Exercise 6? Write a conjecture using logarithmic notation.**

For any real number $x > 0$, $\log\left(\frac{1}{x}\right) = -\log(x)$.

8. **Use the definition of logarithm to justify the conjecture you found in Exercise 7.**

If $\log\left(\frac{1}{x}\right) = a$ for some number a, then $10^a = \frac{1}{x}$. Then, $10^{-a} = x$, and thus, $\log(x) = -a$. We then have $\log\left(\frac{1}{x}\right) = -\log(x)$.

Discussion (3 minutes)

Ask groups to share the conjecture they formed in Exercise 7 with the class. Ensure that all students have the correct conjecture recorded in their notebooks or journals before continuing to the next example.

Example 2 (5 minutes)

Lead the class through these calculations. Let them work either alone or in groups on parts (b)–(d) after leading them through part (a).

Example 2

Use the logarithm tables and the rules we have discovered to estimate the following logarithms to four decimal places:

a. $\log(2100)$

$$\begin{aligned}\log(2100) &= \log(10^2 \cdot 21)\\ &= 2 + \log(21)\\ &= 2 + \log(3) + \log(7)\\ &\approx 2 + 0.4771 + 0.8451\\ &\approx 3.3222\end{aligned}$$

b. $\log(0.00049)$

$$\begin{aligned}\log(0.00049) &= \log(10^{-5} \cdot 49)\\ &= -5 + \log(49)\\ &= -5 + \log(7) + \log(7)\\ &\approx -5 + 0.8451 + 0.8451\\ &\approx -3.3098\end{aligned}$$

c. $\log(42000000)$

$$\begin{aligned}\log(42000000) &= \log(10^6 \cdot 42)\\ &= 6 + \log(42)\\ &= 6 + \log(6) + \log(7)\\ &\approx 6 + 0.7782 + 0.8451\\ &\approx 7.6233\end{aligned}$$

d. $\log\left(\frac{1}{640}\right)$

$$\begin{aligned}\log\left(\frac{1}{640}\right) &= -\log(640)\\ &= -\big(\log(10\cdot 64)\big)\\ &= -\big(1+\log(64)\big)\\ &= -\big(1+\log(8)+\log(8)\big)\\ &\approx -(1+0.9031+0.9031)\\ &\approx -2.8062\end{aligned}$$

Closing (2 minutes)

Ask students to summarize the important parts of the lesson, either in writing, to a partner, or as a class. Use this as an opportunity to informally assess understanding of the lesson. The following are some important summary elements:

Lesson Summary

- The notation $\log(x)$ is used to represent $\log_{10}(x)$.
- The most important property of base-10 logarithms is that for positive real numbers x and y, $\log(xy) = \log(x) + \log(y)$.
- For positive real numbers x,

$$\log\left(\frac{1}{x}\right) = -\log(x).$$

Exit Ticket (4 minutes)

Name ______________________________ Date ______________

Lesson 11: The Most Important Property of Logarithms

Exit Ticket

1. Use the table below to approximate the following logarithms to four decimal places. Do not use a calculator.

   a. $\log(9)$

x	$\log(x)$
2	0.3010
3	0.4771
5	0.6990
7	0.8451

   b. $\log\left(\frac{1}{15}\right)$

   c. $\log(45000)$

2. Suppose that k is an integer, a is a positive real number, and you know the value of $\log(a)$. Explain how to find the value of $\log(10^k \cdot a^2)$.

Exit Ticket Sample Solutions

1. Use the table below to approximate the following logarithms to four decimal places. Do not use a calculator.

x	$\log(x)$
2	0.3010
3	0.4771
5	0.6990
7	0.8451

   a. $\log(9)$

$$\begin{aligned}\log(9) &= \log(3) + \log(3)\\ &\approx 0.4771 + 0.4771\\ &\approx 0.9542\end{aligned}$$

   b. $\log\left(\frac{1}{15}\right)$

$$\begin{aligned}\log\left(\frac{1}{15}\right) &= -\log(15)\\ &= -(\log(3) + \log(5))\\ &\approx -(0.4771 + 0.6990)\\ &\approx -1.1761\end{aligned}$$

   c. $\log(45000)$

$$\begin{aligned}\log(45000) &= \log(10^3 \cdot 45)\\ &= 3 + \log(45)\\ &= 3 + \log(5) + \log(9)\\ &\approx 3 + 0.6990 + 0.9542\\ &\approx 4.6532\end{aligned}$$

2. Suppose that k is an integer, a is a positive real number, and you know the value of $\log(a)$. Explain how to find the value of $\log(10^k \cdot a^2)$.

   Applying the rule for the logarithm of a number multiplied by a power of 10 and then the rule for the logarithm of a product, we have

$$\begin{aligned}\log(10^k \cdot a^2) &= k + \log(a^2)\\ &= k + \log(a) + \log(a)\\ &= k + 2\log(a).\end{aligned}$$

Problem Set Sample Solutions

All of the exercises in this Problem Set should be completed without the use of a calculator.

1. **Use the table of logarithms to the right to estimate the value of the logarithms in parts (a)–(t).**

x	$\log(x)$
2	0.30
3	0.48
5	0.70
7	0.85
11	1.04
13	1.11

a. $\log(25)$

1.40

b. $\log(27)$

1.44

c. $\log(33)$

1.52

d. $\log(55)$

1.74

e. $\log(63)$

1.81

f. $\log(75)$

1.88

g. $\log(81)$

1.92

h. $\log(99)$

2.00

i. $\log(350)$

2.55

j. $\log(0.0014)$

-2.85

k. $\log(0.077)$

-1.11

l. $\log(49000)$

4.70

m. $\log(1.69)$

0.22

n. $\log(6.5)$

0.81

o. $\log\left(\frac{1}{30}\right)$

-1.48

p. $\log\left(\frac{1}{35}\right)$

-1.55

q. $\log\left(\frac{1}{40}\right)$

-1.60

r. $\log\left(\frac{1}{42}\right)$

-1.63

s. $\log\left(\frac{1}{50}\right)$

-1.70

t. $\log\left(\frac{1}{64}\right)$

-1.80

2. **Reduce each expression to a single logarithm of the form $\log(x)$.**

a. $\log(5)+\log(7)$

$\log(35)$

b. $\log(3)+\log(9)$

$\log(27)$

c. $\log(15)-\log(5)$

$\log(3)$

d. $\log(8)+\log\left(\frac{1}{4}\right)$

$\log(2)$

3. Use properties of logarithms to write the following expressions involving logarithms of only prime numbers:

a. $\log(2500)$

$2 + 2\log(5)$

b. $\log(0.00063)$

$-5 + 2\log(3) + \log(7)$

c. $\log(1250)$

$1 + 3\log(5)$

d. $\log(26000000)$

$6 + \log(2) + \log(13)$

4. Use properties of logarithms to show that $\log(2) - \log\left(\frac{1}{13}\right) = \log(26)$.

$$\begin{aligned}\log(2) - \log\left(\frac{1}{13}\right) &= \log(2) - \log(13^{-1})\\ &= \log(2) + \log(13)\\ &= \log(26)\end{aligned}$$

5. Use properties of logarithms to show that $\log(3) + \log(4) + \log(5) - \log(6) = 1$.

There are multiple ways to solve this problem.

$$\begin{aligned}\log(3) + \log(4) + \log(5) - \log(6) &= \log(3) + \log(4) + \log(5) + \log\left(\frac{1}{6}\right)\\ &= \log\left(3 \cdot 4 \cdot 5 \cdot \frac{1}{6}\right)\\ &= \log(10)\\ &= 1\end{aligned}$$

OR

$$\begin{aligned}\log(3) + \log(4) + \log(5) &= \log(60)\\ &= \log(10 \cdot 6)\\ &= \log(10) + \log(6)\\ &= 1 + \log(6)\end{aligned}$$

$$\log(3) + \log(4) + \log(5) - \log(6) = 1$$

6. Use properties of logarithms to show that $\log\left(\frac{1}{2} - \frac{1}{3}\right) + \log(2) = -\log(3)$.

$$\begin{aligned}\log\left(\frac{1}{2} - \frac{1}{3}\right) + \log(2) &= \log\left(\frac{1}{6}\right) + \log(2)\\ &= -\log(6) + \log(2)\\ &= -\big(\log(2) + \log(3)\big) + \log(2)\\ &= -\log(3)\end{aligned}$$

7. Use properties of logarithms to show that $\log\left(\frac{1}{3} - \frac{1}{4}\right) + \left(\log\left(\frac{1}{3}\right) - \log\left(\frac{1}{4}\right)\right) = -2\log(3)$.

$$\begin{aligned}\log\left(\frac{1}{3} - \frac{1}{4}\right) + \left(\log\left(\frac{1}{3}\right) - \log\left(\frac{1}{4}\right)\right) &= \log\left(\frac{1}{12}\right) + \log\left(\frac{1}{3}\right) - \log\left(\frac{1}{4}\right)\\ &= -\log(12) - \log(3) + \log(4)\\ &= -\big(\log(3) + \log(4)\big) - \log(3) + \log(4)\\ &= -2\log(3)\end{aligned}$$

Lesson 12: Properties of Logarithms

Student Outcomes

- Students justify properties of logarithms using the definition and properties already developed.

Lesson Notes

In this lesson, students work exclusively with logarithms base 10; generalization of these results to a generic base b occurs in the next lesson. The opening of this lesson, which echoes homework from Lesson 11, is meant to launch a consideration of some properties of the common logarithm function. The centerpiece of the lesson is the theoretical approach to demonstrating six basic logarithm properties, as opposed to the numerical approach used in previous lessons. In the Problem Set, students apply these properties to calculating logarithms, rewriting logarithmic expressions, and solving base 10 exponential equations (**A-SSE.A.2**, **F-LE.A.4**).

Classwork

Opening Exercise (5 minutes)

Students should work in groups of two or three on this Opening Exercise. This exercise serves to remind students of the "most important property" of logarithms and prepare them for justifying the properties later in the lesson. Verify that students are using the property to break up the logarithm and evaluating the logarithm at known values (e.g., 0.1, 10, 100).

> *Scaffolding:*
> - Prompt students who are struggling with any part of the Opening Exercise with a question, such as "How is the number in parentheses related to 2?" Follow that with the question, "So, how might you find its logarithm given that you know $\log(2)$?"
> - Ask students to factor each number into powers of 10 and factors of 2 before splitting the factors using $\log(xy) = \log(x) + \log(y)$. Students still struggling can be given additional products to break down before finding the approximations of their logarithms.
>
> $$\begin{aligned} 4 &= 2 \cdot 2 \\ 40 &= 10^1 \cdot 2 \cdot 2 \\ 0.4 &= 10^{-1} \cdot 2 \cdot 2 \\ 400 &= 10^2 \cdot 2 \cdot 2 \\ 0.04 &= 10^{-2} \cdot 2 \cdot 2 \end{aligned}$$
>
> Challenge advanced students to find a general formula for $\log(2^k)$ by looking for a pattern in $\log(2^4)$, $\log(2^5)$, $\log(2^6)$, and so on.

Opening Exercise

Use the approximation $\log(2) \approx 0.3010$ to approximate the values of each of the following logarithmic expressions.

a. $\log(20)$

$$\begin{aligned} \log(20) &= \log(10 \cdot 2) \\ &= \log(10) + \log(2) \\ &\approx 1 + 0.3010 \\ &\approx 1.3010 \end{aligned}$$

b. $\log(0.2)$

$$\begin{aligned} \log(0.2) &= \log(0.1 \cdot 2) \\ &= \log(0.1) + \log(2) \\ &\approx -1 + 0.3010 \\ &\approx -0.6990 \end{aligned}$$

c. $\log(2^4)$

$$\begin{aligned}\log(2^4) &= \log(2\cdot 2\cdot 2\cdot 2)\\ &= \log(2\cdot 2)+\log(2\cdot 2)\\ &= \log(2)+\log(2)+\log(2)+\log(2)\\ &\approx 4\cdot(0.3010)\\ &\approx 1.2040\end{aligned}$$

Discussion (4 minutes)

Discuss the properties of logarithms used in the Opening Exercise.

- In all three parts of the Opening Exercise, we used the property $\log(xy) = \log(x) + \log(y)$.
- What are some other properties we used?
  - *We also used* $\log(10) = 1$ *and* $\log(0.1) = -1$.

Example (6 minutes)

Recall that, by definition, $L = \log(x)$ means $10^L = x$. Consider some possible values of x and L, noting that x cannot be a negative number. What is L …

- when $x = 1$?
  - $L = 0$
- when $x = 0$?
  - *The logarithm* L *is not defined. There is no exponent of* 10 *that yields a value of* 0.
- when $x = 10^9$?
  - $L = 9$
- when $x = 10^n$?
  - $L = n$
- when $x = \sqrt[3]{10}$?
  - $L = \frac{1}{3}$

Exercises 1–6 (15 minutes)

Students should work in groups of two or three on each exercise. The first three should be straightforward in view of the definition of base 10 logarithms. Exercise 4 may look somewhat odd, but it, too, follows directly from the definition and presents an important property of logarithms. Exercises 5 and 6 are more difficult, which is why the hints are supplied. When all properties have been established, groups might be asked to show their explanations to the rest of the class as time permits.

Exercises

For Exercises 1–6, explain why each statement below is a property of base-10 logarithms.

1. **Property 1: $\log(1) = 0$**

   Because $L = \log(x)$ means $10^L = x$, then when $x = 1$, $L = 0$.

2. **Property 2: $\log(10) = 1$.**

   Because $L = \log(x)$ means $10^L = x$, then when $x = 10$, $L = 1$.

3. **Property 3: For all real numbers r, $\log(10^r) = r$.**

   Because $L = \log(x)$ means $10^L = x$, then when $x = 10^r$, $L = r$.

4. **Property 4: For any $x > 0$, $10^{\log(x)} = x$.**

   Because $L = \log(x)$ means $10^L = x$, then $x = 10^{\log(x)}$.

MP.3

5. **Property 5: For any positive real numbers x and y, $\log(x \cdot y) = \log(x) + \log(y)$.**
   Hint: Use an exponent rule as well as property 4.

   By the rule $a^b \cdot a^c = a^{b+c}$, $10^{\log(x)} \cdot 10^{\log(y)} = 10^{\log(x)+\log(y)}$.

   By property 4, $10^{\log(x)} \cdot 10^{\log(y)} = x \cdot y$.

   Therefore, $x \cdot y = 10^{\log(x)+\log(y)}$. Again, by property 4, $x \cdot y = 10^{\log(x \cdot y)}$.

   Then, $10^{\log(x \cdot y)} = 10^{\log(x)+\log(y)}$; so, the exponents must be equal, and $\log(x \cdot y) = \log(x) + \log(y)$.

6. **Property 6: For any positive real number x and any real number r, $\log(x^r) = r \cdot \log(x)$.**
   Hint: Use an exponent rule as well as property 4.

   By the rule $(a^b)^c = a^{bc}$, $10^{k\log(x)} = \left(10^{\log(x)}\right)^k$.

   By property 4, $\left(10^{\log(x)}\right)^r = x^r$.

   Therefore, $x^r = 10^{r\log(x)}$. Again, by property 4, $x^r = 10^{\log(x^r)}$.

   Then, $10^{\log(x^r)} = 10^{r\log(x)}$; so, the exponents must be equal, and $\log(x^r) = r \cdot \log(x)$.

Scaffolding:

Establishing the logarithmic properties relies on the exponential laws. Make sure that students have access to the exponential laws either through a poster displayed in the classroom or through notes in their notebooks.

Scaffolding:

Students in groups that struggle with Exercises 3–6 should be encouraged to check the property with numerical values for k, x, m, and n. The check may suggest a general explanation.

Exercises 7–10 (8 minutes)

The next set of exercises bridges the gap between the abstract properties of logarithms and computational problems like those in the Problem Set. Allow students to work alone, in pairs, or in small groups. Circulate to ensure that students are applying the properties correctly. Calculators are not needed for these exercises and should not be used. In Exercises 9 and 10, students need to know that the logarithm is well defined; that is, for positive real numbers X and Y, if $X = Y$, then $\log(X) = \log(Y)$. This is why we can "take the log of both sides" of an equation in order to bring down an exponent and solve the equation. In these last two exercises, students need to choose an appropriate base for the logarithm to use to solve the equation. Any logarithm works to solve the equations if applied properly, so students may find equivalent answers that appear to be different from those listed here.

7. **Apply properties of logarithms to rewrite the following expressions as a single logarithm or number.**

   a. $\frac{1}{2}\log(25) + \log(4)$

   $\log(5) + \log(4) = \log(20)$

   b. $\frac{1}{3}\log(8) + \log(16)$

   $\log(2) + \log(2^4) = \log(32)$

   c. $3\log(5) + \log(0.8)$

   $\log(125) + \log(0.8) = \log(100) = 2$

8. **Apply properties of logarithms to rewrite each expression as a sum of terms involving numbers, $\log(x)$, and $\log(y)$, where x and y are positive real numbers.**

   a. $\log(3x^2y^5)$

   $\log(3) + 2\log(x) + 5\log(y)$

   b. $\log\left(\sqrt{x^7y^3}\right)$

   $\frac{7}{2}\log(x) + \frac{3}{2}\log(y)$

9. **In mathematical terminology, logarithms are *well defined* because if $X = Y$, then $\log(X) = \log(Y)$ for $X, Y > 0$. This means that if you want to solve an equation involving exponents, you can apply a logarithm to both sides of the equation, just as you can take the square root of both sides when solving a quadratic equation. You do need to be careful not to take the logarithm of a negative number or zero.**

   Use the property stated above to solve the following equations.

   a. $10^{10x} = 100$

   $$\log(10^{10x}) = \log(100)$$
   $$10x = 2$$
   $$x = \frac{1}{5}$$

   b. $10^{x-1} = \frac{1}{10^{x+1}}$

   $$\log(10^{x-1}) = -\log(10^{x+1})$$
   $$x - 1 = -(x+1)$$
   $$2x = 0$$
   $$x = 0$$

   c. $100^{2x} = 10^{3x-1}$

   $$\log(100^{2x}) = \log(10^{3x-1})$$
   $$2x\log(100) = (3x-1)$$
   $$4x = 3x - 1$$
   $$x = -1$$

10. **Solve the following equations.**

a. $10^x = 2^7$

$$\log(10^x) = \log(2^7)$$
$$x = 7\log(2)$$

b. $10^{x^2+1} = 15$

$$\log\left(10^{x^2+1}\right) = \log(15)$$
$$x^2 + 1 = \log(15)$$
$$x = \pm\sqrt{\log(15) - 1}$$

c. $4^x = 5^3$

$$\log(4^x) = \log(5^3)$$
$$x\log(4) = 3\log(5)$$
$$x = \frac{3\log(5)}{\log(4)}$$

Closing (2 minutes)

Point out that for each property 1–6, it has been established that the property holds, so these properties can be used in future work with logarithms. The Lesson Summary might be posted in the classroom for at least the rest of the module. Property 7 was established in Lesson 11 through numerical observation; students are now asked to verify both properties 7 and 8 using properties 1–6.

Lesson Summary

We have established the following properties for base-10 logarithms, where x and y are positive real numbers and r is any real number:

1. $\log(1) = 0$
2. $\log(10) = 1$
3. $\log(10^r) = r$
4. $10^{\log(x)} = x$
5. $\log(x \cdot y) = \log(x) + \log(y)$
6. $\log(x^r) = r \cdot \log(x)$

Additional properties not yet established are the following:

7. $\log\left(\frac{1}{x}\right) = -\log(x)$
8. $\log\left(\frac{x}{y}\right) = \log(x) - \log(y)$

Also, logarithms are well defined, meaning that for $X, Y > 0$, if $X = Y$, then $\log(X) = \log(Y)$.

Exit Ticket (5 minutes)

Name ______________________________ Date ________________

Lesson 12: Properties of Logarithms

Exit Ticket

In this lesson, we have established six logarithmic properties for positive real numbers x and y and real numbers r.

1. $\log(1) = 0$
2. $\log(10) = 1$
3. $\log(10^r) = r$
4. $10^{\log(x)} = x$
5. $\log(x \cdot y) = \log(x) + \log(y)$
6. $\log(x^r) = r \cdot \log(x)$

1. Use properties 1–6 of logarithms to establish property 7: $\log\left(\frac{1}{x}\right) = -\log(x)$ for all $x > 0$.

2. Use properties 1–6 of logarithms to establish property 8: $\log\left(\frac{x}{y}\right) = \log(x) - \log(y)$ for $x > 0$ and $y > 0$.

Exit Ticket Sample Solutions

In this lesson, we have established six logarithmic properties for positive real numbers x and y and real numbers r.

1. $\log(1) = 0$
2. $\log(10) = 1$
3. $\log(10^r) = r$
4. $10^{\log(x)} = x$
5. $\log(x \cdot y) = \log(x) + \log(y)$
6. $\log(x^r) = r \cdot \log(x)$

1. Use properties 1–6 of logarithms to establish property 7: $\log\left(\frac{1}{x}\right) = -\log(x)$ for all $x > 0$.

By property 6, $\log(x^k) = k \cdot \log(x)$.

Let $k = -1$; then for $x > 0$, $\log(x^{-1}) = (-1) \cdot \log(x)$, which is equivalent to $\log\left(\frac{1}{x}\right) = -\log(x)$.

Thus, for any $x > 0$, $\log\left(\frac{1}{x}\right) = -\log(x)$.

2. Use properties 1–6 of logarithms to establish property 8: $\log\left(\frac{x}{y}\right) = \log(x) - \log(y)$ for $x > 0$ and $y > 0$.

By property 5, $\log(x \cdot y) = \log(x) + \log(y)$.

By Problem 1 above, for $y > 0$, $\log(y^{-1}) = (-1) \cdot \log(y)$.

Therefore,

$$\begin{aligned}\log\left(\frac{x}{y}\right) &= \log(x) + \log\left(\frac{1}{y}\right) \\ &= \log(x) + (-1)\log(y) \\ &= \log(x) - \log(y).\end{aligned}$$

Thus, for any $x > 0$ and $y > 0$, $\log\left(\frac{x}{y}\right) = \log(x) - \log(y)$.

Problem Set Sample Solutions

Problems 1–7 give students an opportunity to practice using the properties they have established in this lesson. In the remaining problems, students apply base-10 logarithms to solve simple exponential equations.

1. Use the approximate logarithm values below to estimate the value of each of the following logarithms. Indicate which properties you used.

$\log(2) = 0.3010$ $\quad\quad$ $\log(3) = 0.4771$

$\log(5) = 0.6990$ $\quad\quad$ $\log(7) = 0.8451$

a. $\log(6)$

Using property 5,

$\log(6) = \log(3) + \log(2) \approx 0.7781.$

b. $\log(15)$

Using property 5,

$\log(15) = \log(3) + \log(5) \approx 1.1761.$

c. $\log(12)$

Using properties 5 and 6,

$\log(12) = \log(3) + \log(2^2) = \log(3) + 2\log(2) \approx 1.0791.$

d. $\log(10^7)$

Using property 3,

$\log(10^7) = 7.$

e. $\log\left(\frac{1}{5}\right)$

Using property 7,

$\log\left(\frac{1}{5}\right) = -\log(5) \approx -0.6990.$

f. $\log\left(\frac{3}{7}\right)$

Using property 8,

$\log\left(\frac{3}{7}\right) = \log(3) - \log(7) \approx -0.368.$

g. $\log(\sqrt[4]{2})$

Using property 6,

$\log(\sqrt[4]{2}) = \log\left(2^{\frac{1}{4}}\right) = \frac{1}{4}\log(2) \approx 0.0753.$

2. Let $\log(X) = r$, $\log(Y) = s$, and $\log(Z) = t$. Express each of the following in terms of r, s, and t.

a. $\log\left(\frac{X}{Y}\right)$

$r - s$

b. $\log(YZ)$

$s + t$

c. $\log(X^r)$

r^2

d. $\log(\sqrt[3]{Z})$

$\frac{t}{3}$

e. $\log\left(\sqrt[4]{\frac{Y}{Z}}\right)$

$\frac{s-t}{4}$

f. $\log(XY^2Z^3)$

$r + 2s + 3t$

3. Use the properties of logarithms to rewrite each expression in an equivalent form containing a single logarithm.

a. $\log\left(\frac{13}{5}\right) + \log\left(\frac{5}{4}\right)$

$\log\left(\frac{13}{4}\right)$

b. $\log\left(\frac{5}{6}\right) - \log\left(\frac{2}{3}\right)$

$\log\left(\frac{5}{4}\right)$

c. $\frac{1}{2}\log(16) + \log(3) + \log\left(\frac{1}{4}\right)$

$\log(3)$

4. Use the properties of logarithms to rewrite each expression in an equivalent form containing a single logarithm.

a. $\log(\sqrt{x}) + \frac{1}{2}\log\left(\frac{1}{x}\right) + 2\log(x)$

$\log(x^2)$

b. $\log(\sqrt[5]{x}) + \log(\sqrt[5]{x^4})$

$\log(x)$

c. $\log(x) + 2\log(y) - \frac{1}{2}\log(z)$

$\log\left(\frac{xy^2}{\sqrt{z}}\right)$

d. $\frac{1}{3}\left(\log(x) - 3\log(y) + \log(z)\right)$

$\log\left(\sqrt[3]{\frac{xz}{y^3}}\right)$

e. $2(\log(x) - \log(3y)) + 3(\log(z) - 2\log(x))$

$\log\left(\left(\frac{x}{3y}\right)^2\right) + \log\left(\left(\frac{z}{x^2}\right)^3\right) = \log\left(\frac{z^3}{9y^2x^4}\right)$

5. In each of the following expressions, x, y, and z represent positive real numbers. Use properties of logarithms to rewrite each expression in an equivalent form containing only $\log(x)$, $\log(y)$, $\log(z)$, and numbers.

a. $\log\left(\frac{3x^2y^4}{\sqrt{z}}\right)$

$\log(3) + 2\log(x) + 4\log(y) - \frac{1}{2}\log(z)$

b. $\log\left(\frac{42\sqrt[3]{xy^7}}{x^2z}\right)$

$\log(42) - \frac{5}{3}\log(x) + \frac{7}{3}\log(y) - \log(z)$

c. $\log\left(\frac{100x^2}{y^3}\right)$

$2 + 2\log(x) - 3\log(y)$

d. $\log\left(\sqrt{\frac{x^3y^2}{10z}}\right)$

$\frac{1}{2}(3\log(x) + 2\log(y) - 1 - \log(z))$

e. $\log\left(\frac{1}{10x^2z}\right)$

$-1 - 2\log(x) - \log(z)$

6. Express $\log\left(\frac{1}{x} - \frac{1}{x+1}\right) + \left(\log\left(\frac{1}{x}\right) - \log\left(\frac{1}{x+1}\right)\right)$ as a single logarithm for positive numbers x.

$$\begin{aligned}\log\left(\frac{1}{x} - \frac{1}{x+1}\right) + \left(\log\left(\frac{1}{x}\right) - \log\left(\frac{1}{x+1}\right)\right) &= \log\left(\frac{1}{x(x+1)}\right) + \log\left(\frac{1}{x}\right) - \log\left(\frac{1}{x+1}\right)\\ &= -\log(x(x+1)) - \log(x) + \log(x+1)\\ &= -\log(x) - \log(x+1) - \log(x) + \log(x+1)\\ &= -2\log(x)\end{aligned}$$

7. Show that $\log(x + \sqrt{x^2 - 1}) + \log(x - \sqrt{x^2 - 1}) = 0$ for $x \geq 1$.

$$\begin{aligned}\log\left(x + \sqrt{x^2-1}\right) + \log\left(x - \sqrt{x^2-1}\right) &= \log\left(\left(x + \sqrt{x^2-1}\right)\left(x - \sqrt{x^2-1}\right)\right)\\ &= \log\left(x^2 - \left(\sqrt{x^2-1}\right)^2\right)\\ &= \log(x^2 - x^2 + 1)\\ &= \log(1)\\ &= 0\end{aligned}$$

8. If $xy = 10^{3.67}$ for some positive real numbers x and y, find the value of $\log(x) + \log(y)$.

$$\begin{aligned}xy &= 10^{3.67}\\ 3.67 &= \log(xy)\\ \log(xy) &= 3.67\\ \log(x) + \log(y) &= 3.67\end{aligned}$$

9. Solve the following exponential equations by taking the logarithm base 10 of both sides. Leave your answers stated in terms of logarithmic expressions.

a. $10^{x^2} = 320$

$$\log(10^{x^2}) = \log(320)$$
$$x^2 = \log(320)$$
$$x = \pm\sqrt{\log(320)}$$

b. $10^{\frac{x}{8}} = 300$

$$\log\left(10^{\frac{x}{8}}\right) = \log(300)$$
$$\frac{x}{8} = \log(10^2 \cdot 3)$$
$$\frac{x}{8} = 2 + \log(3)$$
$$x = 16 + 8\log(3)$$

c. $10^{3x} = 400$

$$\log(10^{3x}) = \log(400)$$
$$3x \cdot \log(10) = \log(10^2 \cdot 4)$$
$$3x \cdot 1 = 2 + \log(4)$$
$$x = \frac{1}{3}(2 + \log(4))$$

d. $5^{2x} = 200$

$$\log(5^{2x}) = \log(200)$$
$$2x \cdot \log(5) = \log(100) + \log(2)$$
$$2x = \frac{2 + \log(2)}{\log(5)}$$
$$x = \frac{2 + \log(2)}{2\log(5)}$$

e. $3^x = 7^{-3x+2}$

$$\log(3^x) = \log\left(7^{-3x+2}\right)$$
$$x\log(3) = (-3x + 2)\log(7)$$
$$x\log(3) + 3x\log(7) = 2\log(7)$$
$$x\left(\log(3) + 3\log(7)\right) = 2\log(7)$$
$$x = \frac{2\log(7)}{\log(3) + 3\log(7)} = \frac{\log(49)}{\log(3) + \log(343)} = \frac{\log(49)}{\log(1029)}$$

(Any of the three equivalent forms given above are acceptable answers.)

10. Solve the following exponential equations.

a. $10^x = 3$

$$x = \log(3)$$

b. $10^y = 30$

$y = \log(30)$

c. $10^z = 300$

$z = \log(300)$

d. Use the properties of logarithms to justify why x, y, and z form an arithmetic sequence whose constant difference is 1.

Since $y = \log(30)$, $y = \log(10 \cdot 3) = 1 + \log(3) = 1 + x$.
Similarly, $z = 2 + \log(3) = 2 + x$.

Thus, the sequence x, y, z is the sequence $\log(3)$, $1 + \log(3)$, $2 + \log(3)$, and these numbers form an arithmetic sequence whose first term is $\log(3)$ with constant difference 1.

11. Without using a calculator, explain why the solution to each equation must be a real number between 1 and 2.

a. $11^x = 12$

12 is greater than 11^1 and less than 11^2, so the solution is between 1 and 2.

b. $21^x = 30$

30 is greater than 21^1 and less than 21^2, so the solution is between 1 and 2.

c. $100^x = 2000$

$100^2 = 10000$, so $2,000$ is between 100^1 and 00^2 , so the solution is between 1 and 2.

d. $\left(\frac{1}{11}\right)^x = 0.01$

$\frac{1}{100}$ is between $\frac{1}{11}$ and $\frac{1}{121}$, so the solution is between 1 and 2.

e. $\left(\frac{2}{3}\right)^x = \frac{1}{2}$

$\left(\frac{2}{3}\right)^2 = \frac{4}{9}$, and $\frac{1}{2}$ is between $\frac{4}{9}$ and $\frac{2}{3}$, so the solution is between 1 and 2.

f. $99^x = 9000$

$99^2 = 9801$. Since $9,000$ is less than $9,801$ and greater than 99, the solution is between 1 and 2.

12. Express the exact solution to each equation as a base-10 logarithm. Use a calculator to approximate the solution to the nearest 1000^{th}.

a. $11^x = 12$

$$\log(11^x) = \log(12)$$
$$x\log(11) = \log(12)$$
$$x = \frac{\log(12)}{\log(11)}$$
$$x \approx 1.036$$

b. $21^x = 30$

$x = \frac{\log(30)}{\log(21)}$

$x \approx 1.117$

c. $100^x = 2000$

$x = \frac{\log(2000)}{\log(100)}$

$x \approx 1.651$

d. $\left(\frac{1}{11}\right)^x = 0.01$

$x = -\frac{2}{\log\left(\frac{1}{11}\right)}$

$x \approx 1.921$

e. $\left(\frac{2}{3}\right)^x = \frac{1}{2}$

$x = \frac{\log\left(\frac{1}{2}\right)}{\log\left(\frac{2}{3}\right)}$

$x \approx 1.710$

f. $99^x = 9000$

$x = \frac{\log(9000)}{\log(99)}$

$x \approx 1.981$

13. Show that for any real number r, the solution to the equation $10^x = 3 \cdot 10^r$ is $\log(3) + r$.

Substituting $x = \log(3) + r$ into 10^x and using properties of exponents and logarithms gives

$$\begin{aligned} 10^x &= 10^{\log(3)+r} \\ &= 10^{\log(3)}10^r \\ &= 3 \cdot 10^r. \end{aligned}$$

Thus, $x = \log(3) + r$ is a solution to the equation $10^x = 3 \cdot 10^r$.

14. Solve each equation. If there is no solution, explain why.

a. $3 \cdot 5^x = 21$

$$\begin{aligned} 5^x &= 7 \\ \log(5^x) &= \log(7) \\ x\log(5) &= \log(7) \\ x &= \frac{\log(7)}{\log(5)} \end{aligned}$$

b. $10^{x-3} = 25$

$\log(10^{x-3}) = \log(25)$
$x = 3 + \log(25)$

c. $10^x + 10^{x+1} = 11$

$10^x(1 + 10) = 11$
$10^x = 1$
$x = 0$

d. $8 - 2^x = 10$

$-2^x = 2$
$2^x = -2$

There is no solution because 2^x is always positive for all real x.

15. Solve the following equation for n: $A = P(1 + r)^n$.

$$A = P(1 + r)^n$$
$$\log(A) = \log[(P(1 + r)^n]$$
$$\log(A) = \log(P) + \log[(1 + r)^n]$$
$$\log(A) - \log(P) = n \log(1 + r)$$
$$n = \frac{\log(A) - \log(P)}{\log(1 + r)}$$
$$n = \frac{\log\left(\frac{A}{P}\right)}{\log(1 + r)}$$

The remaining questions establish a property for the logarithm of a sum. Although this is an application of the logarithm of a product, the formula does have some applications in information theory and can help with the calculations necessary to use tables of logarithms, which are explored further in Lesson 15.

16. In this exercise, we will establish a formula for the logarithm of a sum. Let $L = \log(x + y)$, where $x, y > 0$.

a. **Show $\log(x) + \log\left(1 + \frac{y}{x}\right) = L$. State as a property of logarithms after showing this is a true statement.**

$$\begin{aligned}\log(x) + \log\left(1 + \frac{y}{x}\right) &= \log\left(x\left(1 + \frac{y}{x}\right)\right) \\ &= \log\left(x + \frac{xy}{x}\right) \\ &= \log(x + y) \\ &= L\end{aligned}$$

Therefore, for $x, y > 0$, $\log(x + y) = \log(x) + \log\left(1 + \frac{y}{x}\right)$.

b. Use part (a) and the fact that $\log(100) = 2$ to rewrite $\log(365)$ as a sum.

$$\begin{aligned}\log(365) &= \log(100 + 265) \\ &= \log(100) + \log\left(1 + \frac{265}{100}\right) \\ &= \log(100) + \log(3.65) \\ &= 2 + \log(3.65)\end{aligned}$$

c. Rewrite 365 in scientific notation, and use properties of logarithms to express $\log(365)$ as a sum of an integer and a logarithm of a number between 0 and 10.

$$365 = 3.65 \times 10^2$$

$$\begin{aligned}\log(365) &= \log(3.65 \times 10^2) \\ &= \log(3.65) + \log(10^2) \\ &= 2 + \log(3.65)\end{aligned}$$

d. What do you notice about your answers to (b) and (c)?

Separating 365 into $100 + 265$ and using the formula for the logarithm of a sum is the same as writing 365 in scientific notation and using the formula for the logarithm of a product.

e. Find two integers that are upper and lower estimates of $\log(365)$.

Since $1 < 3.65 < 10$, we know that $0 < \log(3.65) < 1$. This tells us that $2 < 2 + \log(3.65) < 3$, so $2 < \log(365) < 3$.

Lesson 13: Changing the Base

Student Outcomes

- Students understand how to change logarithms from one base to another.
- Students calculate logarithms with any base using a calculator that computes only logarithms base 10 and base e.
- Students justify properties of logarithms with any base.

Lesson Notes

The first example in this lesson demonstrates how to use a base-10 logarithm to calculate a base-2 logarithm, leading to the change of base formula for logarithms. The change of base formula allows students to generalize the properties of base-10 logarithms developed in the previous few lessons to logarithms with general base b. This lesson introduces the natural logarithm $\ln(x) = \log_e(x)$. Calculators are used briefly in this lesson to compute both common and natural logarithms, and one of the goals of the lesson is to explain why the calculator only has a LOG and an LN key. Students solve exponential equations by applying the appropriate logarithm (**F-LE.A.4**).

Materials

Students need access either to graphing calculators or computer software capable of computing logarithms with base 10 and base e, such as the Wolfram|Alpha engine.

Classwork

Example 1 (5 minutes)

The purpose of this example is to show how to find $\log_2(x)$ using $\log(x)$.

- We have been working primarily with base-10 logarithms, but in Lesson 7 we defined logarithms for any base b. For example, the number 2 might be the base. When logarithms have bases other than 10, it often helps to be able to rewrite the logarithm in terms of base 10 logarithms. Let $L = \log_2(x)$, and show that $L = \frac{\log(x)}{\log(2)}$.
  - *Let* $L = \log_2(x)$.
    Then $2^L = x$.
    Taking the logarithm of each side, we get
    $$\log(2^L) = \log(x)$$
    $$L \cdot \log(2) = \log(x)$$
    $$L = \frac{\log(x)}{\log(2)}.$$
    Therefore, $\log_2(x) = \frac{\log(x)}{\log(2)}$.

> *Scaffolding:*
> - Students who struggle with the first step of this example might need to be reminded of the definition of *logarithm* from Lesson 7: $L = \log_b(x)$ means $b^L = x$. Therefore, $L = \log_2(x)$ means $2^L = x$.
> - Advanced learners may want to immediately start with the second part of the example, converting $\log_b(x)$ into $\frac{\log(x)}{\log(b)}$.

Remember that $\log(2)$ is a number, so this shows that $\log_2(x)$ is a rescaling of $\log(x)$.

- The example shows how we can convert $\log_2(x)$ to an expression involving $\log(x)$. More generally, suppose we are given a logarithm with base b. What is $\log_b(x)$ in terms of $\log(x)$?
  - *Let* $L = \log_b(x)$.
    Then $b^L = x$.
    Taking the logarithm of each side, we get
    $$\begin{aligned} \log(b^L) &= \log(x) \\ L \cdot \log(b) &= \log(x) \\ L &= \frac{\log(x)}{\log(b)}. \end{aligned}$$
    Therefore, $\log_b(x) = \frac{\log(x)}{\log(b)}$.
- This equation not only allows us to change from $\log_b(x)$ to $\log(x)$ but to change the base in the other direction as well: $\log(x) = \log_b(x) \cdot \log(b)$.

Exercise 1 (3 minutes)

The first exercise deals with the general formula for changing the base of a logarithm. It follows the same pattern as Example 1. Take time for students to share their results from Exercise 1 in a class discussion before moving on to Exercise 2 so that all students understand how the base of a logarithm is changed. Ask students to work in pairs on this exercise.

> *Scaffolding:*
> If students have difficulty with Exercise 1, they should review the argument in Example 1, noting that it deals with base 10, whereas this exercise generalizes that base to a.

Exercises

1. **Assume that x, a, and b are all positive real numbers, so that $a \neq 1$ and $b \neq 1$. What is $\log_b(x)$ in terms of $\log_a(x)$? The resulting equation allows us to change the base of a logarithm from a to b.**

   Let $L = \log_b(x)$. Then $b^L = x$. Taking the logarithm base a of each side, we get
   $$\begin{aligned} \log_a(b^L) &= \log_a(x) \\ L \cdot \log_a(b) &= \log_a(x) \\ L &= \frac{\log_a(x)}{\log_a(b)}. \end{aligned}$$
   Therefore, $\log_b(x) = \frac{\log_a(x)}{\log_a(b)}$.

Discussion (2 minutes)

Ask a student to present the solution to Exercise 1 to the class to ensure that all students understand how to change the base of a logarithm and how the formula comes from the definition of the logarithm as an exponential equation. Be sure that students record the formula in their notebooks.

> Change of Base Formula for Logarithms
> If x, a, and b are all positive real numbers with $a \neq 1$ and $b \neq 1$, then
> $$\log_b(x) = \frac{\log_a(x)}{\log_a(b)}.$$

Exercise 2 (2 minutes)

In the second exercise, students practice changing bases. They need a calculator with the ability to calculate logarithms base 10. Later in the lesson, students need to calculate natural logarithms as well. Students should work in pairs on this exercise, with one student using the calculator and the other keeping track of the computation. Students should share their results for Exercise 2 in a class discussion before moving on to Exercise 3.

2. Approximate each of the following logarithms to four decimal places. Use the LOG key on your calculator rather than logarithm tables, first changing the base of the logarithm to 10 if necessary.

a. $\log(3^2)$

$\log(3^2) = \log(9) \approx 0.9542$

Therefore, $\log(3^2) \approx 0.9542$.

OR

$\log(3^2) = 2\log(3) \approx 2 \cdot 0.4771 \approx 0.9542$

Therefore, $\log(3^2) \approx 0.9542$.

b. $\log_3(3^2)$

$\log_3(3^2) = \dfrac{2\log(3)}{\log(3)} = 2$

Therefore, $\log_3(3^2) = 2.0000$.

c. $\log_2(3^2)$

$\log_2(3^2) = \log_2(9) = \dfrac{\log(9)}{\log(2)} \approx 3.1699$

Therefore, $\log_2(3^2) \approx 3.1699$.

Scaffolding:

Students who are not familiar with the LOG key on the calculator can check how it works by evaluating the following expressions:

$\log(1)$,

$\log(10)$,

$\log(10^3)$.

Exercise 3 (8 minutes)

3. In Lesson 12, we justified a number of properties of base-10 logarithms. Working in pairs, justify the following properties of base-b logarithms:

a. $\log_b(1) = 0$

Because $L = \log_b(x)$ *means* $b^L = x$*, then when* $x = 1$, $L = 0$.

b. $\log_b(b) = 1$

Because $L = \log_b(x)$ *means* $b^L = x$*, then when* $x = b$, $L = 1$.

c. $\log_b(\mathrm{b}^r) = r$

Because $L = \log_b(x)$ *means* $b^L = x$*, then when* $x = b^r$, $L = r$.

Scaffolding:

By working in pairs, students should be able to reconstruct the arguments they used in Lesson 10. If they have trouble, they should be encouraged to use the definition and properties already justified.

d. $b^{\log_b(x)} = x$

Because $L = \log_b(x)$ means $b^L = x$, then $x = b^{\log_b(x)}$.

e. $\log_b(x \cdot y) = \log_b(x) + \log_b(y)$

By the rule $a^q \cdot a^r = a^{q+r}$, $b^{\log_b(x)} \cdot b^{\log_b(y)} = b^{\log_b(x)+\log_b(y)}$.

By property 4, $b^{\log_b(x)} \cdot b^{\log_b(y)} = x \cdot y$.

Therefore, $x \cdot y = b^{\log_b(x)+\log_b(y)}$. By property 4 again, $x \cdot y = b^{\log_b(x \cdot y)}$.

So, the exponents must be equal, and $\log_b(x \cdot y) = \log_b(x) + \log_b(y)$.

f. $\log_b(x^r) = r \cdot \log_b(x)$

By the rule $(a^q)^r = a^{qr}$, $b^{r \log_b(x)} = \left(b^{\log_b(x)}\right)^r$.

By property 4, $\left(b^{\log_b(x)}\right)^r = x^r$.

Therefore, $x^r = b^{r\log_b(x)}$. By property 4 again, $x^r = b^{\log_b(x^r)}$.

So, the exponents must be equal, and $\log_b(x^r) = r \cdot \log_b(x)$.

g. $\log_b\left(\frac{1}{x}\right) = -\log_b(x)$

By property 6, $\log_b(x^k) = k \cdot \log_b(x)$.

Let $k = -1$; then for $x \neq 0$, $\log_b(x^{-1}) = (-1) \cdot \log_b(x)$.

Thus, $\log_b\left(\frac{1}{x}\right) = -\log_b(x)$.

h. $\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$

By property 5, $\log_b(x \cdot y) = \log_b(x) + \log_b(y)$.

By property 7, for $y \neq 0$, $\log_b(y^{-1}) = (-1) \cdot \log_b(y)$.

Therefore, $\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$.

Discussion (2 minutes)

Define the natural logarithm in this Discussion. Because students often misinterpret the symbol ln as the word *in,* take the time to emphasize that the notation is an *L* followed by an *N*, which comes from the French for natural logarithm: *le logarithme naturel*.

- Recall Euler's number e from Lesson 5, which is an irrational number approximated by $e \approx 2.718\,28 \ldots$. This number plays an important role in many parts of mathematics, and it is frequently used as the base of logarithms. Because exponential functions with base e are used to model growth and change of natural phenomena, a logarithm with base e is called a *natural logarithm*. The notation for the natural logarithm of a positive number x is $\ln(x) = \log_e(x)$.
- What is the value of $\ln(1)$?
  - $\ln(1) = 0$

- What is the value of $\ln(e)$? The value of $\ln(e^2)$? Of $\ln(e^3)$?
  - $\ln(e) = 1$, $\ln(e^2) = 2$, *and* $\ln(e^3) = 3$
- Because scientists primarily use logarithms base 10 and base e, calculators often only have two logarithm buttons: LOG for calculating $\log(x)$ and LN for calculating $\ln(x)$. With the change of base formula, you can use either the common logarithm (base 10) or the natural logarithm (base e) to calculate the value of a logarithm with any allowable base b, so technically we only need one of those two buttons. However, each base has important uses, so most calculators are able to calculate logarithms in either base.

Exercises 4–6 (5 minutes)

Exercises 4–6 allow students to compare the values of $\ln(x)$ to the more familiar values of $\log(x)$ for a few values of x and to conclude that for any $x \geq 1$, $\log(x) \leq \ln(x)$. Students need a calculator with an LN key. They should work in pairs on these exercises, with one student using the calculator and the other recording the result. They should share their results for Exercise 4 in a class discussion before moving on.

Scaffolding:

Students who are not familiar with the LN key on the calculator can check how it works by evaluating the following expressions:

$\ln(1)$,
$\ln(e)$,
$\ln(e^3)$.

4. Use the LN and LOG keys on your calculator to find the value of each logarithm to four decimal places.

a.	$\ln(1)$	0.0000	$\log(1)$	0.0000
b.	$\ln(3)$	1.0986	$\log(3)$	0.4771
c.	$\ln(10)$	2.3026	$\log(10)$	1.0000
d.	$\ln(25)$	3.2189	$\log(25)$	1.3979
e.	$\ln(100)$	4.6052	$\log(100)$	2.0000

5. Make a conjecture that compares values of $\log(x)$ to $\ln(x)$ for $x \geq 1$.

It appears that for $x \geq 1$, $\log(x) \leq \ln(x)$.

6. Justify your conjecture in Exercise 5 using the change of base formula.

By the change of base formula, $\log(x) = \frac{\ln(x)}{\ln(10)}$. Then $\ln(10) \cdot \log(x) = \ln(x)$. Since $\ln(10) \approx 2.3$, $\log(x) \leq \ln(10) \cdot \log(x)$, and thus $\log(x) \leq \ln(x)$.

Example 2 (3 minutes)

This example introduces more complicated expressions involving logarithms and showcases the power of logarithms in rearranging logarithmic expressions. Students have done similar exercises in their homework in prior lessons for base-10 logarithms, so this example and the following exercises demonstrate how the same procedures apply to natural logarithms. Remind students that Exercise 3 established that the logarithm properties developed for base-10 logarithms apply for logarithms of any base, including base e.

- Write as an expression containing only one logarithm: $\ln(k^2) + \ln\left(\frac{1}{k^2}\right) - \ln\left(\sqrt{k}\right)$.
  - $\ln(k^2) + \ln\left(\frac{1}{k^2}\right) - \ln\left(\sqrt{k}\right) = 2\ln(k) - 2\ln(k) - \frac{1}{2}\cdot\ln(k) = -\frac{1}{2}\ln(k)$
- Therefore, $\ln(k^2) + \ln\left(\frac{1}{k^2}\right) - \ln\left(\sqrt{k}\right) = -\frac{1}{2}\ln(k)$.

Exercises 7–8 (6 minutes)

Exercise 7 follows Example 2 by asking students to simplify more complicated logarithmic expressions. In Exercise 7, students condense a sum of logarithmic expressions to an expression containing only one logarithm, while in Exercise 8, students take a single complicated logarithm and break it up into simpler parts. Students should work in pairs on these exercises, sharing their results in a class discussion before the Closing.

7. **Write as a single logarithm.**

   a. $\ln(4) - 3\ln\left(\frac{1}{3}\right) + \ln(2)$

$$\begin{aligned}\ln(4) - 3\ln\left(\frac{1}{3}\right) + \ln(2) &= \ln(4) + \ln(3^3) + \ln(2)\\ &= \ln(4\cdot 3^3\cdot 2)\\ &= \ln(216)\\ &= \ln(6^3)\\ &= 3\ln(6)\end{aligned}$$

   Any of the last three expressions is an acceptable final answer.

   b. $\ln(5) + \frac{3}{5}\ln(32) - \ln(4)$

$$\begin{aligned}\ln(5) + \frac{3}{5}\ln(32) - \ln(4) &= \ln(5) + \ln\left(32^{\frac{3}{5}}\right) - \ln(4)\\ &= \ln(5) + \ln(8) - \ln(4)\\ &= \ln(5\cdot 8) - \ln(4)\\ &= \ln\left(\frac{40}{4}\right)\\ &= \ln(10)\end{aligned}$$

   Therefore, $\ln(5) + \frac{3}{5}\ln(32) - \ln(4) = \ln(10)$.

8. **Write each expression as a sum or difference of constants and logarithms of simpler terms.**

   a. $\ln\left(\frac{\sqrt{5x^3}}{e^2}\right)$

$$\begin{aligned}\ln\left(\frac{\sqrt{5x^3}}{e^2}\right) &= \ln(\sqrt{5}) + \ln\left(\sqrt{x^3}\right) - \ln(e^2)\\ &= \frac{1}{2}\ln(5) + \frac{3}{2}\ln(x) - 2\end{aligned}$$

   b. $\ln\left(\frac{(x+y)^2}{x^2+y^2}\right)$

$$\begin{aligned}\ln\left(\frac{(x+y)^2}{x^2+y^2}\right) &= \ln(x+y)^2 - \ln(x^2+y^2)\\ &= 2\ln(x+y) - \ln(x^2+y^2)\end{aligned}$$

   The point of this simplification is that neither of these terms can be simplified further.

Closing (4 minutes)

Have students summarize the lesson by discussing the following questions and coming to a consensus before students record the answers in their notebooks:

- What is the definition of the logarithm base b?
  - *If there exist numbers b, L, and x so that $b^L = x$, then $L = \log_b(x)$.*
- What does $\ln(x)$ represent?
  - *The notation $\ln(x)$ represents the logarithm of x base e; that is, $\ln(x) = \log_e(x)$.*
- How can we use a calculator to approximate a logarithm to a base other than 10 or e?
  - *Use the change of base formula to convert a logarithm with base b to one with base 10 or base e; then, use the appropriate calculator function.*

Lesson Summary

We have established a formula for changing the base of logarithms from b to a:

$$\log_b(x) = \frac{\log_a(x)}{\log_a(b)}.$$

In particular, the formula allows us to change logarithms base b to common or natural logarithms, which are the only two kinds of logarithms that most calculators compute:

$$\log_b(x) = \frac{\log(x)}{\log(b)} = \frac{\ln(x)}{\ln(b)}.$$

We have also established the following properties for base b logarithms. If x, y, a, and b are all positive real numbers with $a \neq 1$ and $b \neq 1$ and r is any real number, then:

1. $\log_b(1) = 0$
2. $\log_b(b) = 1$
3. $\log_b(b^r) = r$
4. $b^{\log_b(x)} = x$
5. $\log_b(x \cdot y) = \log_b(x) + \log_b(y)$
6. $\log_b(x^r) = r \cdot \log_b(x)$
7. $\log_b\left(\frac{1}{x}\right) = -\log_b(x)$
8. $\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$.

Exit Ticket (5 minutes)

Name ____ Date ____________________

Lesson 13: Changing the Base

Exit Ticket

1. Are there any properties that hold for base-10 logarithms that would not be valid for the logarithm base e? Why? Are there any properties that hold for base-10 logarithms that would not be valid for some positive base b, such that $b \neq 1$?

2. Write each logarithm as an equivalent expression involving only logarithms base 10.
   a. $\log_3(25)$
   b. $\log_{100}(x^2)$

3. Rewrite each expression as an equivalent expression containing only one logarithm.
   a. $3\ln(p+q) - 2\ln(q) - 7\ln(p)$
   b. $\ln(xy) - \ln\left(\frac{x}{y}\right)$

Exit Ticket Sample Solutions

1. Are there any properties that hold for base-10 logarithms that would not be valid for the logarithm base e? Why? Are there any properties that hold for base-10 logarithms that would not be valid for some positive base b, such that $b \neq 1$?

   No. Any property that is true for a base-10 logarithm will be true for a base-e logarithm. The only difference between a common logarithm and a natural logarithm is a scale change because $\log(x) = \frac{\ln(x)}{\ln(10)}$, and $\ln(x) = \frac{\log(x)}{\log(e)}$.

   Since $\log_b(x) = \frac{\log(x)}{\log(b)}$, we would only encounter a problem if $\log(b) = 0$, but this only happens when $b = 1$, and 1 is not a valid base for logarithms.

2. Write each logarithm as an equivalent expression involving only logarithms base 10.

   a. $\log_3(25)$

   $$\log_3(25) = \frac{\log(25)}{\log(3)}$$

   b. $\log_{100}(x^2)$

   $$\begin{aligned}\log_{100}(x^2) &= \frac{\log(x^2)}{\log(100)}\\ &= \frac{2\log(x)}{2}\\ &= \log(x)\end{aligned}$$

3. Rewrite each expression as an equivalent expression containing only one logarithm.

   a. $3\ln(p+q) - 2\ln(q) - 7\ln(p)$

   $$\begin{aligned}3\ln(p+q) - 2\ln(q) - 7\ln(p) &= \ln((p+q)^3) - \left(\ln(q^2) + \ln(p^7)\right)\\ &= \ln((p+q)^3) - \ln(q^2p^7)\\ &= \ln\left(\frac{(p+q)^3}{q^2p^7}\right)\end{aligned}$$

   b. $\ln(xy) - \ln\left(\frac{x}{y}\right)$

   $$\begin{aligned}\ln(xy) - \ln\left(\frac{x}{y}\right) &= \ln(x) + \ln(y) - \ln(x) + \ln(y)\\ &= 2\ln(y)\\ &= \ln(y^2)\end{aligned}$$

   Therefore, $\ln(xy) - \ln\left(\frac{x}{y}\right)$ is equivalent to both $2\ln(y)$ and $\ln(y^2)$.

Problem Set Sample Solutions

1. Evaluate each of the following logarithmic expressions, approximating to four decimal places if necessary. Use the LN or LOG key on your calculator rather than a table.

   a. $\log_8(16)$

   $$\log_8(16) = \frac{\log(16)}{\log(8)} = \frac{\log(2^4)}{\log(2^3)} = \frac{4 \cdot \log(2)}{3 \cdot \log(2)} = \frac{4}{3}$$

   Therefore, $\log_8(16) = \frac{4}{3}$.

   b. $\log_7(11)$

   $$\log_7(11) = \frac{\log(11)}{\log(7)} \approx 1.2323$$

   Therefore, $\log_7(11) \approx 1.2323$.

   c. $\log_3(2) + \log_2(3)$

   $$\log_3(2) + \log_2(3) = \frac{\log(2)}{\log(3)} + \frac{\log(3)}{\log(2)} \approx 2.2159$$

   Therefore, $\log_3(2) + \log_2(3) \approx 2.2159$.

2. Use logarithmic properties and the fact that $\ln(2) \approx 0.69$ and $\ln(3) \approx 1.10$ to approximate the value of each of the following logarithmic expressions. Do not use a calculator.

   a. $\ln(e^4)$

   $$\ln(e^4) = 4\ln(e) = 4$$

   Therefore, $\ln(e^4) = 4$.

   b. $\ln(6)$

   $$\ln(6) = \ln(2) + \ln(3) \approx 0.69 + 1.10 \approx 1.79$$

   Therefore, $\ln(6) \approx 1.79$.

c. $\ln(108)$

$$\begin{aligned}\ln(108) &= \ln(4\cdot 27)\\ &= \ln(4)+\ln(27)\\ &\approx 2\ln(2)+3\ln(3)\\ &\approx 1.38+3.30\\ &\approx 4.68\end{aligned}$$

Therefore, $\ln(108)\approx 4.68$.

d. $\ln\left(\frac{8}{3}\right)$

$$\begin{aligned}\ln\left(\frac{8}{3}\right) &= \ln(8)-\ln(3)\\ &= \ln(2^3)-\ln(3)\\ &\approx 3(0.69)-1.10\\ &\approx 0.97\end{aligned}$$

Therefore, $\ln\left(\frac{8}{3}\right)\approx 0.97$.

3. Compare the values of $\log_{\frac{1}{9}}(10)$ and $\log_9\left(\frac{1}{10}\right)$ without using a calculator.

Using the change of base formula,

$$\begin{aligned}\log_{\frac{1}{9}}(10) &= \frac{\log_9(10)}{\log_9\left(\frac{1}{9}\right)}\\ &= \frac{\log_9(10)}{-1}\\ &= -\log_9(10)\\ &= \log_9\left(\frac{1}{10}\right).\end{aligned}$$

Thus, $\log_{\frac{1}{9}}(10)=\log_9\left(\frac{1}{10}\right)$.

4. Show that for any positive numbers a and b with $a\neq 1$ and $b\neq 1$, $\log_a(b)\cdot\log_b(a)=1$.

Using the change of base formula,

$$\log_a(\mathrm{b})=\frac{\log_b(\mathrm{b})}{\log_b(\mathrm{a})}=\frac{1}{\log_b(\mathrm{a})}.$$

Thus,

$$\log_a(b)\cdot\log_b(a)=\frac{1}{\log_b(a)}\cdot\log_b(a)=1.$$

5. Express x in terms of a, e, and y if $\ln(x)-\ln(y)=2a$.

$$\begin{aligned}\ln(x)-\ln(y) &= 2a\\ \ln\left(\frac{x}{y}\right) &= 2a\\ \frac{x}{y} &= e^{2a}\\ x &= y\,e^{2a}\end{aligned}$$

6. **Rewrite each expression in an equivalent form that only contains one base-10 logarithm.**

   a. $\log_2(800)$

   $$\log_2(800) = \frac{\log(800)}{\log(2)} = \frac{\log(2^3)+2}{\log(2)} = \frac{3\log(2)+2}{\log(2)} = \frac{3\log(2)}{\log(2)} + \frac{2}{\log(2)} = 3 + \frac{2}{\log(2)}$$

   b. $\log_x\left(\frac{1}{10}\right)$, **for positive real values of** $x \neq 1$

   $$\log_x\left(\frac{1}{10}\right) = \frac{\log\left(\frac{1}{10}\right)}{\log(x)} = -\frac{1}{\log(x)}$$

   c. $\log_5(12500)$

   $$\log_5(12500) = \frac{\log(5^3 \cdot 10^2)}{\log(5)} = \frac{\log(5^3)+\log(10^2)}{\log(5)} = \frac{3\log(5)+2\log(10)}{\log(5)} = 3 + \frac{2}{\log(5)}$$

   d. $\log_3(0.81)$

   $$\log_3(0.81) = \frac{\log\left(\frac{81}{100}\right)}{\log(3)} = \frac{\log(81)-\log(100)}{\log(3)} = \frac{4\log(3)-2}{\log(3)} = 4 - \frac{2}{\log(3)}$$

7. **Write each number in terms of natural logarithms, and then use the properties of logarithms to show that it is a rational number.**

   a. $\log_9(\sqrt{27})$

   $$\frac{\ln(\sqrt{27})}{\ln(9)} = \frac{\ln\left(3^{\frac{3}{2}}\right)}{\ln(3^2)} = \frac{\frac{3}{2}\ln(3)}{2\ln(3)} = \frac{3}{4}$$

   b. $\log_8(32)$

   $$\frac{\ln(32)}{\ln(8)} = \frac{\ln(2^5)}{\ln(2^3)} = \frac{5}{3}$$

   c. $\log_4\left(\frac{1}{8}\right)$

   $$\frac{\ln\left(\frac{1}{8}\right)}{\ln(4)} = \frac{\ln(2^{-3})}{\ln(2^2)} = -\frac{3}{2}$$

8. **Write each expression as an equivalent expression with a single logarithm. Assume x, y, and z are positive real numbers.**

   a. $\ln(x) + 2\ln(y) - 3\ln(z)$

   $$\ln\left(\frac{xy^2}{z^3}\right)$$

b. $\frac{1}{2}\left(\ln(x+y) - \ln(z)\right)$

$\ln\left(\sqrt{\frac{x+y}{z}}\right)$

c. $(x+y) + \ln(z)$

$(x+y)\ln(e) + \ln(z) = \ln(e^{x+y}) + \ln(z) = \ln(e^{x+y} \cdot z)$

9. Rewrite each expression as sums and differences in terms of $\ln(x)$, $\ln(y)$, and $\ln(z)$.

a. $\ln(xyz^3)$

$\ln(x) + \ln(y) + 3\ln(z)$

b. $\ln\left(\frac{e^3}{xyz}\right)$

$3 - \ln(x) - \ln(y) - \ln(z)$

c. $\ln\left(\sqrt{\frac{x}{y}}\right)$

$\frac{1}{2}\left(\ln(x) - \ln(y)\right)$

10. Use base-5 logarithms to rewrite each exponential equation as a logarithmic equation, and solve the resulting equation. Use the change of base formula to convert to a base-10 logarithm that can be evaluated on a calculator. Give each answer to 4 decimal places. If an equation has no solution, explain why.

a. $5^{2x} = 20$

$$2x = \log_5(20)$$
$$x = \frac{1}{2}\log_5(20)$$
$$x = \frac{\log(20)}{2\log(5)}$$
$$x \approx 0.9307$$

b. $75 = 10 \cdot 5^{x-1}$

$$7.5 = 5^{x-1}$$
$$x = \log_5(7.5) + 1$$
$$x = \frac{\log(7.5)}{\log(5)} + 1$$
$$x \approx 2.2519$$

c. $5^{2+x} - 5^x = 10$

$5^x(5^2 - 1) = 10$

$5^x = \frac{10}{24}$

$x = \log_5\left(\frac{10}{24}\right)$

$x = \frac{\log\left(\frac{10}{24}\right)}{\log(5)}$

$x = \frac{\log(10) - \log(24)}{\log(5)}$

$x \approx -0.5440$

d. $5^{x^2} = 0.25$

$x^2 = \log_5(0.25)$

$x^2 = \frac{\log(0.25)}{\log(5)}$

$x^2 \approx -0.8614$

This equation has no real solution because x^2 cannot be negative for any real number x.

11. In Lesson 6, you discovered that $\log(x \cdot 10^k) = k + \log(x)$ by looking at a table of logarithms. Use the properties of logarithms to justify this property for an arbitrary base $b > 0$ with $b \neq 1$. That is, show that $\log_b(x \cdot b^k) = k + \log_b(x)$.

$$\begin{aligned}\log_b(x \cdot b^k) &= \log_b(x) + \log_b(b^k)\\ &= k + \log_b(x)\end{aligned}$$

MP.3

12. Larissa argued that since $\log_2(2) = 1$ and $\log_2(4) = 2$, then it must be true that $\log_2(3) = 1.5$. Is she correct? Explain how you know.

Larissa is not correct. If $\log_2(x) = 1.5$, then $2^{1.5} = x$, so $x = 2^{\frac{3}{2}} = 2\sqrt{2}$. Since $3 \neq 2\sqrt{2}$, Larissa's calculation is not correct.

13. Extension: Suppose that there is some positive number b so that

$$\log_b(2) = 0.36$$

$$\log_b(3) = 0.57$$

$$\log_b(5) = 0.84.$$

a. Use the given values of $\log_b(2)$, $\log_b(3)$, and $\log_b(5)$ to evaluate the following logarithms.

i. $\log_b(6)$

$$\begin{aligned}\log_b(6) &= \log_b(2 \cdot 3)\\ &= \log_b(2) + \log_b(3)\\ &= 0.36 + 0.57\\ &= 0.93\end{aligned}$$

ii. $\log_b(8)$

$$\begin{aligned}\log_b(8) &= \log_b(2^3)\\ &= 3\cdot\log_b(2)\\ &= 3\cdot 0.36\\ &= 1.08\end{aligned}$$

iii. $\log_b(10)$

$$\begin{aligned}\log_b(10) &= \log_b(2\cdot 5)\\ &= \log_b(2)+\log_b(5)\\ &= 0.36+0.84\\ &= 1.20\end{aligned}$$

iv. $\log_b(600)$

$$\begin{aligned}\log_b(600) &= \log_b(6\cdot 100)\\ &= \log_b(6)+\log_b(100)\\ &= 0.93+2\log_b(10)\\ &= 0.93+2(1.20)\\ &= 0.93+2.40\\ &= 3.33\end{aligned}$$

b. Use the change of base formula to convert $\log_b(10)$ to base 10, and solve for b. Give your answer to four decimal places.

From part (iii) above, $\log_b(10) = 1.20$. *Then,*

$$\begin{aligned}1.20 &= \log_b(10)\\ 1.20 &= \frac{\log_{10}(10)}{\log_{10}(b)}\\ 1.20 &= \frac{1}{\log_{10}(b)}\\ \frac{1}{1.20} &= \log_{10}(b)\\ b &= 10^{\frac{1}{1.20}}\\ b &\approx 6.8129.\end{aligned}$$

14. Use a logarithm with an appropriate base to solve the following exponential equations.

a. $2^{3x} = 16$

$$\begin{aligned}\log_2(2^{3x}) &= \log_2(16)\\ 3x &= 4\\ x &= \frac{4}{3}\end{aligned}$$

b. $2^{x+3} = 4^{3x}$

$$\log_2(2^{x+3}) = \log_2(4^{3x})$$
$$x + 3 = 3x \cdot \log_2(4)$$
$$x + 3 = 3x \cdot 2$$
$$5x = 3$$
$$x = \frac{3}{5}$$

c. $3^{4x-2} = 27^{x+2}$

$$\log_3(3^{4x-2}) = \log_3(27^{x+2})$$
$$(4x - 2)\log_3(3) = (x + 2)\log_3(27)$$
$$4x - 2 = 3(x + 2)$$
$$4x - 2 = 3x + 6$$
$$x = 8$$

d. $4^{2x} = \left(\frac{1}{4}\right)^{3x}$

$$\log_4(4^{2x}) = \log_4\left(\left(\frac{1}{4}\right)^{3x}\right)$$
$$2x\log_4(4) = 3x\log_4\left(\frac{1}{4}\right)$$
$$2x = 3x(-1)$$
$$5x = 0$$
$$x = 0$$

e. $5^{0.2x+3} = 625$

$$\log_5(5^{0.2x+3}) = \log_5(625)$$
$$(0.2x + 3)\log_5(5) = \log_5(5^4)$$
$$0.2x + 3 = 4$$
$$0.2x = 1$$
$$x = 5$$

15. **Solve each exponential equation.**

a. $3^{2x} = 81$

$x = 2$

b. $6^{3x} = 36^{x+1}$

$x = 2$

c. $625 = 5^{3x}$

$x = \frac{4}{3}$

d. $25^{4-x} = 5^{3x}$

$x = \frac{8}{5}$

e. $32^{x-1} = \frac{1}{2}$

$x = \frac{4}{5}$

f. $\frac{4^{2x}}{2^{x-3}} = 1$

$x = -1$

g. $\frac{1}{8^{2x-4}} = 1$

$x = 2$

h. $2^x = 81$

$x = \frac{\ln(81)}{\ln(2)}$

i. $8 = 3^x$

$x = \frac{\ln(8)}{\ln(3)}$

j. $6^{x+2} = 12$

$x = -2 + \frac{\log(12)}{\log(6)}$

k. $10^{x+4} = 27$

$x = -4 + \log(27)$

l. $2^{x+1} = 3^{1-x}$

$x = \frac{\log(3) - \log(2)}{\log(2) + \log(3)}$

m. $3^{2x-3} = 2^{x+4}$

$x = \frac{4\log(2) + 3\log(3)}{3\log(3) - \log(2)}$

n. $e^{2x} = 5$

$x = \frac{\ln(5)}{2}$

o. $e^{x-1} = 6$

$x = 1 + \ln(6)$

16. In Problem 9(e) of Lesson 12, you solved the equation $3^x = 7^{-3x+2}$ using the logarithm base 10.

a. Solve $3^x = 7^{-3x+2}$ using the logarithm base 3.

$$\log_3(3^x) = \log_3(7^{-3x+2})$$
$$x = (-3x+2)\log_3(7)$$
$$x = -3x\log_3(7) + 2\log_3(7)$$
$$x + 3x\log_3(7) = 2\log_3(7)$$
$$x\left(1 + 3\log_3(7)\right) = 2\log_3(7)$$
$$x = \frac{2\log_3(7)}{1 + 3\log_3(7)}$$

b. Apply the change of base formula to show that your answer to part (a) agrees with your answer to Problem 9(e) of Lesson 12.

Changing from base 3 to base 10, we see that

$$\log_3(7) = \frac{\log(7)}{\log(3)}.$$

Then,

$$\frac{2\log_3(7)}{1 + 3\log_3(7)} = \frac{2\left(\frac{\log(7)}{\log(3)}\right)}{1 + 3\left(\frac{\log(7)}{\log(3)}\right)}$$
$$= \frac{2\log(7)}{\log(3) + 3\log(7)},$$

which was the answer from Problem 9(e) of Lesson 12.

c. **Solve $3^x = 7^{-3x+2}$ using the logarithm base 7.**

$$\begin{aligned} \log_7(3^x) &= \log_7(7^{-3x+2}) \\ x\log_7(3) &= -3x+2 \\ 3x + x\log_7(3) &= 2 \\ x\big(3+\log_7(3)\big) &= 2 \\ x &= \frac{2}{3+\log_7(3)} \end{aligned}$$

d. **Apply the change of base formula to show that your answer to part (c) also agrees with your answer to Problem 9(e) of Lesson 12.**

Changing from base 7 to base 10, we see that

$$\log_7(3) = \frac{\log(3)}{\log(7)}.$$

Then,

$$\begin{aligned} \frac{2}{3+\log_7(3)} &= \frac{2}{3+\frac{\log(3)}{\log(7)}} \\ &= \frac{2\log(7)}{3\log(7)+\log(3)}, \end{aligned}$$

which was the answer from Problem 9(e) of Lesson 12.

17. **Pearl solved the equation $2^x = 10$ as follows:**

$$\begin{aligned} \log(2^x) &= \log(10) \\ x\log(2) &= 1 \\ x &= \frac{1}{\log(2)}. \end{aligned}$$

Jess solved the equation $2^x = 10$ as follows:

$$\begin{aligned} \log_2(2^x) &= \log_2(10) \\ x\log_2(2) &= \log_2(10) \\ x &= \log_2(10). \end{aligned}$$

Is Pearl correct? Is Jess correct? Explain how you know.

MP.3

Both Pearl and Jess are correct. If we take Jess's solution and apply the change of base formula, we have

$$\begin{aligned} x &= \log_2(10) \\ &= \frac{\log(10)}{\log(2)} \\ &= \frac{1}{\log(2)}. \end{aligned}$$

Thus, the two solutions are equivalent, and both students are correct.

Lesson 14: Solving Logarithmic Equations

Student Outcomes

- Students solve simple logarithmic equations using the definition of logarithm and logarithmic properties.

Lesson Notes

In this lesson, students solve simple logarithmic equations by first putting them into the form $\log_b(Y) = L$, where b is either 2, 10, or e, Y is an expression, and L is a number, and then using the definition of logarithm to rewrite the equation in the form $b^L = Y$. Students are able to evaluate logarithms without technology by selecting an appropriate base; solutions are provided with this in mind. In Lesson 15, students learn the technique of solving exponential equations using logarithms of any base without relying on the definition. Students need to use the properties of logarithms developed in prior lessons to rewrite the equations in an appropriate form before solving (**A-SSE.A.2**, **F-LE.A.4**). The lesson starts with a few fluency exercises to reinforce the logarithmic properties before moving on to solving equations.

Classwork

Opening Exercise (3 minutes)

The following exercises provide practice with the definition of the logarithm and prepare students to solve logarithmic equations using the methods outlined later in the lesson. Encourage students to work alone on these exercises, but allow students to work in pairs if necessary.

Scaffolding:

- Remind students of the main properties that they use by writing the following on the board:
  $\log_b(x) = L$ means $b^L = x$;
  $\log_b(xy) = \log_b(x) + \log_b(y)$;
  $\log_b\left(\dfrac{x}{y}\right) = \log_b(x) - \log_b(y)$;
  $\log_b(x^r) = r \cdot \log_b(x)$;
  $\log_b\left(\dfrac{1}{x}\right) = -\log(x)$.
- Consistently using a visual display of these properties throughout the module is helpful.

Opening Exercise

Convert the following logarithmic equations to equivalent exponential equations

a. $\log(10{,}000) = 4$ $\quad$ $10^4 = 10{,}000$

b. $\log(\sqrt{10}) = \frac{1}{2}$ $\quad$ $10^{\frac{1}{2}} = \sqrt{10}$

c. $\log_2(256) = 8$ $\quad$ $2^8 = 256$

d. $\log_4(256) = 4$ $\quad$ $4^4 = 256$

e. $\ln(1) = 0$ $\quad$ $e^0 = 1$

f. $\log(x+2) = 3$ $\quad$ $x + 2 = 10^3$

Examples (6 minutes)

Students should be ready to take the next step from converting logarithmic equations to an equivalent exponential expression to solving the resulting equation. Decide whether or not students need to see a teacher-led example or can attempt to solve these equations in pairs. Anticipate that students neglect to check for extraneous solutions in these examples. After the examples, lead the discussion to the existence of an extraneous solution in Example 3.

Examples

Write each of the following equations as an equivalent exponential equation, and solve for x.

1. $\log(3x+7) = 0$

$$\begin{aligned} \log(3x+7) &= 0 \\ 10^0 &= 3x+7 \\ 1 &= 3x+7 \\ x &= -2 \end{aligned}$$

2. $\log_2(x+5) = 4$

$$\begin{aligned} \log_2(x+5) &= 4 \\ 2^4 &= x+5 \\ 16 &= x+5 \\ x &= 11 \end{aligned}$$

3. $\log(x+2) + \log(x+5) = 1$

$$\begin{aligned} \log(x+2) + \log(x+5) &= 1 \\ \log\big((x+2)(x+5)\big) &= 1 \\ (x+2)(x+5) &= 10^1 \\ x^2 + 7x + 10 &= 10 \\ x^2 + 7x &= 0 \\ x(x+7) &= 0 \\ x = 0 \text{ or } x &= -7 \end{aligned}$$

However, if $x = -7$, then $(x+2) = -5$, and $(x+5) = -2$, so both logarithms in the equation are undefined. Thus, -7 is an extraneous solution, and only 0 is a valid solution to the equation.

Discussion (4 minutes)

Ask students to volunteer their solutions to the equations in the Opening Exercise. This line of questioning is designed to allow students to determine that there is an extraneous solution to Example 3. If the class has already discovered this fact, opt to accelerate or skip this discussion.

- What is the solution to the equation in Example 1?
  - -2
- What is the result if you evaluate $\log(3x+7)$ at $x = -2$? Did you find a solution?
  - $\log(3(-2)+7) = \log(1) = 0$, *so* -2 *is a solution to* $\log(3x+7) = 0$.

- What is the solution to the equation in Example 2?
  - 11
- What is the result if you evaluate $\log_2(x+5)$ at $x = 11$? Did you find a solution?
  - $\log_2(11+5) = \log_2(16) = 4$, *so* 11 *is a solution to* $\log_2(x+5) = 4$.
- What is the solution to the equation in Example 3?
  - *There were two solutions:* 0 *and* -7.
- What is the result if you evaluate $\log(x+2) + \log(x+5)$ at $x = 0$? Did you find a solution?
  - $\log(2) + \log(5) = \log(2 \cdot 5) = \log(10) = 1$, *so* 0 *is a solution to* $\log(x+2) + \log(x+5) = 1$.
- What is the result if you evaluate $\log(x+2) + \log(x+5)$ at $x = -7$? Did you find a solution?
  - $\log(-7+2)$ *and* $\log(-7+5)$ *are not defined because* $-7+2$ *and* $-7+5$ *are negative. Thus,* -7 *is not a solution to the original equation.*
- What is the term we use for an apparent solution to an equation that fails to solve the original equation?
  - *It is called an extraneous solution.*
- Remember to look for extraneous solutions and to exclude them when you find them.

Exercise 1 (4 minutes)

Allow students to work in pairs or small groups to think about the exponential equation below. This equation can be solved rather simply by an application of the logarithmic property $\log_b(x^r) = r\log_b(x)$. However, if students do not see to apply this logarithmic property, it can become algebraically difficult.

MP.3

Exercises

1. **Drew said that the equation $\log_2[(x+1)^4] = 8$ cannot be solved because he expanded $(x+1)^4 = x^4 + 4x^3 + 6x^2 + 4x + 1$ and realized that he cannot solve the equation $x^4 + 4x^3 + 6x^2 + 4x + 1 = 2^8$. Is he correct? Explain how you know.**

   If we apply the logarithmic properties, this equation is solvable.

$$\log_2[(x+1)^4] = 8$$
$$4\log_2(x+1) = 8$$
$$\log_2(x+1) = 2$$
$$x+1 = 2^2$$
$$x = 3$$

   Check: If $x = 3$, then $\log_2[(3+1)^4] = 4\log_2(4) = 4 \cdot 2 = 8$, so 3 is a solution to the original equation.

Exercises 2–4 (6 minutes)

Students should work on these three exercises independently or in pairs to help develop fluency with these types of problems. Circulate around the room, and remind students to check for extraneous solutions as necessary.

Solve the equations in Exercises 2–4 for x.

2. $\ln((4x)^5) = 15$

$$5 \cdot \ln(4x) = 15$$
$$\ln(4x) = 3$$
$$e^3 = 4x$$
$$x = \frac{e^3}{4}$$

Check: Since $4\left(\frac{e^3}{4}\right) > 0$*, we know that* $\ln\left(\left(4 \cdot \frac{e^3}{5}\right)^5\right)$ *is defined. Thus,* $\frac{e^3}{4}$ *is the solution to the equation.*

3. $\log((2x+5)^2) = 4$

$$2 \cdot \log(2x+5) = 4$$
$$\log(2x+5) = 2$$
$$10^2 = 2x+5$$
$$100 = 2x+5$$
$$95 = 2x$$
$$x = \frac{95}{2}$$

Check: Since $2\left(\frac{95}{2}\right) + 5 \neq 0$*, we know that* $\log\left(\left(2 \cdot \frac{95}{2} + 5\right)^2\right)$ *is defined.*

Thus, $\frac{95}{2}$ *is the solution to the equation.*

4. $\log_2((5x+7)^{19}) = 57$

$$19 \cdot \log_2(5x+7) = 57$$
$$\log_2(5x+7) = 3$$
$$2^3 = 5x+7$$
$$8 = 5x+7$$
$$1 = 5x$$
$$x = \frac{1}{5}$$

Check: Since $5\left(\frac{1}{5}\right) + 7 > 0$*, we know that* $\log_2\left(5 \cdot \frac{1}{5} + 7\right)$ *is defined.*

Thus, $\frac{1}{5}$ *is the solution to this equation.*

Example 4 (4 minutes)

Students encountered the first extraneous solution in Example 3, but there were no extraneous solutions in Exercises 2–4. After working through Example 4, debrief students to informally assess their understanding, and provide guidance to align their understanding with the concepts. Remind students that they generally need to apply logarithmic properties before being able to solve a logarithmic equation. Some sample questions are included with likely student responses. Remember to have students check for extraneous solutions in all cases.

$$\log(x+10) - \log(x-1) = 2$$
$$\log\left(\frac{x+10}{x-1}\right) = 2$$
$$\frac{x+10}{x-1} = 10^2$$
$$x + 10 = 100(x-1)$$
$$99x = 110$$
$$x = \frac{10}{9}$$

- Is $\frac{10}{9}$ a valid solution? Explain how you know.
  - *Yes;* $\log\left(\frac{10}{9}+10\right)$ *and* $\log\left(\frac{10}{9}-1\right)$ *are both defined, so* $\frac{10}{9}$ *is a valid solution.*
- Why could we not rewrite the original equation in exponential form using the definition of the logarithm immediately?
  - *The equation needs to be in the form* $\log_b(Y) = L$ *before using the definition of a logarithm to rewrite it in exponential form, so we had to use the logarithmic properties to combine terms first.*

Example 5 (3 minutes)

Make sure students verify the solutions in Example 5 because there is an extraneous solution.

$$\log_2(x+1) + \log_2(x-1) = 3$$
$$\log_2\big((x+1)(x-1)\big) = 3$$
$$\log_2(x^2-1) = 3$$
$$2^3 = x^2 - 1$$
$$0 = x^2 - 9$$
$$0 = (x-3)(x+3)$$

Thus, $x = 3$ or $x = -3$. These solutions need to be checked to see if they are valid.

- Is 3 a valid solution?
  - $\log_2(3+1) + \log_2(3-1) = \log_2(4) + \log_2(2) = 2 + 1 = 3$*, so 3 is a valid solution.*

- Is -3 a valid solution?
  - *Because $-3+1=-2$, $\log_2(-3+1)=\log_2(-2)$ is undefined, so -3 not a valid solution. The value -3 is an extraneous solution, and this equation has only one solution: 3.*
- What should we look for when determining whether or not a solution to a logarithmic equation is extraneous?
  - *We cannot take the logarithm of a negative number or 0, so any solution that would result in the input to a logarithm being negative or 0 cannot be included in the solution set for the equation.*

Exercises 5–9 (8 minutes)

Have students work on these exercises individually to develop fluency with solving logarithmic equations. Circulate throughout the classroom to informally assess understanding and provide assistance as needed.

Solve the logarithmic equations in Exercises 5–9, and identify any extraneous solutions.

5. $\log(x^2+7x+12)-\log(x+4)=0$

$$\log\left(\frac{x^2+7x+12}{x+4}\right)=0$$
$$\frac{x^2+7x+12}{x+4}=10^0$$
$$\frac{x^2+7x+12}{x+4}=1$$
$$x^2+7x+12=x+4$$
$$0=x^2+6x+8$$
$$0=(x+4)(x+2)$$
$$x=-4 \text{ or } x=-2$$

Check: If $x=-4$, then $\log(x+4)=\log(0)$, which is undefined. Thus, -4 is an extraneous solution. Therefore, the only solution is -2.

6. $\log_2(3x)+\log_2(4)=4$

$$\log_2(3x)+2=4$$
$$\log_2(3x)=2$$
$$2^2=3x$$
$$4=3x$$
$$x=\frac{4}{3}$$

Check: Since $\frac{4}{3}>0$, $\log_2\left(3\cdot\frac{4}{3}\right)$ is defined.

Therefore, $\frac{4}{3}$ is a valid solution.

7. $2\ln(x+2) - \ln(-x) = 0$

$$\ln((x+2)^2) - \ln(-x) = 0$$
$$\ln\left(\frac{(x+2)^2}{-x}\right) = 0$$
$$1 = \frac{(x+2)^2}{-x}$$
$$-x = x^2 + 4x + 4$$
$$0 = x^2 + 5x + 4$$
$$0 = (x+4)(x+1)$$
$$x = -4 \text{ or } x = -1$$

Check: Thus, we get $x = -4$ or $x = -1$ as solutions to the quadratic equation. However, if $x = -4$, then $\ln(x+2) = \ln(-2)$, so -4 is an extraneous solution. Therefore, the only solution is -1.

8. $\log(x) = 2 - \log(x)$

$$\log(x) + \log(x) = 2$$
$$2 \cdot \log(x) = 2$$
$$\log(x) = 1$$
$$x = 10$$

Check: Since $10 > 0$, $\log(10)$ is defined.

Therefore, 10 is a valid solution to this equation.

9. $\ln(x+2) = \ln(12) - \ln(x+3)$

$$\ln(x+2) + \ln(x+3) = \ln(12)$$
$$\ln\big((x+2)(x+3)\big) = \ln(12)$$
$$(x+2)(x+3) = 12$$
$$x^2 + 5x + 6 = 12$$
$$x^2 + 5x - 6 = 0$$
$$(x-1)(x+6) = 0$$
$$x = 1 \text{ or } x = -6$$

Check: If $= -6$, then the expressions $\ln(x+2)$ and $\ln(x+3)$ are undefined.
Therefore, the only valid solution to the original equation is 1.

Closing (3 minutes)

Have students summarize the process they use to solve logarithmic equations in writing. Circulate around the classroom to informally assess student understanding.

- *If an equation can be rewritten in the form $\log_b(Y) = L$ for an expression Y and a number L, then apply the definition of the logarithm to rewrite as $b^L = Y$. Solve the resulting exponential equation, and check for extraneous solutions.*
- *If an equation can be rewritten in the form $\log_b(Y) = \log_b(Z)$ for expressions Y and Z, then the fact that the logarithmic functions are one-to-one gives $Y = Z$. Solve this resulting equation, and check for extraneous solutions.*

Exit Ticket (4 minutes)

Name ______________________________ Date ______________

Lesson 14: Solving Logarithmic Equations

Exit Ticket

Find all solutions to the following equations. Remember to check for extraneous solutions.

1. $\log_2(3x + 7) = 4$

2. $\log(x - 1) + \log(x - 4) = 1$

Exit Ticket Sample Solutions

Find all solutions to the following equations. Remember to check for extraneous solutions.

1. $\log_2(3x+7) = 4$

$$\begin{aligned} \log_2(3x+7) &= 4 \\ 3x+7 &= 2^4 \\ 3x &= 16-7 \\ x &= 3 \end{aligned}$$

Since $3(3)+7 > 0$, we know 3 is a valid solution to the equation.

2. $\log(x-1) + \log(x-4) = 1$

$$\begin{aligned} \log\big((x-1)(x-4)\big) &= 1 \\ \log(x^2-5x+4) &= 1 \\ x^2-5x+4 &= 10 \\ x^2-5x-6 &= 0 \\ (x-6)(x+1) &= 0 \\ x &= 6 \text{ or } x = -1 \end{aligned}$$

Check: Since the left side of the equation is not defined for $x = -1$, this is an extraneous solution.

Therefore, the only valid solution is 6.

Problem Set Sample Solutions

1. Solve the following logarithmic equations.

a. $\log(x) = \frac{5}{2}$

$$\begin{aligned} \log(x) &= \frac{5}{2} \\ x &= 10^{\frac{5}{2}} \\ x &= 100\sqrt{10} \end{aligned}$$

Check: Since $100\sqrt{10} > 0$, we know $\log\big(100\sqrt{10}\big)$ is defined.

Therefore, the solution to this equation is $100\sqrt{10}$.

b. $5\log(x+4) = 10$

$$\begin{aligned} \log(x+4) &= 2 \\ x+4 &= 10^2 \\ x+4 &= 100 \\ x &= 96 \end{aligned}$$

Check: Since $96+4 > 0$, we know $\log(96+4)$ is defined.

Therefore, the solution to this equation is 96.

c. $\log_2(1-x)=4$

$$1-x=2^4$$
$$x=-15$$

Check: Since $1-(-15)>0$*, we know* $\log_2(1-(-15))$ *is defined.*

Therefore, the solution to this equation is -15*.*

d. $\log_2(49x^2)=4$

$$\log_2[(7x)^2]=4$$
$$2\cdot\log_2(7x)=4$$
$$\log_2(7x)=2$$
$$7x=2^2$$
$$x=\frac{4}{7}$$

Check: Since $49\left(\frac{4}{7}\right)^2>0$*, we know* $\log_2\left(49\left(\frac{4}{7}\right)^2\right)$ *is defined.*

Therefore, the solution to this equation is $\frac{4}{7}$*.*

e. $\log_2(9x^2+30x+25)=8$

$$\log_2[(3x+5)^2]=8$$
$$2\cdot\log_2(3x+5)=8$$
$$\log_2(3x+5)=4$$
$$3x+5=2^4$$
$$3x+5=16$$
$$3x=11$$
$$x=\frac{11}{3}$$

Check: Since $9\left(\frac{11}{3}\right)^2+30\left(\frac{11}{3}\right)+25=256$*, and* $256>0$*,* $\log_2\left(9\left(\frac{11}{3}\right)^2+30\left(\frac{11}{3}\right)+25\right)$ *is defined.*

Therefore, the solution to this equation is $\frac{11}{3}$*.*

2. Solve the following logarithmic equations.

a. $\ln(x^6)=36$

$$6\cdot\ln(x)=36$$
$$\ln(x)=6$$
$$x=e^6$$

Check: Since $e^6>0$*, we know* $\ln((e^6)^6)$ *is defined.*

Therefore, the only solution to this equation is e^6*.*

b. $\log[(2x^2 + 45x - 25)^5] = 10$

$$5 \cdot \log(2x^2 + 45x - 25) = 10$$
$$\log(2x^2 + 45x - 25) = 2$$
$$2x^2 + 45x - 25 = 10^2$$
$$2x^2 + 45x - 125 = 0$$
$$2x^2 + 50x - 5x - 125 = 0$$
$$2x(x + 25) - 5(x + 25) = 0$$
$$(2x - 5)(x + 25) = 0$$

Check: Since $2x^2 + 45x - 25 > 0$ for both $x = -25$ and $x = \frac{5}{2}$, we know the left side of the original equation is defined at these values.

Therefore, the two solutions to this equation are -25 and $\frac{5}{2}$.

c. $\log[(x^2 + 2x - 3)^4] = 0$

$$4\log(x^2 + 2x - 3) = 0$$
$$\log(x^2 + 2x - 3) = 0$$
$$x^2 + 2x - 3 = 10^0$$
$$x^2 + 2x - 3 = 1$$
$$x^2 + 2x - 4 = 0$$

$$x = \frac{-2 \pm \sqrt{4 + 16}}{2}$$
$$= -1 \pm \sqrt{5}$$

Check: Since $x^2 + 2x - 3 = 1$ when $x = -1 + \sqrt{5}$ or $x = -1 - \sqrt{5}$, we know the logarithm is defined for these values of x.

Therefore, the two solutions to the equation are $-1 + \sqrt{5}$ and $-1 - \sqrt{5}$.

3. Solve the following logarithmic equations.

a. $\log(x) + \log(x - 1) = \log(3x + 12)$

$$\log(x) + \log(x - 1) = \log(3x + 12)$$
$$\log\big(x(x - 1)\big) = \log(3x + 12)$$
$$x(x - 1) = 3x + 12$$
$$x^2 - 4x - 12 = 0$$
$$(x + 2)(x - 6) = 0$$

Check: Since $\log(-2)$ is undefined, -2 is an extraneous solution.

Therefore, the only solution to this equation is 6.

b. $\ln(32x^2) - 3\ln(2) = 3$

$$\ln(32x^2) - \ln(2^3) = 3$$
$$\ln\left(\frac{32x^2}{8}\right) = 3$$
$$4x^2 = e^3$$
$$x^2 = \frac{e^3}{4}$$
$$x = \frac{\sqrt{e^3}}{2} \text{ or } x = -\frac{\sqrt{e^3}}{2}$$

Check: Since the value of x in the logarithmic expression is squared, $\ln(32x^2)$ is defined for any nonzero value of x.

Therefore, both $\frac{\sqrt{e^3}}{2}$ and $-\frac{\sqrt{e^3}}{2}$ are valid solutions to this equation.

c. $\log(x) + \log(-x) = 0$

$$\log(x(-x)) = 0$$
$$\log(-x^2) = 0$$
$$-x^2 = 10^0$$
$$x^2 = -1$$

Since there is no real number x so that $x^2 = -1$, there is no solution to this equation.

d. $\log(x+3) + \log(x+5) = 2$

$$\log((x+3)(x+5)) = 2$$
$$(x+3)(x+5) = 10^2$$
$$x^2 + 8x + 15 - 100 = 0$$
$$x^2 + 8x - 85 = 0$$

$$x = \frac{-8 \pm \sqrt{64+340}}{2}$$
$$= -4 \pm \sqrt{101}$$

Check: The left side of the equation is not defined for $x = -4 - \sqrt{101}$, but it is for $x = -4 + \sqrt{101}$.

Therefore, the only solution to this equation is $x = -4 + \sqrt{101}$.

e. $\log(10x+5) - 3 = \log(x-5)$

$$\log(10x+5) - \log(x-5) = 3$$
$$\log\left(\frac{10x+5}{x-5}\right) = 3$$
$$\frac{10x+5}{x-5} = 10^3$$
$$\frac{10x+5}{x-5} = 1000$$
$$10x+5 = 1000x - 5000$$
$$5005 = 990x$$
$$x = \frac{91}{18}$$

Check: Both sides of the equation are defined for $x = \frac{91}{18}$.

Therefore, the solution to this equation is $\frac{91}{18}$.

f. $\log_2(x) + \log_2(2x) + \log_2(3x) + \log_2(36) = 6$

$$\log_2(x \cdot 2x \cdot 3x \cdot 36) = 6$$
$$\log_2(6^3x^3) = 6$$
$$\log_2[(6x)^3] = 6$$
$$3 \cdot \log_2(6x) = 6$$
$$\log_2(6x) = 2$$
$$6x = 2^2$$
$$x = \frac{2}{3}$$

Check: Since $\frac{2}{3} > 0$*, all logarithmic expressions in this equation are defined for* $x = \frac{2}{3}$.

Therefore, the solution to this equation is $\frac{2}{3}$.

4. **Solve the following equations.**

a. $\log_2(x) = 4$

16

b. $\log_6(x) = 1$

6

c. $\log_3(x) = -4$

$\frac{1}{81}$

d. $\log_{\sqrt{2}}(x) = 4$

4

e. $\log_{\sqrt{5}}(x) = 3$

$5\sqrt{5}$

f. $\log_3(x^2) = 4$

$9, -9$

g. $\log_2(x^{-3}) = 12$

$\frac{1}{16}$

h. $\log_3(8x + 9) = 4$

9

i. $2 = \log_4(3x - 2)$

6

j. $\log_5(3 - 2x) = 0$

1

k. $\ln(2x) = 3$

$\frac{e^3}{2}$

l. $\log_3(x^2 - 3x + 5) = 2$

$4, -1$

m. $\log((x^2 + 4)^5) = 10$

$4\sqrt{6}, -4\sqrt{6}$

n. $\log(x) + \log(x + 21) = 2$

4

o. $\log_4(x - 2) + \log_4(2x) = 2$

4

p. $\log(x) - \log(x + 3) = -1$

$\frac{1}{3}$

q. $\log_4(x + 3) - \log_4(x - 5) = 2$

$\frac{83}{15}$

r. $\log(x) + 1 = \log(\mathrm{x} + 9)$

1

s. $\log_3(x^2 - 9) - \log_3(x + 3) = 1$

6

t. $1 - \log_8(x - 3) = \log_8(2x)$

4

u. $\log_2(x^2 - 16) - \log_2(x - 4) = 1$

No solution

v. $\log\left(\sqrt{(x + 3)^3}\right) = \frac{3}{2}$

7

w. $\ln(4x^2 - 1) = 0$

$\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}$

x. $\ln(x + 1) - \ln(2) = 1$

$2e - 1$

Lesson 15: Why Were Logarithms Developed?

Student Outcomes

- Students use logarithm tables to calculate products and quotients of multi-digit numbers without technology.
- Students understand that logarithms were developed to speed up arithmetic calculations by reducing multiplication and division to the simpler operations of addition and subtraction.
- Students solve logarithmic equations of the form $\log(X) = \log(Y)$ by equating X and Y.

Lesson Notes

This final lesson in Topic B includes two procedures that appear to be different but are closely related mathematically. First, students work with logarithm tables to see how applying logarithms simplified calculations in the days before computing machines and electronic technology. They also learn a bit of the history of how and why logarithms first appeared—a history often obscured when logarithmic functions are introduced as inverses of exponential functions. The last two pages of this document contain a base 10 table of logarithms that can be copied and distributed; such tables are also available on the Internet.

Second, students learn to solve the final type of logarithmic equation, $\log(X) = \log(Y)$, where X and Y are either real numbers or expressions that take on positive real values (**A.SSE.A.2**, **F-LE.A.4**). Using either technique requires knowing that the logarithm is a one-to-one function; that is, if $\log(X) = \log(Y)$, then $X = Y$. Students do not yet have the vocabulary to be told this directly, but it is stated as a fact in this lesson, and they further explore the idea of one-to-one functions in Precalculus and Advanced Topics. As with Lessons 10 and 12, this lesson involves only base-10 logarithms, but the Problem Set does require that students do some work with logarithms base e and base 2. Remind students to check for extraneous solutions when solving logarithmic equations.

Classwork

Discussion (4 minutes): How to Read a Table of Logarithms

- For this lesson, we will pretend that we live in the time when logarithms were discovered, before there were calculators or computing machines. In this time, scientists, merchants, and sailors needed to make calculations for both astronomical observation and navigation. Logarithms made these calculations much easier, faster, and more accurate than calculation with large numbers. In fact, noted mathematician Pierre-Simon LaPlace (France, circa 1800) said that "[logarithms are an] admirable artifice which, by reducing to a few days the labour of many months, doubles the life of the astronomer, and spares him the errors and disgust inseparable from long calculations."
- A typical table of common logarithms, like the table at the end of this document, has many rows of numbers arranged in ten columns. The numbers in the table are decimals. In our table, they are given to four decimal places, and there are 90 rows of them (some tables of logarithms have 900 rows). Down the left-hand side of the table are the numbers from 1.0 to 9.9. Across the top of the table are the numbers from 0 to 9. To read the table, you locate the number whose logarithm you want using the numbers down the left of the table followed by the numbers across the top.

MP.5 & MP.6

- What does the number in the third row and second column represent (the entry for 1.21)?
  - *The logarithm of* 1.21*, which is approximately* 0.0828
- The logarithm of numbers larger than 9.9 and smaller than 1.0 can also be found using this table. Suppose you want to find $\log(365)$. Is there any way we can rewrite 365 to include a number between 1.0 and 9.9?
  - *Rewrite* 365 *in scientific notation:* 3.65×10^2.
- Can we simplify $\log(3.65 \times 10^2)$?
  - *We can apply the formula for the logarithm of a product. Then, we have* $\log(10^2) + \log(3.65) = 2 + \log(3.65)$.
- Now, all that is left is to find the value of $\log(3.65)$ using the table. What is the value of $\log(365)$?
  - *The table entry is* 0.5623*. That means* $\log(365) \approx 2 + 0.5623$*, so* $\log(365) \approx 2.5623$.
- How would you find $\log(0.365)$?
  - *In scientific notation,* $0.365 = 3.65 \times 10^{-1}$*. So, once again, you would find the row for* 3.6 *and the column for* 5*, and you would again find the number* 0.5623*. But this time, you would have* $\log(0.365) \approx -1 + 0.5623$*, so* $\log(0.365) \approx 0.4377$.

Example 1 (7 minutes)

Students multiply multi-digit numbers without technology and then use a table of logarithms to find the same product using logarithms.

- Find the product 3.42×2.47 without using a calculator.
  - *Using paper and pencil, and without any rounding, students should get* 8.4474*. The point is to show how much time the multiplication of multi-digit numbers can take.*
- How could we use logarithms to find this product?
  - *If we take the logarithm of the product, we can rewrite the product as a sum of logarithms.*
- Rewrite the logarithm of the product as the sum of logarithms.
  - $\log(3.42 \times 2.47) = \log(3.42) + \log(2.47)$
- Use the table of logarithms to look up the values of $\log(3.42)$ and $\log(2.47)$.
  - *According to the table,* $\log(3.42) \approx 0.5340$, and $\log(2.47) \approx 0.3927$.
- Approximate the logarithm $\log(3.42 \times 2.47)$.
  - *The approximate sum is*

$$\begin{aligned}\log(3.42 \times 2.47) &\approx 0.5340 + 0.3927\\ &\approx 0.9267.\end{aligned}$$

- What if there is more than one number that has a logarithm of 0.9267? Suppose that there are two numbers X and Y that satisfy $\log(X) = 0.9267$ and $\log(Y) = 0.9267$. Then $10^{0.9267} = X$, and $10^{0.9267} = Y$, so that $X = Y$. This means that there is only one number that has the logarithm 0.9267. So, what is that number?

Scaffolding:

- Students may need to be reminded that if the logarithm is greater than 1, a power of 10 greater than 1 is involved, and only the decimal part of the number is found in the table.
- Struggling students should attempt a simpler product such as 1.20×6.00 to illustrate the process.
- Advanced students may use larger or more precise numbers as a challenge. To multiply a product such as 34.293×107.9821, students have to employ scientific notation and the property for the logarithm of a product.

- Can we find the exact number that has logarithm 0.9267 using the table?
  - *The table says that* $\log(8.44) \approx 0.9263$, *and* $\log(8.45) \approx 0.9269$.
- Which is closer?
  - $\log(8.45) \approx 0.9267$
- Since $\log(3.42 \times 2.47) \approx \log(8.45)$, what can we conclude is an approximate value for 3.42×2.47?
  - *Since* $\log(3.42 \times 2.47) \approx \log(8.45)$, *we know that* $3.42 \times 2.47 \approx 8.45$.
- Does this agree with the product you found when you did the calculation by hand?
  - *Yes. By hand, we found that the product is* 8.4474, *which is approximately* 8.45.

Discussion (2 minutes)

In Example 1, we showed that there was only one number that had logarithm 0.9267. This result generalizes to any number and any base of the logarithm: If $\log_b(X) = \log_b(Y)$, then $X = Y$. We need to know this property both to use a logarithm table to look up values that produce a certain logarithmic value and to solve logarithmic equations later in the lesson.

> If X and Y are positive real numbers, or expressions that take on the value of positive real numbers, and $\log_b(X) = \log_b(Y)$, then $X = Y$.

Example 2 (4 minutes)

This example is a continuation of the first example, with the addition of scientific notation to further explain the power of logarithms. Because much of the reasoning was explained in Example 1, this should take much less time to work through.

- Now, what if we needed to calculate $(3.42 \times 10^{14}) \times (5.76 \times 10^{12})$?
- Take the logarithm of this product, and find its approximate value using the logarithm table.
  - $$\begin{aligned}\log\big((3.42 \times 10^{14}) \times (5.76 \times 10^{12})\big) &= \log(3.42) + \log(10^{14}) + \log(5.76) + \log(10^{12})\\ &= \log(3.42) + \log(5.76) + 14 + 12\\ &\approx 0.5340 + 0.7604 + 26\\ &\approx 27.2944\end{aligned}$$
- Look up 0.2944 in the logarithm table.
  - *Since* $\log(1.97) \approx 0.2945$, *we can say that* $0.2944 \approx \log(1.97)$.
- How does that tell us which number has a logarithm approximately equal to 27.2944?
  - $\log(1.97 \times 10^{27}) = 27 + \log(1.97)$, *so* $\log(1.97 \times 10^{27}) \approx 27.2944$.
- Finally, what is an approximate value of the product $(3.42 \times 10^{14}) \times (5.76 \times 10^{12})$?
  - $(3.42 \times 10^{14}) \times (5.76 \times 10^{12}) \approx 1.97 \times 10^{27}$

Example 3 (6 minutes)

- According to one estimate, the mass of the earth is roughly 5.28×10^{24} kg, and the mass of the moon is about 7.35×10^{22} kg. Without using a calculator but using the table of logarithms, find how many times greater the mass of Earth is than the mass of the moon.
  - *Let R be the ratio of the two masses. Then $R = \frac{5.28 \times 10^{24}}{7.35 \times 10^{22}} = \frac{5.28}{7.35} \cdot 10^2$.*

    Taking the logarithm of each side,

$$\begin{aligned}\log(R) &= \log\left(\frac{5.28}{7.35} \cdot 10^2\right)\\ &= 2 + \log\left(\frac{5.28}{7.35}\right)\\ &= 2 + \log(5.28) - \log(7.35)\\ &\approx 2 + 0.7226 - 0.8663\\ &\approx 1.8563.\end{aligned}$$

- Find 0.8563 in the table entries to estimate R.
  - *In the table, 0.8563 is closest to $\log(7.18)$.*

    So, $\log(71.8) \approx 1.8563$, and therefore, the mass of Earth is approximately 71.8 times that of the moon.
- Logarithms turn out to be very useful in dealing with especially large or especially small numbers. How does it help to have those numbers expressed in scientific notation if we are going to use a logarithm table to perform multiplication or division?
  - *Again, answers will differ, but students should at least recognize that scientific notation is helpful in working with very large or very small numbers. Using scientific notation, we can express each number as the product of a number between 1 and 10, and a power of 10. Taking the logarithm of the number allows us to use properties of logarithms base 10 to handle more easily any number n where $0 < n < 1$ or $n \geq 10$. The logarithm of the number between 1 and 10 can be read from the table, and the exponent of the power of 10 can then be added to it.*
- Whenever we have a number of the form $k \times 10^n$ where n is an integer and k is a number between 1.0 and 9.9, the logarithm of this number will always be $n + \log(k)$ and can be evaluated using a table of logarithms like the one included in this lesson.

Discussion (4 minutes)

Logarithms were devised by the Scottish mathematician John Napier (1550–1617) with the help of the English mathematician Henry Briggs (1561–1630) to simplify arithmetic computations with multi-digit numbers by turning multiplication and division into addition and subtraction. The basic idea is that while a sequence of powers like $2^0, 2^1, 2^2, 2^3, 2^4, 2^5, \ldots$ is increasing multiplicatively, the sequence of its exponents is increasing additively. If numbers can be represented as the powers of a base, they can be multiplied by adding their exponents and divided by subtracting their exponents. Napier and Briggs published the first tables of what came to be called base-10 or common logarithms.

- It was Briggs's idea to base the logarithms on the number 10. Why do you think he made that choice?
  - *The number* 10 *is the base of our number system. So, taking* 10 *as the base of common logarithms makes hand calculations with logarithms easier. It is really the same argument that makes scientific notation helpful: Powers of* 10 *are easy to use in calculations.*

Exercises (12 minutes)

Now that students know that if two logarithmic expressions with the same base are equal, then the arguments inside of the logarithms are equal, and students can solve a wider variety of logarithmic equations without invoking the definition each time. Due to the many logarithmic properties that students now know, there are multiple approaches to solving these equations. Discuss different approaches with students and their responses to Exercise 2.

Exercises

1. Solve the following equations. Remember to check for extraneous solutions because logarithms are only defined for positive real numbers.

a. $\log(x^2) = \log(49)$

$x^2 = 49$
$x = 7$ *or* $x = -7$

Check: Both solutions are valid since 7^2 *and* $(-7)^2$ *are both positive numbers.*

The two solutions are 7 *and* -7.

b. $\log(x+1) + \log(x+2) = \log(7x-17)$

$\log\big((x+1)(x-2)\big) = \log(7x-17)$
$(x+1)(x-2) = 7x-17$
$x^2 - x - 2 = 7x - 17$
$x^2 - 8x + 15 = 0$
$(x-5)(x-3) = 0$
$x = 3$ *or* $x = 5$

Check: Since $x+1$, $x-2$, *and* $7x-17$ *are all positive for either* $x = 3$ *or* $x = 5$, *both solutions are valid.*

Thus, the solutions to this equation are 3 *and* 5.

c. $\log(x^2+1) = \log\big(x(x-2)\big)$

$x^2 + 1 = x(x-2)$
$= x^2 - 2x$
$1 = -2x$
$x = -\frac{1}{2}$

Check: Both $\left(-\frac{1}{2}\right)^2 + 1 > 0$ *and* $-\frac{1}{2}\left(-\frac{1}{2} - 2\right) > 0$, *so the solution* $-\frac{1}{2}$ *is valid.*

Thus, $-\frac{1}{2}$ *is the only valid solution to this equation.*

Scaffolding:

If the class seems to be struggling with the process to solve logarithmic equations, then either encourage them to create a graphic organizer that summarizes the types of problems and approaches that they should use in each case, or hang one on the board for reference. A sample graphic organizer is included.

Rewrite Problem in the Form …	
$\log_b(Y) = L$	$\log_b(Y) = \log_b(Z)$
Then …	
$b^L = Y$	$Y = Z$

d. $\log(x+4)+\log(x-1)=\log(3x)$

$$\log\big((x+4)(x-1)\big)=\log(3x)$$
$$(x+4)(x-1)=3x$$
$$x^2+3x-4=3x$$
$$x^2-4=0$$
$$x=2 \text{ or } x=-2$$

Check: Since $\log(3x)$ is undefined when $x=-2$, there is an extraneous solution of $x=-2$.

The only valid solution to this equation is 2.

e. $\log(x^2-x)-\log(x-2)=\log(x-3)$

$$\log(x-2)+\log(x-3)=\log(x^2-x)$$
$$\log\big((x-2)(x-3)\big)=\log(x^2-x)$$
$$(x-2)(x-3)=x^2-x$$
$$x^2-5x+6=x^2-x$$
$$4x=6$$
$$x=\frac{3}{2}$$

Check: When $x=\frac{3}{2}$, we have $x-2<0$, so $\log(x-2)$, $\log(x-3)$, and $\log(x^2-x)$ are all undefined. So, the solution $x=\frac{3}{2}$ is extraneous.

There are no valid solutions to this equation.

f. $\log(x)+\log(x-1)+\log(x+1)=3\log(x)$

$$\log\big(x(x-1)(x+1)\big)=\log(x^3)$$
$$\log(x^3-x)=\log(x^3)$$
$$x^3-x=x^3$$
$$x=0$$

Since $\log(0)$ is undefined, $x=0$ is an extraneous solution.

There are no valid solutions to this equation.

g. $\log(x-4)=-\log(x-2)$

Two possible approaches to solving this equation are shown.

$$\log(x-4)=\log\left(\frac{1}{x-2}\right)$$
$$x-4=\frac{1}{x-2}$$
$$(x-4)(x-2)=1$$
$$x^2-6x+8=1$$
$$x^2-6x+7=0$$
$$x=3\pm\sqrt{2}$$

$$\log(x-4)+\log(x-2)=0$$
$$\log\big((x-4)(x-2)\big)=\log(1)$$
$$(x-4)(x-2)=1$$
$$x^2-6x+8=1$$
$$x^2-6x+7=0$$
$$x=3\pm\sqrt{2}$$

Check: If $x=3-\sqrt{2}$, then $x<2$, so $\log(x-2)$ is undefined. Thus, $3-\sqrt{2}$ is an extraneous solution.

The only valid solution to this equation is $3+\sqrt{2}$.

2. **How do you know if you need to use the definition of logarithm to solve an equation involving logarithms as we did in Lesson 15 or if you can use the methods of this lesson?**

   If the equation involves only logarithmic expressions, then it can be reorganized to be of the form $\log(X) = \log(Y)$ and then solved by equating $X = Y$. If there are constants involved, then the equation can be solved using the definition and properties of logarithms.

Closing (2 minutes)

Ask students the following questions, and after coming to a consensus, have students record the answers in their notebooks:

- How do we use a table of logarithms to compute a product of two numbers x and y?
  - *We look up approximations to $\log(x)$ and $\log(y)$ in the table, add those logarithms, and then look up the sum in the table to extract the approximate product.*
- Does this process provide an exact answer? Explain how you know.
  - *It is only an approximation because the table only allows us to look up x to two decimal places and $\log(x)$ to four decimal places.*
- How do we solve an equation in which every term contains a logarithm?
  - *We rearrange the terms to get an equation of the form $\log(X) = \log(Y)$, then equate X and Y, and solve from there.*
- How does that differ from solving an equation that contains constant terms?
  - *If an equation has constant terms, then we rearrange the equation to the form $\log(X) = c$, apply the definition of the logarithm, and solve from there.*

Lesson Summary

A table of base-10 logarithms can be used to simplify multiplication of multi-digit numbers:

1. **To compute $A \cdot B$ for positive real numbers A and B, look up $\log(A)$ and $\log(B)$ in the logarithm table.**
2. **Add $\log(A)$ and $\log(B)$. The sum can be written as $k + d$, where k is an integer and $0 \leq d < 1$ is the decimal part.**
3. **Look back at the table, and find the entry closest to the decimal part, d.**
4. **The product of that entry and 10^k is an approximation to $A \cdot B$.**

A similar process simplifies division of multi-digit numbers:

1. **To compute $\frac{A}{B}$ for positive real numbers A and B, look up $\log(A)$ and $\log(B)$ in the logarithm table.**
2. **Calculate $\log(A) - \log(B)$. The difference can be written as $k + d$, where k is an integer and $0 \leq d < 1$ is the decimal part.**
3. **Look back at the table to find the entry closest to the decimal part, d.**
4. **The product of that entry and 10^k is an approximation to $\frac{A}{B}$.**

For any positive values X and Y, if $\log_b(X) = \log_b(Y)$, we can conclude that $X = Y$. This property is the essence of how a logarithm table works, and it allows us to solve equations with logarithmic expressions on both sides of the equation.

Exit Ticket (4 minutes)

Name ____ Date ____________________

Lesson 15: Why Were Logarithms Developed?

Exit Ticket

The surface area of Jupiter is $6.14 \times 10^{10}\ \mathrm{km}^2$, and the surface area of Earth is $5.10 \times 10^{8}\ \mathrm{km}^2$. Without using a calculator but using the table of logarithms, find how many times greater the surface area of Jupiter is than the surface area of Earth.

Exit Ticket Sample Solutions

The surface area of Jupiter is $6.14 \times 10^{10}\ \text{km}^2$, and the surface area of Earth is $5.10 \times 10^{8}\ \text{km}^2$. Without using a calculator but using the table of logarithms, find how many times greater the surface area of Jupiter is than the surface area of Earth.

Let R be the ratio of the two surface areas. Then $R = \frac{6.14 \times 10^{10}}{5.10 \times 10^{8}} = \frac{6.14}{5.10} \cdot 10^2$.

Taking the logarithm of each side,

$$\begin{aligned} \log(R) &= \log\left(\frac{6.14}{5.10} \cdot 10^2\right) \\ &= 2 + \log\left(\frac{6.14}{5.10}\right) \\ &= 2 + \log(6.14) - \log(5.10) \\ &\approx 2 + 0.7882 - 0.7076 \\ &\approx 2.0806. \end{aligned}$$

Find 0.0806 in the table entries to estimate R.

Look up 0.0806, which is closest to $\log(1.20)$. Note that $2 + 0.0806 \approx \log(100) + \log(1.20)$, so

$\log(120) \approx 2.0806$. Therefore, the surface area of Jupiter is approximately 120 times that of Earth.

Problem Set Sample Solutions

These problems give students additional practice using base-10 logarithms to perform arithmetic calculations and solve equations.

1. **Use the table of logarithms to approximate solutions to the following logarithmic equations:**

   a. $\log(x) = 0.5044$

   In the table, 0.5044 is closest to $\log(3.19)$, so $\log(x) \approx \log(3.19)$.
   Therefore, $x \approx 3.19$.

   b. $\log(x) = -0.5044$ **(Hint: Begin by writing -0.5044 as $[(-0.5044) + 1] - 1$.)**

   $$\begin{aligned} \log(x) &= [(-0.5044) + 1] - 1 \\ &= 0.4956 - 1 \end{aligned}$$

   In the table, 0.4956 is closest to $\log(3.13)$, so

   $$\begin{aligned} \log(x) &\approx \log(3.13) - 1 \\ &\approx \log(3.13) - \log(10) \\ &\approx \log\left(\frac{3.13}{10}\right) \\ &\approx \log(0.313). \end{aligned}$$

   Therefore, $x \approx 0.313$.

   Alternatively, -0.5044 is the opposite of 0.5044, so x is the reciprocal of the answer in part (a). Thus, $x \approx 3.19^{-1} \approx 0.313$.

c. $\log(x) = 35.5044$

$$\begin{aligned}\log(x) &= 35 + 0.5044\\ &= \log(10^{35}) + 0.5044\\ &\approx \log(10^{35}) + \log(3.19)\\ &\approx \log(3.19 \times 10^{35})\end{aligned}$$

Therefore, $x \approx 3.19 \times 10^{35}$.

d. $\log(x) = 4.9201$

$$\begin{aligned}\log(x) &= 4 + 0.9201\\ &= \log(10^{4}) + 0.9201\\ &\approx \log(10^{4}) + \log(8.32)\\ &\approx \log(8.32 \times 10^{4})\end{aligned}$$

Therefore, $x \approx 83200$.

2. Use logarithms and the logarithm table to evaluate each expression.

a. $\sqrt{2.33}$

$$\log\left((2.33)^{\frac{1}{2}}\right) = \frac{1}{2}\log(2.33) \approx \frac{1}{2}(0.3674) \approx 0.1837$$

Thus, $\log(\sqrt{2.33}) \approx 0.1837$*, and locating* 0.1837 *in the logarithm table gives a value of approximately* 1.53*. Therefore,* $\sqrt{2.33} \approx 1.53$.

b. $13500 \cdot 3600$

$$\begin{aligned}\log(1.35 \cdot 10^4 \cdot 3.6 \cdot 10^3) &= \log(1.35) + \log(3.6) + 4 + 3\\ &\approx 0.1303 + 0.5563 + 7\\ &\approx 7.6866\end{aligned}$$

Thus, $\log(13500 \cdot 3600) \approx 7.6866$*. Locating* 0.6866 *in the logarithm table gives a value of* 4.86*. Therefore, the product is approximately* 4.86×10^7.

c. $\dfrac{7.2\times 10^9}{1.3\times 10^5}$

$$\log(7.2) + \log(10^9) - \log(1.3) - \log(10^5) \approx 0.8573 + 9 - 0.1139 - 5 \approx 4.7434$$

Locating 0.7434 *in the logarithm table gives* 5.54*. So, the quotient is approximately* 5.54×10^4.

3. Solve for x: $\log(3) + 2\log(x) = \log(27)$.

$$\begin{aligned}\log(3) + 2\log(x) &= \log(27)\\ \log(3) + \log(x^2) &= \log(27)\\ \log(3x^2) &= \log(27)\\ 3x^2 &= 27\\ x^2 &= 9\\ x &= \pm 3\end{aligned}$$

Because $\log(x)$ *is only defined for positive real numbers* x*, the only solution to the equation is* 3.

4. Solve for x: $\log(3x) + \log(x+4) = \log(15)$.

$$\begin{aligned} \log(3x^2 + 12x) &= \log(15) \\ 3x^2 + 12x &= 15 \\ 3x^2 + 12x - 15 &= 0 \\ 3x^2 + 15x - 3x - 15 &= 0 \\ 3x(x+5) - 3(x+5) &= 0 \\ (3x-3)(x+5) &= 0 \end{aligned}$$

Thus, 1 and -5 solve the quadratic equation, but -5 is an extraneous solution to the logarithmic equation. Hence, 1 is the only solution.

5. Solve for x.

a. $\log(x) = \log(y+z) + \log(y-z)$

$$\begin{aligned} \log(x) &= \log(y+z) + \log(y-z) \\ \log(x) &= \log\big((y+z)(y-z)\big) \\ \log(x) &= \log(y^2 - z^2) \\ x &= y^2 - z^2 \end{aligned}$$

b. $\log(x) = \big(\log(y) + \log(z)\big) + \big(\log(y) - \log(z)\big)$

$$\begin{aligned} \log(x) &= \big(\log(y) + \log(z)\big) + \big(\log(y) - \log(z)\big) \\ \log(x) &= 2\log(y) \\ \log(x) &= \log(y^2) \\ x &= y^2 \end{aligned}$$

6. If x and y are positive real numbers, and $\log(y) = 1 + \log(x)$, express y in terms of x.

Since $\log(10x) = 1 + \log(x)$, we see that $\log(y) = \log(10x)$. Then $y = 10x$.

7. If x, y, and z are positive real numbers, and $\log(x) - \log(y) = \log(y) - \log(z)$, express y in terms of x and z.

$$\begin{aligned} \log(x) - \log(y) &= \log(y) - \log(z) \\ \log(x) + \log(z) &= 2\log(y) \\ \log(xz) &= \log(y^2) \\ xz &= y^2 \\ y &= \sqrt{xz} \end{aligned}$$

8. If x and y are positive real numbers, and $\log(x) = y\big(\log(y+1) - \log(y)\big)$, express x in terms of y.

$$\begin{aligned} \log(x) &= y\big(\log(y+1) - \log(y)\big) \\ \log(x) &= y\left(\log\left(\frac{y+1}{y}\right)\right) \\ \log(x) &= \log\left(\left(\frac{y+1}{y}\right)^y\right) \\ x &= \left(\frac{y+1}{y}\right)^y \end{aligned}$$

9. If x and y are positive real numbers, and $\log(y) = 3 + 2\log(x)$, express y in terms of x.

Since $\log(1000x^2) = 3 + \log(x^2) = 3 + 2\log(x)$, we see that $\log(y) = \log(1000x^2)$. Thus, $y = 1000x^2$.

10. If x, y, and z are positive real numbers, and $\log(z) = \log(y) + 2\log(x) - 1$, express z in terms of x and y.

Since $\log\left(\frac{x^2y}{10}\right) = \log(y) + 2\log(x) - 1$, we see that $\log(z) = \log\left(\frac{x^2y}{10}\right)$. Thus, $z = \frac{x^2y}{10}$.

11. Solve the following equations.

a. $\ln(10) - \ln(7 - x) = \ln(x)$

$$\begin{aligned} \ln\left(\frac{10}{7-x}\right) &= \ln(x) \\ \frac{10}{7-x} &= x \\ 10 &= x(7-x) \\ x^2 - 7x + 10 &= 0 \\ (x-5)(x-2) &= 0 \\ x = 2 \text{ or } x &= 5 \end{aligned}$$

Check: If $x = 2$ or $x = 5$, then the expressions x and $7 - x$ are positive.

Thus, both 2 and 5 are valid solutions to this equation.

b. $\ln(x + 2) + \ln(x - 2) = \ln(9x - 24)$

$$\begin{aligned} \ln\big((x+2)(x-2)\big) &= \ln(9x - 24) \\ x^2 - 4 &= 9x - 24 \\ x^2 - 9x + 20 &= 0 \\ (x-4)(x-5) &= 0 \\ x = 4 \text{ or } x &= 5 \end{aligned}$$

Check: If $x = 4$ or $x = 5$, then the expressions $x + 2$, $x - 2$, and $9x - 24$ are all positive.

Thus, both 4 and 5 are valid solutions to this equation.

c. $\ln(x + 2) + \ln(x - 2) = \ln(-2x - 1)$

$$\begin{aligned} \ln\big((x+2)(x-2)\big) &= \ln(-2x - 1) \\ \ln(x^2 - 4) &= \ln(-2x - 1) \\ x^2 - 4 &= -2x - 1 \\ x^2 + 2x - 3 &= 0 \\ (x+3)(x-1) &= 0 \\ x = -3 \text{ or } x &= 1 \end{aligned}$$

So, $x = -3$ or $x = 1$, but $x = -3$ makes the input to both logarithms on the left-hand side negative, and $x = 1$ makes the input to the second and third logarithms negative. Thus, there are no solutions to the original equation.

12. Suppose the formula $P = P_0(1+r)^t$ gives the population of a city P growing at an annual percent rate r, where P_0 is the population t years ago.

    a. Find the time t it takes this population to double.

    Let $P = 2P_0$.

    Then,

$$\begin{aligned} 2P_0 &= P_0(1+r)^t \\ 2 &= (1+r)^t \\ \log(2) &= \log(1+r)^t \\ \log(2) &= t\log(1+r) \\ t &= \frac{\log(2)}{\log(1+r)}. \end{aligned}$$

    b. Use the structure of the expression to explain why populations with lower growth rates take a longer time to double.

    If r is a decimal between 0 and 1, then the denominator will be a number between 0 and $\log(2)$. Thus, the value of t will be large for small values of r and getting closer to 1 as r increases.

    c. Use the structure of the expression to explain why the only way to double the population in one year is if there is a 100 percent growth rate.

    For the population to double, we need to have $t = 1$. This happens if $\log(2) = \log(1+r)$, and then we have $2 = 1 + r$ and $r = 1$.

13. If $x > 0$, $a + b > 0$, $a > b$, and $\log(x) = \log(a+b) + \log(a-b)$, find x in terms of a and b.

    Applying properties of logarithms, we have

$$\begin{aligned} \log(x) &= \log(a+b) + \log(a-b) \\ &= \log\big((a+b)(a-b)\big) \\ &= \log(a^2 - b^2). \end{aligned}$$

    So, $x = a^2 - b^2$.

MP.3

14. Jenn claims that because $\log(1) + \log(2) + \log(3) = \log(6)$, then $\log(2) + \log(3) + \log(4) = \log(9)$.

    a. Is she correct? Explain how you know.

    Jenn is not correct. Even though $\log(1) + \log(2) + \log(3) = \log(1 \cdot 2 \cdot 3) = \log(6)$, the logarithm properties give $\log(2) + \log(3) + \log(4) = \log(2 \cdot 3 \cdot 4) = \log(24)$. Since $9 \neq 24$, we know that $\log(9) \neq \log(24)$.

MP.3

b. If $\log(a) + \log(b) + \log(c) = \log(a + b + c)$, express c in terms of a and b. Explain how this result relates to your answer to part (a).

$$\begin{aligned} \log(a) + \log(b) + \log(c) &= \log(a+b+c) \\ \log(abc) &= \log(a+b+c) \\ abc &= a+b+c \\ abc - c &= a+b \\ c(ab-1) &= a+b \\ c &= \frac{a+b}{ab-1} \end{aligned}$$

If $\log(2) + \log(3) + \log(4)$ were equal to $\log(9)$, then we would have $4 = \frac{2+3}{2\cdot 3-1}$. However, $\frac{2+3}{2\cdot 3-1} = \frac{5}{5} = 1 \neq 4$, so we know that $\log(2) + \log(3) + \log(4) \neq \log(9)$.

c. Find other values of a, b, and c so that $\log(a) + \log(b) + \log(c) = \log(a + b + c)$.

Many answers are possible; in fact, any positive values of a and b where $ab \neq 1$ and $c = \frac{a+b}{ab-1}$ will satisfy $\log(a) + \log(b) + \log(c) = \log(a + b + c)$. One such answer is $a = 3$, $b = 7$, and $c = \frac{1}{2}$.

15. In Problem 7 of the Lesson 12 Problem Set, you showed that for $x \geq 1$, $\log\left(x + \sqrt{x^2-1}\right) + \log\left(x - \sqrt{x^2-1}\right) = 0$. It follows that $\log\left(x + \sqrt{x^2-1}\right) = -\log\left(x - \sqrt{x^2-1}\right)$. What does this tell us about the relationship between the expressions $x + \sqrt{x^2-1}$ and $x - \sqrt{x^2-1}$?

Since we know $\log\left(x + \sqrt{x^2-1}\right) = -\log\left(x - \sqrt{x^2-1}\right)$, and $-\log\left(x - \sqrt{x^2-1}\right) = \log\left(\frac{1}{x-\sqrt{x^2-1}}\right)$, we know that $\log\left(x + \sqrt{x^2-1}\right) = \log\left(\frac{1}{x-\sqrt{x^2-1}}\right)$. Then $x + \sqrt{x^2-1} = \frac{1}{x-\sqrt{x^2-1}}$. We can verify that these expressions are reciprocals by multiplying them together:

$$\begin{aligned} \left(x + \sqrt{x^2-1}\right)\left(x - \sqrt{x^2-1}\right) &= x^2 + x\sqrt{x^2-1} - x\sqrt{x^2-1} - \left(\sqrt{x^2-1}\right)^2 \\ &= x^2 - (x^2 - 1) \\ &= 1. \end{aligned}$$

16. Use the change of base formula to solve the following equations.

a. $\log(x) = \log_{100}(x^2 - 2x + 6)$

$$\begin{aligned} \log(x) &= \frac{\log(x^2-2x+6)}{\log(100)} \\ \log(x) &= \frac{1}{2}\log(x^2-2x+6) \\ 2\log(x) &= \log(x^2-2x+6) \\ \log(x^2) &= \log(x^2-2x+6) \\ x^2 &= x^2 - 2x + 6 \\ 2x &= 6 \\ x &= 3 \end{aligned}$$

Since both sides of the equation are defined for $x = 3$, the only solution to this equation is 3.

b. $\log(x-2) = \log_{100}(14-x)$

$$\log(x-2) = \frac{\log(14-x)}{\log(100)}$$
$$\log(x-2) = \frac{1}{2}\log(14-x)$$
$$2\log(x-2) = \log(14-x)$$
$$\log((x-2)^2) = \log(14-x)$$
$$(x-2)^2 = 14-x$$
$$x^2-4x+4 = 14-x$$
$$x^2-3x-10 = 0$$
$$(x-5)(x+2) = 0$$

Thus, either $x=5$ or $x=-2$. Since the left side of the equation is undefined when $x=-2$, but both sides are defined for $x=5$, the only solution to the equation is 5.

c. $\log_2(x+1) = \log_4(x^2+3x+4)$

$$\log_2(x+1) = \log_4(x^2+3x+4)$$
$$\log_2(x+1) = \frac{\log_2(x^2+3x+4)}{\log_2(4)}$$
$$2\log_2(x+1) = \log_2(x^2+3x+4)$$
$$\log_2((x+1)^2) = \log_2(x^2+3x+4)$$
$$(x+1)^2 = x^2+3x+4$$
$$x^2+2x+1 = x^2+3x+4$$
$$2x+1 = 3x+4$$
$$x = -3$$

Since the left side of the equation is undefined for $x=-3$, there is no solution to this equation.

d. $\log_2(x-1) = \log_8(x^3-2x^2-2x+5)$

$$\log_2(x-1) = \frac{\log_2(x^3-2x^2-2x+5)}{\log_2(8)}$$
$$3\log_2(x-1) = \log_2(x^3-2x^2-2x+5)$$
$$\log_2((x-1)^3) = \log_2(x^3-2x^2-2x+5)$$
$$(x-1)^3 = x^3-2x^2-2x+5$$
$$x^3-3x^2+3x-1 = x^3-2x^2-2x+5$$
$$x^2-5x+6 = 0$$
$$(x-3)(x-2) = 0$$

Since both sides of the equation are defined for $x=3$ and $x=2$, 2 and 3 are both valid solutions to this equation.

17. **Solve the following equation:** $\log(9x) = \frac{2\ln(3) + \ln(x)}{\ln(10)}$.

Rewrite the left-hand side using the change of base formula:

$$\begin{aligned}\log(9x) &= \frac{\ln(9x)}{\ln(10)} \\ &= \frac{\ln(9) + \ln(x)}{\ln(10)} \\ &= \frac{\ln(3^2) + \ln(x)}{\ln(10)}.\end{aligned}$$

Thus, the equation is true for all $x > 0$.

Common Logarithm Table

N	0	1	2	3	4	5	6	7	8	9
1.0	0.0000	0.0043	0.0086	0.0128	0.0170	0.0212	0.0253	0.0294	0.0334	0.0374
1.1	0.0414	0.0453	0.0492	0.0531	0.0569	0.0607	0.0645	0.0682	0.0719	0.0755
1.2	0.0793	0.0828	0.0864	0.0899	0.0934	0.0969	0.1004	0.1038	0.1072	0.1106
1.3	0.1142	0.1173	0.1206	0.1239	0.1271	0.1303	0.1335	0.1367	0.1399	0.1430
1.4	0.1465	0.1492	0.1523	0.1553	0.1584	0.1614	0.1644	0.1673	0.1703	0.1732
1.5	0.1765	0.1790	0.1818	0.1847	0.1875	0.1903	0.1931	0.1959	0.1987	0.2014
1.6	0.2046	0.2068	0.2095	0.2122	0.2148	0.2175	0.2201	0.2227	0.2253	0.2279
1.7	0.2310	0.2330	0.2355	0.2380	0.2405	0.2430	0.2455	0.2480	0.2504	0.2529
1.8	0.2558	0.2577	0.2601	0.2625	0.2648	0.2672	0.2695	0.2718	0.2742	0.2765
1.9	0.2793	0.2810	0.2833	0.2856	0.2878	0.2900	0.2923	0.2945	0.2967	0.2989
2.0	0.3016	0.3032	0.3054	0.3075	0.3096	0.3118	0.3139	0.3160	0.3181	0.3201
2.1	0.3228	0.3243	0.3263	0.3284	0.3304	0.3324	0.3345	0.3365	0.3385	0.3404
2.2	0.3431	0.3444	0.3464	0.3483	0.3502	0.3522	0.3541	0.3560	0.3579	0.3598
2.3	0.3624	0.3636	0.3655	0.3674	0.3692	0.3711	0.3729	0.3747	0.3766	0.3784
2.4	0.3809	0.3820	0.3838	0.3856	0.3874	0.3892	0.3909	0.3927	0.3945	0.3962
2.5	0.3986	0.3997	0.4014	0.4031	0.4048	0.4065	0.4082	0.4099	0.4116	0.4133
2.6	0.4156	0.4166	0.4183	0.4200	0.4216	0.4232	0.4249	0.4265	0.4281	0.4298
2.7	0.4320	0.4330	0.4346	0.4362	0.4378	0.4393	0.4409	0.4425	0.4440	0.4456
2.8	0.4478	0.4487	0.4502	0.4518	0.4533	0.4548	0.4564	0.4579	0.4594	0.4609
2.9	0.4631	0.4639	0.4654	0.4669	0.4683	0.4698	0.4713	0.4728	0.4742	0.4757
3.0	0.4778	0.4786	0.4800	0.4814	0.4829	0.4843	0.4857	0.4871	0.4886	0.4900
3.1	0.4920	0.4928	0.4942	0.4955	0.4969	0.4983	0.4997	0.5011	0.5024	0.5038
3.2	0.5058	0.5065	0.5079	0.5092	0.5105	0.5119	0.5132	0.5145	0.5159	0.5172
3.3	0.5192	0.5198	0.5211	0.5224	0.5237	0.5250	0.5263	0.5276	0.5289	0.5302
3.4	0.5321	0.5328	0.5340	0.5353	0.5366	0.5378	0.5391	0.5403	0.5416	0.5428
3.5	0.5447	0.5453	0.5465	0.5478	0.5490	0.5502	0.5514	0.5527	0.5539	0.5551
3.6	0.5570	0.5575	0.5587	0.5599	0.5611	0.5623	0.5635	0.5647	0.5658	0.5670
3.7	0.5689	0.5694	0.5705	0.5717	0.5729	0.5740	0.5752	0.5763	0.5775	0.5786
3.8	0.5804	0.5809	0.5821	0.5832	0.5843	0.5855	0.5866	0.5877	0.5888	0.5899
3.9	0.5917	0.5922	0.5933	0.5944	0.5955	0.5966	0.5977	0.5988	0.5999	0.6010
4.0	0.6027	0.6031	0.6042	0.6053	0.6064	0.6075	0.6085	0.6096	0.6107	0.6117
4.1	0.6134	0.6138	0.6149	0.6160	0.6170	0.6180	0.6191	0.6201	0.6212	0.6222
4.2	0.6239	0.6243	0.6253	0.6263	0.6274	0.6284	0.6294	0.6304	0.6314	0.6325
4.3	0.6341	0.6345	0.6355	0.6365	0.6375	0.6385	0.6395	0.6405	0.6415	0.6425
4.4	0.6441	0.6444	0.6454	0.6464	0.6474	0.6484	0.6493	0.6503	0.6513	0.6522
4.5	0.6538	0.6542	0.6551	0.6561	0.6571	0.6580	0.6590	0.6599	0.6609	0.6618
4.6	0.6634	0.6637	0.6646	0.6656	0.6665	0.6675	0.6684	0.6693	0.6702	0.6712
4.7	0.6727	0.6730	0.6739	0.6749	0.6758	0.6767	0.6776	0.6785	0.6794	0.6803
4.8	0.6818	0.6821	0.6830	0.6839	0.6848	0.6857	0.6866	0.6875	0.6884	0.6893
4.9	0.6908	0.6911	0.6920	0.6928	0.6937	0.6946	0.6955	0.6964	0.6972	0.6981
5.0	0.6996	0.6998	0.7007	0.7016	0.7024	0.7033	0.7042	0.7050	0.7059	0.7067
5.1	0.7082	0.7084	0.7093	0.7101	0.7110	0.7118	0.7126	0.7135	0.7143	0.7152
5.2	0.7166	0.7168	0.7177	0.7185	0.7193	0.7202	0.7210	0.7218	0.7226	0.7235
5.3	0.7249	0.7251	0.7259	0.7267	0.7275	0.7284	0.7292	0.7300	0.7308	0.7316
5.4	0.7330	0.7332	0.7340	0.7348	0.7356	0.7364	0.7372	0.7380	0.7388	0.7396

N	0	1	2	3	4	5	6	7	8	9
5.5	0.7404	0.7412	0.7419	0.7427	0.7435	0.7443	0.7451	0.7459	0.7466	0.7474
5.6	0.7482	0.7490	0.7497	0.7505	0.7513	0.7520	0.7528	0.7536	0.7543	0.7551
5.7	0.7559	0.7566	0.7574	0.7582	0.7589	0.7597	0.7604	0.7612	0.7619	0.7627
5.8	0.7634	0.7642	0.7649	0.7657	0.7664	0.7672	0.7679	0.7686	0.7694	0.7701
5.9	0.7709	0.7716	0.7723	0.7731	0.7738	0.7745	0.7752	0.7760	0.7767	0.7774
6.0	0.7782	0.7789	0.7796	0.7803	0.7810	0.7818	0.7825	0.7832	0.7839	0.7846
6.1	0.7853	0.7860	0.7868	0.7875	0.7882	0.7889	0.7896	0.7903	0.7910	0.7917
6.2	0.7924	0.7931	0.7938	0.7945	0.7952	0.7959	0.7966	0.7973	0.7980	0.7987
6.3	0.7993	0.8000	0.8007	0.8014	0.8021	0.8028	0.8035	0.8041	0.8048	0.8055
6.4	0.8062	0.8069	0.8075	0.8082	0.8089	0.8096	0.8102	0.8109	0.8116	0.8122
6.5	0.8129	0.8136	0.8142	0.8149	0.8156	0.8162	0.8169	0.8176	0.8182	0.8189
6.6	0.8195	0.8202	0.8209	0.8215	0.8222	0.8228	0.8235	0.8241	0.8248	0.8254
6.7	0.8261	0.8267	0.8274	0.8280	0.8287	0.8293	0.8299	0.8306	0.8312	0.8319
6.8	0.8325	0.8331	0.8338	0.8344	0.8351	0.8357	0.8363	0.8370	0.8376	0.8382
6.9	0.8388	0.8395	0.8401	0.8407	0.8414	0.8420	0.8426	0.8432	0.8439	0.8445
7.0	0.8451	0.8457	0.8463	0.8470	0.8476	0.8482	0.8488	0.8494	0.8500	0.8506
7.1	0.8513	0.8519	0.8525	0.8531	0.8537	0.8543	0.8549	0.8555	0.8561	0.8567
7.2	0.8573	0.8579	0.8585	0.8591	0.8597	0.8603	0.8609	0.8615	0.8621	0.8627
7.3	0.8633	0.8639	0.8645	0.8651	0.8657	0.8663	0.8669	0.8675	0.8681	0.8686
7.4	0.8692	0.8698	0.8704	0.8710	0.8716	0.8722	0.8727	0.8733	0.8739	0.8745
7.5	0.8751	0.8756	0.8762	0.8768	0.8774	0.8779	0.8785	0.8791	0.8797	0.8802
7.6	0.8808	0.8814	0.8820	0.8825	0.8831	0.8837	0.8842	0.8848	0.8854	0.8859
7.7	0.8865	0.8871	0.8876	0.8882	0.8887	0.8893	0.8899	0.8904	0.8910	0.8915
7.8	0.8921	0.8927	0.8932	0.8938	0.8943	0.8949	0.8954	0.8960	0.8965	0.8971
7.9	0.8976	0.8982	0.8987	0.8993	0.8998	0.9004	0.9009	0.9015	0.9020	0.9025
8.0	0.9031	0.9036	0.9042	0.9047	0.9053	0.9058	0.9063	0.9069	0.9074	0.9079
8.1	0.9085	0.9090	0.9096	0.9101	0.9106	0.9112	0.9117	0.9122	0.9128	0.9133
8.2	0.9138	0.9143	0.9149	0.9154	0.9159	0.9165	0.9170	0.9175	0.9180	0.9186
8.3	0.9191	0.9196	0.9201	0.9206	0.9212	0.9217	0.9222	0.9227	0.9232	0.9238
8.4	0.9243	0.9248	0.9253	0.9258	0.9263	0.9269	0.9274	0.9279	0.9284	0.9289
8.5	0.9294	0.9299	0.9304	0.9309	0.9315	0.9320	0.9325	0.9330	0.9335	0.9340
8.6	0.9345	0.9350	0.9355	0.9360	0.9365	0.9370	0.9375	0.9380	0.9385	0.9390
8.7	0.9395	0.9400	0.9405	0.9410	0.9415	0.9420	0.9425	0.9430	0.9435	0.9440
8.8	0.9445	0.9450	0.9455	0.9460	0.9465	0.9469	0.9474	0.9479	0.9484	0.9489
8.9	0.9494	0.9499	0.9504	0.9509	0.9513	0.9518	0.9523	0.9528	0.9533	0.9538
9.0	0.9542	0.9547	0.9552	0.9557	0.9562	0.9566	0.9571	0.9576	0.9581	0.9586
9.1	0.9590	0.9595	0.9600	0.9605	0.9609	0.9614	0.9619	0.9624	0.9628	0.9633
9.2	0.9638	0.9643	0.9647	0.9652	0.9657	0.9661	0.9666	0.9671	0.9675	0.9680
9.3	0.9685	0.9689	0.9694	0.9699	0.9703	0.9708	0.9713	0.9717	0.9722	0.9727
9.4	0.9731	0.9736	0.9741	0.9745	0.9750	0.9754	0.9759	0.9763	0.9768	0.9773
9.5	0.9777	0.9782	0.9786	0.9791	0.9795	0.9800	0.9805	0.9809	0.9814	0.9818
9.6	0.9823	0.9827	0.9832	0.9836	0.9841	0.9845	0.9850	0.9854	0.9859	0.9863
9.7	0.9868	0.9872	0.9877	0.9881	0.9886	0.9890	0.9894	0.9899	0.9903	0.9908
9.8	0.9912	0.9917	0.9921	0.9926	0.9930	0.9934	0.9939	0.9943	0.9948	0.9952
9.9	0.9956	0.9961	0.9965	0.9969	0.9974	0.9978	0.9983	0.9987	0.9991	0.9996

Lesson 16: Rational and Irrational Numbers

Student Outcomes

- Students interpret addition and multiplication of two irrational numbers in the context of logarithms and find better-and-better decimal approximations of the sum and product, respectively.
- Students work with and interpret logarithms with irrational values in preparation for graphing logarithmic functions.

Lesson Notes

This foundational lesson revisits the fundamental differences between rational and irrational numbers. We begin by reviewing how to locate an irrational number on the number line by squeezing its infinite decimal expansion between two rational numbers, a process students may recall from Grade 8 Module 7 Lesson 7. In preparation for graphing logarithmic functions, the main focus of this lesson is to understand the process of locating values of logarithms on the number line (**F-IF.C.7e**). We then go a step further to understand this process in the context of adding two irrational logarithmic expressions (**N-RN.B.3**). Although students have addressed **N-RN.B.3** in Algebra I and have worked with irrational numbers in previous lessons in this module, such as Lesson 5, students need to fully understand how to sum two irrational logarithmic expressions in preparation for graphing logarithmic functions in the next lesson, in alignment with **F-IF.C.7e**. Students have been exposed to two approaches to adding rational numbers: a geometric approach by placing the numbers on the number line, as reviewed in Module 1 Lesson 24, and a numerical approach by applying an addition algorithm. In the Opening Exercise, students are asked to recall both methods for adding rational numbers. Both approaches fail with irrational numbers, and we locate the sum of two irrational numbers (or a rational and an irrational number) by squeezing its infinite decimal expansion between two rational numbers. Emphasize to students that since they have been performing addition for many, many years, we are more interested in the process of addition than in the result.

This lesson emphasizes mathematical practice standard MP.3 as students develop and then justify conjectures about sums and products of irrational logarithmic expressions.

Classwork

Opening Exercise (4 minutes)

Have students work in pairs or small groups on the sums below. Remind them that we are interested in the process of addition as much as obtaining the correct result. Ask groups to volunteer their responses at the end of the allotted time.

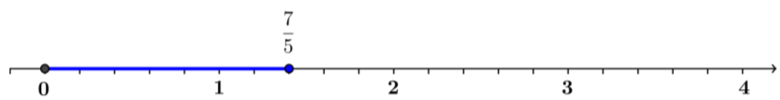
Scaffolding:

- Ask struggling students to represent a simpler sum, such as $\frac{1}{5}+\frac{1}{4}$.
- Ask advanced students to explain how to represent the sum of two generic rational numbers, such as $\frac{a}{b}+\frac{c}{d}$, on the number line.

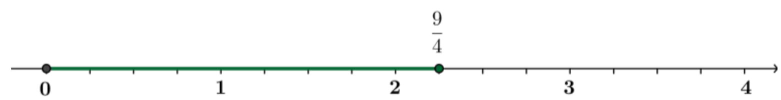
Opening Exercise

a. **Explain how to use a number line to add the fractions $\frac{7}{5}+\frac{9}{4}$.**

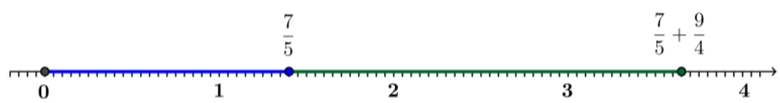
First, we locate the point $\frac{7}{5}$ on the number line by dividing each unit into 5 intervals of length $\frac{1}{5}$.



Then, we locate the point $\frac{9}{4}$ on the number line by dividing each unit into 4 intervals of length $\frac{1}{4}$.



To find the sum, we place the green segment of length $\frac{9}{4}$ end-to-end with the blue segment of length $\frac{7}{5}$, and the right end point of the green segment is the sum. Since the tick marks at units of $\frac{1}{4}$ and $\frac{1}{5}$ do not align, we make new tick marks that are $\frac{1}{20}$ apart. Then $\frac{7}{5}=\frac{28}{20}$ and $\frac{9}{4}=\frac{45}{20}$, so the sum is located at point $\frac{73}{20}$.



b. **Convert $\frac{7}{5}$ and $\frac{9}{4}$ to decimals, and explain the process for adding them together.**

We know that $\frac{7}{5}=1.4$ and $\frac{9}{4}=2.25$. To add these numbers, we add a zero placeholder to 1.4 to get 1.40 so that each number has the same number of decimal places. Then, we line them up at the decimal place and add from right to left, carrying over a power of 10 if needed (we do not need to carry for this sum).

Step 1: Working from right to left, we first add 0 hundredths $+$ 5 hundredths $=$ 5 hundredths.

$$\begin{array}{r} 1.40 \\ \underline{+2.25} \\ 5 \end{array}$$

Step 2: Then we add 4 tenths $+2$ tenths $=$ 6 tenths.

$$\begin{array}{r} 1.40 \\ \underline{+2.25} \\ 65 \end{array}$$

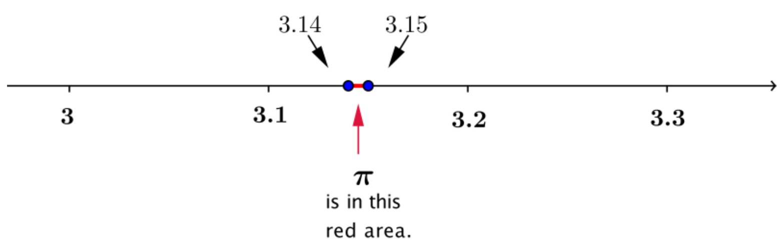
Step 3: And, finally, we add 1 one $+$ 2 ones $=$ 3 ones.

$$\begin{array}{r} 1.40 \\ \underline{+2.25} \\ 3.65 \end{array}$$

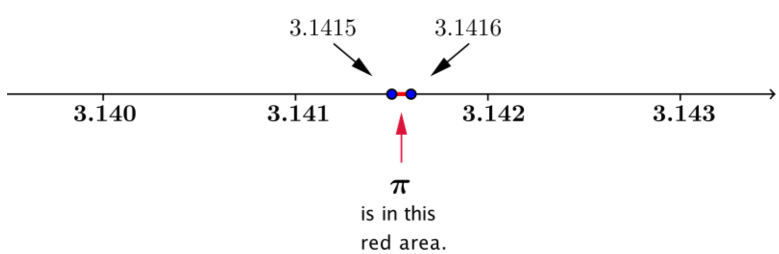
Discussion (8 minutes)

- How would we add two numbers such as $\pi+\frac{7}{5}$?
  - *Student answers will vary, but the point is that our algorithm that adds rational numbers in decimal form does not apply to sums that involve irrational numbers.*

- Many of the numbers we have been working with are irrational numbers, such as $\sqrt{2}$, e, $\log(2)$, and π, meaning that we cannot write them as quotients of integers. One of the key distinctions between rational and irrational numbers is that rational numbers can be expressed as a decimal that either terminates (such as $\frac{1}{8} = 0.125$) or repeats infinitely (such as $\frac{1}{9} = 0.11111\ldots$) Irrational numbers cannot be exactly represented by a decimal expansion because the digits to the right of the decimal point never end and never repeat predictably. The best we can do to represent irrational numbers with a decimal expansion is to find an approximation.
- For example, consider the number π. What is the value of π?
  - 3.14 *(Students may answer this question with varying degrees of accuracy.)*
- Is that the exact value of π?
  - *No. We cannot write a decimal expansion for the exact value of π.*
- What is the *last digit* of π?
  - *Since the decimal expansion for π never terminates, there is no last digit of π.*



- Where is π on the number line?
  - *The number π is between 3.14 and 3.15.*
- What kind of numbers are 3.14 and 3.15?
  - *Rational numbers*
- Can we get better under and over estimates than that?
  - *Yes, since $\pi \approx 3.14159$, we know that $3.1415 < \pi < 3.1416$.*



- What kind of numbers are 3.1415 and 3.1416?
  - *Rational numbers*
- Can we find rational numbers with 10 digits that are good upper and lower bounds for π? How?
  - *Yes. Just take the expansion for π on the calculator and round up and down.*
- So, how can we add two irrational numbers? Let's turn our attention back to logarithms.

- The numbers $\log(3)$ and $\log(4)$ are examples of irrational numbers. The calculator says that $\log(3) = 0.4771212547$ and $\log(4) = 0.6020599913$. What is wrong with those two statements?
  - *Since* $\log(3)$ *and* $\log(4)$ *are irrational numbers, their decimal expansions do not terminate so the calculator gives approximations and not exact values.*
- So, we really should write $\log(3) = 0.4771212547 \ldots$ and $\log(4) = 0.6020599913 \ldots$ This means there are more digits that we cannot see. What happens when we try to add these decimal expansions together numerically?
  - *There is no last digit for us to use to start our addition algorithm, which moves from right to left. We cannot even start adding these together using our usual method.*
- So, our standard algorithm for adding numbers fails. How can we add $\log(3) + \log(4)$?

Example 1 (8 minutes)

Begin these examples with direct instruction and teacher modeling, and gradually release responsibility to the students when they are ready to tackle these questions in pairs or small groups.

- Since we do not have a direct method to add $\log(3) + \log(4)$, we should try another approach. Remember that according to the calculator, $\log(3) = 0.4771212547 \ldots$ and $\log(4) = 0.6020599913 \ldots$ While we could use the calculator to add these approximations to find an estimate of $\log(3) + \log(4)$, we are interested in making sense of the operation.
- What if we just need an approximation of $\log(3) + \log(4)$ to one decimal place, that is, to an accuracy of 10^{-1}? If we do not need more accuracy than that, we can use $\log(3) \approx 0.477$ and $\log(4) \approx 0.602$. Then,

$$0.47 \leq \log(3) \leq 0.48;$$
$$0.60 \leq \log(4) \leq 0.61.$$

MP.3

- Based on these inequalities, what statement can we make about the sum $\log(3) + \log(4)$? Explain why you believe your statement is correct.
  - *Adding terms together, we have*

    $$1.07 \leq \log(3) + \log(4) \leq 1.09.$$

    Rounding to one decimal place, we have

    $$1.1 \leq \log(3) + \log(4) \leq 1.1.$$

    So, to one decimal place, $\log(3) + \log(4) \approx 1.1$.
- What if we wanted to find the value of $\log(3) + \log(4)$ to two decimal places? What are the under and over estimates for $\log(3)$ and $\log(4)$ that we should start with before we add? What do we get when we add them together?
  - *We should start with*

    $$0.477 \leq \log(3) \leq 0.478;$$
    $$0.602 \leq \log(4) \leq 0.603.$$

    Then we have

    $$1.079 \leq \log(3) + \log(4) \leq 1.081.$$

    To two decimal places, $\log(3) + \log(4) \approx 1.08$.

- Now, find the value of $\log(3) + \log(4)$ to five decimal places, that is, to an accuracy of 10^{-5}.
  - *We should start with*

$$0.477121 \leq \log(3) \leq 0.477122;$$
$$0.602059 \leq \log(4) \leq 0.602060.$$

Then we have

$$1.079180 \leq \log(3) + \log(4) \leq 1.079182.$$

To five decimal places, $\log(3) + \log(4) \approx 1.07918$.

- Now, find the value of $\log(3) + \log(4)$ to eight decimal places, that is, to an accuracy of 10^{-8}.
  - *We should start with*

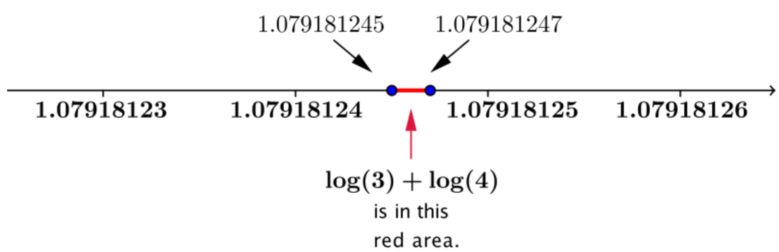
$$0.477121254 \leq \log(3) \leq 0.477121255$$
$$0.602059991 \leq \log(4) \leq 0.602059992.$$

Then we have

$$1.079181245 \leq \log(3) + \log(4) \leq 1.079181247.$$

To eight decimal places, $\log(3) + \log(4) \approx 1.07918125$.

- Notice that we are *squeezing* the actual value of $\log(3) + \log(4)$ between two rational numbers. Since we know how to plot a rational number on the number line, we can get really close to the location of the irrational number $\log(3) + \log(4)$ by squeezing it between two rational numbers.



- Could we keep going? If we knew enough digits of the decimal expansions of $\log(3)$ and $\log(4)$, could we find an approximation of $\log(3) + \log(4)$ to 20 decimal places? Or 50 decimal places? Or 1,000 decimal places?
  - *Yes. There is no reason that we cannot continue this process, provided we know enough digits of the original irrational numbers* $\log(3)$ *and* $\log(4)$.
- Summarize what you have learned from this example in your notebook. (Allow students a minute or so to record the main points of this example.)

Exercises 1–5 (8 minutes)

Have students complete these exercises in pairs or small groups. Expect students to react with surprise to the results: In the previous example, the sum $\log(3) + \log(4)$ was irrational, so the decimal expansion never terminated. In this set of exercises, the sum $\log(4) + \log(25)$ is rational with the exact value 2, and at each step, their estimate of the sum will be exact.

Exercises

1. According to the calculator, $\log(4) = 0.6020599913\ldots$ and $\log(25) = 1.3979400087\ldots$. Find an approximation of $\log(4) + \log(25)$ to one decimal place, that is, to an accuracy of 10^{-1}.

$$0.60 < \log(4) < 0.61$$
$$1.39 < \log(25) < 1.40$$
$$1.99 < \log(4) + \log(25) < 2.01$$
$$\log(4) + \log(25) \approx 2.0$$

2. Find the value of $\log(4) + \log(25)$ to an accuracy of 10^{-2}.

$$0.602 < \log(4) < 0.603$$
$$1.397 < \log(25) < 1.398$$
$$1.999 < \log(4) + \log(25) < 2.001$$
$$\log(4) + \log(25) \approx 2.00$$

3. Find the value of $\log(4) + \log(25)$ to an accuracy of 10^{-8}.

$$0.602059991 \le \log(4) \le 0.602059992$$
$$1.397940008 \le \log(25) \le 1.397940009$$
$$1.999999999 \le \log(4) + \log(25) \le 2.000000001$$
$$\log(4) + \log(25) \approx 2.00000000$$

MP.3

4. Make a conjecture: Is $\log(4) + \log(25)$ a rational or an irrational number?

It appears that $\log(4) + \log(25) = 2$ exactly, so my conjecture is that $\log(4) + \log(25)$ is a rational number.

5. Why is your conjecture in Exercise 4 true?

The logarithm rule that says $\log(x) + \log(y) = \log(xy)$ applies here.

$$\begin{aligned}\log(4) + \log(25) &= \log(4 \cdot 25)\\ &= \log(100)\\ &= \log(10^2)\\ &= 2\end{aligned}$$

Discussion (3 minutes)

- We have seen how we can squeeze the sum of two irrational numbers between rational numbers and get an approximation of the sum to whatever accuracy we want. What about multiplication?

MP.3

- Make a conjecture: Without actually calculating it, what is the value of $\log(4) \cdot \log(25)$?

Allow students time to discuss this question with a partner and write down a response before allowing students to put forth their ideas to the class. This would be an ideal time to use personal white boards, if you have them, for students to record and display their conjectures. Ensure that students have a written record of this conjecture that will be disproven in the next set of exercises.

  - *Conjectures will vary; some might be $\log(4) \cdot \log(25) = \log(29)$ or $\log(4) \cdot \log(25) = \log(100)$.*

Exercises 6–8 (6 minutes)

Remember that the calculator gives the following values: $\log(4) = 0.6020599913\ldots$ and $\log(25) = 1.3979400087\ldots$

6. Find the value of $\log(4) \cdot \log(25)$ to three decimal places.

$$0.6020 \le \log(4) \le 0.6021$$
$$1.3979 \le \log(25) \le 1.3980$$
$$0.8415358 \le \log(4) \cdot \log(25) \le 0.8417358$$
$$\log(4) \cdot \log(25) \approx 0.842$$

7. Find the value of $\log(4) \cdot \log(25)$ to five decimal places.

$$0.602059 \le \log(4) \le 0.602060$$
$$1.397940 \le \log(25) \le 1.397941$$
$$0.8416423585 \le \log(4) \cdot \log(25) \le 0.8416443585$$
$$\log(4) \cdot \log(25) \approx 0.84164$$

8. Does your conjecture from the above discussion appear to be true?

No. The work from Exercise 6 shows that $\log(4) \cdot \log(25) \neq \log(29)$, and $\log(4) \cdot \log(25) \neq \log(100)$. (Answers will vary based on student conjectures.)

Closing (3 minutes)

Ask students to respond to the following prompts independently, and then have them share their responses with a partner. After students have a chance to write and discuss, go through the key points in the Lesson Summary below.

- List five rational numbers.
  - *Student responses will vary; possible responses include $1, 10, \frac{3}{5}, 17$, and $-\frac{42}{13}$.*
- List five irrational numbers.
  - *Student responses will vary; possible responses include $\sqrt{3}, \pi, e, 1 + \sqrt{2}$, and π^3.*
- Is 0 a rational or irrational number? Explain how you know.
  - *Since 0 is an integer, we can write $0 = \frac{0}{1}$, which is a quotient of integers. Thus, 0 is a rational number.*
- If a number is given as a decimal, how can you tell if it is a rational or an irrational number?
  - *If the decimal representation terminates or repeats at some point, then the number is rational and can be expressed as the quotient of two integers. Otherwise, the number is irrational.*

Lesson Summary

- **Irrational numbers occur naturally and frequently.**
- **The n th roots of most integers and rational numbers are irrational.**
- **Logarithms of most positive integers or positive rational numbers are irrational.**
- **We can locate an irrational number on the number line by trapping it between lower and upper estimates. The infinite process of squeezing the irrational number in smaller and smaller intervals locates exactly where the irrational number is on the number line.**
- **We can perform arithmetic operations such as addition and multiplication with irrational numbers using lower and upper approximations and squeezing the result of the operation in smaller and smaller intervals between two rational approximations to the result.**

Exit Ticket (5 minutes)

Name ______________________________ Date ____________________

Lesson 16: Rational and Irrational Numbers

Exit Ticket

The decimal expansions of e and $\sqrt{5}$ are given below.

$$e \approx 2.71828182 \ldots$$
$$\sqrt{5} \approx 2.23606797 \ldots$$

a. Find an approximation of $\sqrt{5} + e$ to three decimal places. Do not use a calculator.

b. Explain how you can locate $\sqrt{5} + e$ on the number line. How is this different from locating $2.6 + 2.7$ on the number line?

Exit Ticket Sample Solutions

The decimal expansions of e and $\sqrt{5}$ are given below.

$$e \approx 2.71828182 \ldots$$
$$\sqrt{5} \approx 2.23606797 \ldots$$

a. Find an approximation of $\sqrt{5} + e$ to three decimal places. Do not use a calculator.

$$2.2360 \leq \sqrt{5} \leq 2.2361$$
$$2.7182 \leq e \leq 2.7183$$
$$4.9542 \leq \sqrt{5} + e \leq 4.9544$$

Thus, to three decimal places, $\sqrt{5} + e \approx 4.954$.

b. Explain how you can locate $\sqrt{5} + e$ on the number line. How is this different from locating $2.6 + 2.7$ on the number line?

We cannot locate $\sqrt{5} + e$ precisely on the number line because the sum is irrational, but we can get as close to it as we want by squeezing it between two rational numbers, r_1 and r_2, that differ only in the last decimal place, $r_1 \leq \sqrt{5} + e \leq r_2$. Since we can locate rational numbers on the number line, we can get arbitrarily close to the true location of $\sqrt{5} + e$ by starting with more and more accurate decimal representations of $\sqrt{5}$ and e. This differs from pinpointing the location of sums of rational numbers because we can precisely locate the sum $2.6 + 2.7 = 5.3$ by dividing the interval $[5, 6]$ into 10 parts of equal length 0.1. Then, the point 5.3 is located exactly at the point between the third and fourth parts.

Problem Set Sample Solutions

1. Given that $\sqrt{5} \approx 2.2360679775$ and $\pi \approx 3.1415926535$, find the sum $\sqrt{5} + \pi$ to an accuracy of 10^{-8} without using a calculator.

   From the estimations we are given, we know that

$$2.236067977 < \sqrt{5} < 2.236067978$$
$$3.141592653 < \pi < 3.141592654.$$

   Adding these together gives

$$5.377660630 < \sqrt{5} + \pi < 5.377660632.$$

   Then, to an accuracy of 10^{-8}, we have

$$\sqrt{5} + \pi \approx 5.37766063.$$

2. Put the following numbers in order from least to greatest.

$$\sqrt{2}, \pi, 0, e, \frac{22}{7}, \frac{\pi^2}{3}, 3.14, \sqrt{10}$$

$$0, \sqrt{2}, e, 3.14, \pi, \frac{22}{7}, \sqrt{10}, \frac{\pi^2}{3}$$

3. **Find a rational number between the specified two numbers.**

   a. $\frac{4}{13}$ and $\frac{5}{13}$

   Many answers are possible. Since $\frac{4}{13} = \frac{8}{26}$ *and* $\frac{5}{13} = \frac{10}{26}$, *we know that* $\frac{4}{13} < \frac{9}{26} < \frac{5}{13}$.

   b. $\frac{3}{8}$ and $\frac{5}{9}$

   Many answers are possible. Since $\frac{3}{8} = \frac{27}{72}$ *and* $\frac{5}{9} = \frac{40}{72}$, *we know that* $\frac{30}{72} = \frac{5}{12}$ *is between* $\frac{3}{8}$ *and* $\frac{5}{9}$.

   c. 1.7299999 and 1.73

   Many answers are possible. $1.7299999 < 1.72999995 < 1.73$.

   d. $\frac{\sqrt{2}}{7}$ and $\frac{\sqrt{2}}{9}$

   Many answers are possible. Since $\frac{\sqrt{2}}{9} \approx 0.157135$ *and* $\frac{\sqrt{2}}{7} \approx 0.202031$, *we know* $\frac{\sqrt{2}}{9} < 0.2 < \frac{\sqrt{2}}{7}$.

   e. π and $\sqrt{10}$

   Many answers are possible. Since $\pi \approx 3.14159$ *and* $\sqrt{10} \approx 3.16228$, *we know* $\pi < 3.15 < \sqrt{10}$.

4. **Knowing that $\sqrt{2}$ is irrational, find an irrational number between $\frac{1}{2}$ and $\frac{5}{9}$.**

   One such number is $r\sqrt{2}$, *for some rational number* r. *Then* $\frac{1}{2} < r\sqrt{2} < \frac{5}{9}$, *so we have* $\frac{1}{2\sqrt{2}} < r < \frac{5}{9\sqrt{2}}$. *Since* $\frac{1}{2\sqrt{2}} \approx 0.3536$ *and* $\frac{5}{9\sqrt{2}} \approx 0.3929$, *we can let* $r = 0.36$. *Then,* $0.36\sqrt{2}$ *is an irrational number between* $\frac{1}{2}$ *and* $\frac{5}{9}$.

5. **Give an example of an irrational number between e and π.**

   Many answers are possible, such as $\frac{\pi+e}{2}$, $\sqrt{\pi e}$, *or* $\frac{10}{11}\pi$.

6. **Given that $\sqrt{2}$ is irrational, which of the following numbers are irrational?**

$$\frac{\sqrt{2}}{2}, 2+\sqrt{2}, \frac{\sqrt{2}}{2\sqrt{2}}, \frac{2}{\sqrt{2}}, \left(\sqrt{2}\right)^2$$

   Note that $\frac{\sqrt{2}}{2\sqrt{2}} = \frac{1}{2}$, $\frac{2}{\sqrt{2}} = \sqrt{2}$, *and* $\left(\sqrt{2}\right)^2 = 2$. *The numbers* $\frac{\sqrt{2}}{2}$, $2+\sqrt{2}$, *and* $\frac{2}{\sqrt{2}}$ *are irrational.*

7. **Given that π is irrational, which of the following numbers are irrational?**

$$\frac{\pi}{2}, \frac{\pi}{2\pi}, \sqrt{\pi}, \pi^2$$

   The numbers $\frac{\pi}{2}$, $\sqrt{\pi}$, *and* π^2 *are irrational.*

8. Which of the following numbers are irrational?

$$1, 0, \sqrt{5}, \sqrt[3]{64}, e, \pi, \frac{\sqrt{2}}{2}, \frac{\sqrt{8}}{\sqrt{2}}, \cos\left(\frac{\pi}{3}\right), \sin\left(\frac{\pi}{3}\right)$$

The numbers $\sqrt{5}$, e, π, $\frac{\sqrt{2}}{2}$, and $\sin\left(\frac{\pi}{3}\right)$ are irrational.

9. Find two irrational numbers x and y so that their average is rational.

If $x = 1 + \sqrt{2}$ and $y = 3 - \sqrt{2}$, then $\frac{x+y}{2} = \frac{1}{2}\left(\left(1+\sqrt{2}\right) + \left(3 - \sqrt{2}\right)\right) = 2$. So, the average of x and y is rational.

10. Suppose that $\frac{2}{3}x$ is an irrational number. Explain how you know that x must be an irrational number. (Hint: What would happen if there were integers a and b so that $x = \frac{a}{b}$?)

If x is rational, then there are integers a and b so that $x = \frac{a}{b}$. Then $\frac{2a}{3b}$ is rational, so $\frac{2}{3}x$ is also rational. This contradicts the given fact that $\frac{2}{3}x$ is irrational, so it is not possible for x to be rational. Thus, x must be an irrational number.

11. If r and s are rational numbers, prove that $r + s$ and $r - s$ are also rational numbers.

If r and s are rational numbers, then there exist integers a, b, c, d with $b \neq 0$ and $d \neq 0$ so that $r = \frac{a}{b}$ and $s = \frac{c}{d}$.

Then,

$$\begin{aligned} r + s &= \frac{a}{b} + \frac{c}{d} \\ &= \frac{ad}{bd} + \frac{bc}{bd} \\ &= \frac{ad + bc}{bd} \end{aligned}$$

$$\begin{aligned} r - s &= \frac{a}{b} - \frac{c}{d} \\ &= \frac{ad - bc}{bd}. \end{aligned}$$

Since $ad + bc$, $ad - bc$, and bd are integers, $r + s$ and $r - s$ are rational numbers.

12. If r is a rational number and x is an irrational number, determine whether the following numbers are always rational, sometimes rational, or never rational. Explain how you know.

a. $r + x$

If $r + x = y$ and y is rational, then $r - y = -x$ would be rational by Problem 11. Since x is irrational, we know $-x$ is irrational, so y cannot be rational. Thus, the sum $r + x$ is never rational.

b. $r - x$

If $r - x = y$ and y is rational, then $r - y = x$ would be rational by Problem 11. Since x is irrational, y cannot be rational. Thus, the difference $r - x$ is never rational.

c. rx

If $rx = y$, $r \neq 0$, and y is rational, then there are integers a, b, c, d with $a \neq 0$, $b \neq 0$, and $d \neq 0$ so that $r = \frac{a}{b}$ and $y = \frac{c}{d}$. Then $x = \frac{y}{r} = \frac{cb}{ad}$, so x is rational. Since x was not rational, the only way that rx can be rational is if $r = 0$. Thus, rx is sometimes rational (in only one case).

d. x^r

If $x = \sqrt[r]{k}$ for some rational number k, then $x^r = k$ is rational. For example, $\left(\sqrt{5}\right)^2 = 5$ is rational. But, π^r is never rational for any exponent r, so x^r is sometimes rational.

13. If x and y are irrational numbers, determine whether the following numbers are always rational, sometimes rational, or never rational. Explain how you know.

a. $x + y$

This is sometimes rational. For example, $\pi + \sqrt{2}$ is irrational, but $\left(1 + \sqrt{2}\right) + \left(1 - \sqrt{2}\right) = 2$ is rational.

b. $x - y$

This is sometimes rational. For example, $\pi - \sqrt{2}$ is irrational, but $\left(5 + \sqrt{3}\right) - \left(1 + \sqrt{3}\right) = 4$ is rational.

c. xy

This is sometimes rational. For example, $\pi\sqrt{2}$ is irrational, but $\sqrt{2} \cdot \sqrt{8} = 4$ is rational.

d. $\frac{x}{y}$

This is sometimes rational. For example, $\frac{\pi}{\sqrt{2}}$ is irrational, but $\frac{\sqrt{8}}{\sqrt{2}} = 2$ is rational.

Lesson 17: Graphing the Logarithm Function

Student Outcomes

- Students graph the functions $f(x) = \log(x)$, $g(x) = \log_2(x)$, and $h(x) = \ln(x)$ by hand and identify key features of the graphs of logarithmic functions.

Lesson Notes

In this lesson, students work in pairs or small groups to generate graphs of $f(x) = \log(x)$, $g(x) = \log_2(x)$, or $h(x) = \log_5(x)$. Students compare the graphs of these three functions to derive the key features of graphs of general logarithmic functions for bases $b > 1$. Tables of function values are provided so that calculators are not needed in this lesson; all graphs should be drawn by hand. Students relate the domain of the logarithmic functions to the graph in accordance with **F-IF.B.5**. After the graphs are generated and conclusions drawn about their properties, students use properties of logarithms to find additional points on the graphs. Continue to rely on the definition of the logarithm, which was stated in Lesson 8, and properties of logarithms developed in Lessons 12 and 13:

LOGARITHM: If three numbers, L, b, and x are related by $x = b^L$, then L is the *logarithm base* b of x, and we write $\log_b(x)$. That is, the value of the expression $L = \log_b(x)$ is the power of b needed to obtain x. Valid values of b as a base for a logarithm are $0 < b < 1$ and $b > 1$.

Classwork

Opening (1 minute)

Divide the students into pairs or small groups; ideally, the number of groups formed is a multiple of three. Assign the function $f(x) = \log(x)$ to one-third of the groups, and refer to these groups as the 10-team. Assign the function $g(x) = \log_2(x)$ to the second third of the groups, and refer to these groups as the 2-team. Assign the function $h(x) = \log_5(x)$ to the remaining third of the groups, and refer to these groups as the 5-team.

Scaffolding:

- Struggling students may benefit from watching the teacher model the process of plotting points.
- Consider assigning struggling students to the 2-team because the function values are integers.
- Alternatively, assign advanced students to the 2-team and ask them to generate the graph of $y = \log_2(x)$ without the given table.

Opening Exercise (8 minutes)

While student groups are creating graphs and responding to the prompts that follow, circulate and observe student work. Select three groups to present their graphs and results at the end of the exercise.

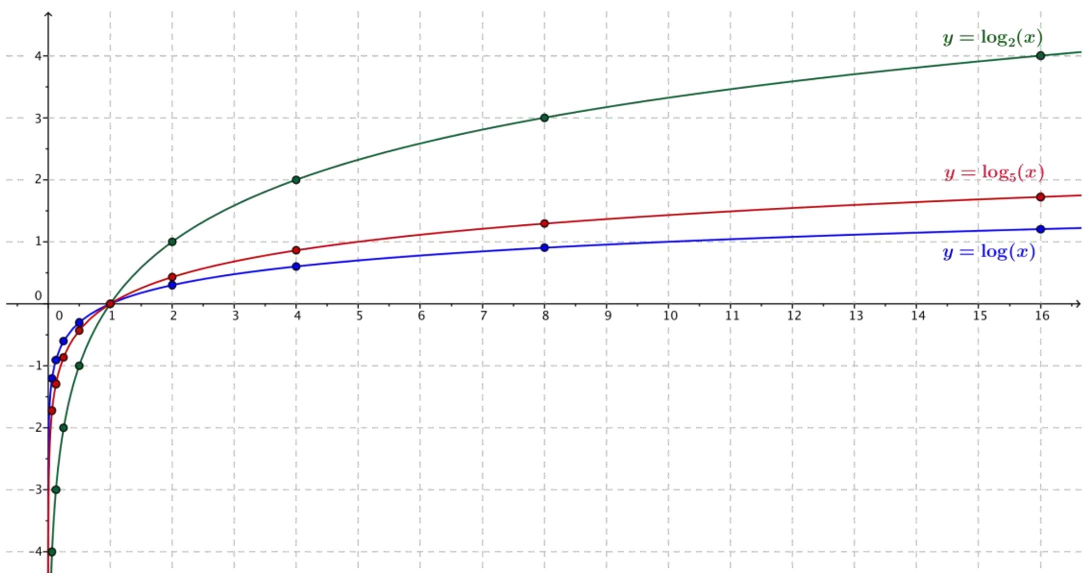
Opening Exercise

Graph the points in the table for your assigned function $f(x) = \log(x)$, $g(x) = \log_2(x)$, or $h(x) = \log_5(x)$ for $0 < x \leq 16$. Then, sketch a smooth curve through those points and answer the questions that follow.

10-team $f(x) = \log(x)$	
x	$f(x)$
0.0625	-1.20
0.125	-0.90
0.25	-0.60
0.5	-0.30
1	0
2	0.30
4	0.60
8	0.90
16	1.20

2-team $g(x) = \log_2(x)$	
x	$g(x)$
0.0625	-4
0.125	-3
0.25	-2
0.5	-1
1	0
2	1
4	2
8	3
16	4

5-team $h(x) = \log_5(x)$	
x	$h(x)$
0.0625	-1.72
0.125	-1.29
0.25	-0.86
0.5	-0.43
1	0
2	0.43
4	0.86
8	1.29
16	1.72



a. **What does the graph indicate about the domain of your function?**

The domain of each of these functions is the positive real numbers, which can be stated as $(0, \infty)$.

b. **Describe the x-intercepts of the graph.**

There is one x-intercept at 1.

c. **Describe the y-intercepts of the graph.**

There are no y-intercepts of this graph.

d. **Find the coordinates of the point on the graph with y-value 1.**

For the 10-team, this is $(10, 1)$. For the 2-team, this is $(2, 1)$. For the 5-team, this is $(5, 1)$.

e. **Describe the behavior of the function as $x \to 0$.**

As $x \to 0$, the function values approach negative infinity; that is, $f(x) \to -\infty$. The same is true for the functions g and h.

f. **Describe the end behavior of the function as $x \to \infty$.**

As $x \to \infty$, the function values slowly increase. That is, $f(x) \to \infty$. The same is true for the functions g and h.

g. **Describe the range of your function.**

The range of each of these functions is all real numbers, $(-\infty, \infty)$.

h. **Does this function have any relative maxima or minima? Explain how you know.**

Since the function values continue to increase, and there are no peaks or valleys in the graph, the function has no relative maxima or minima.

Presentations (5 minutes)

Select three groups of students to present each of the three graphs, projecting each graph through a document camera or copying the graph onto a transparency sheet and displaying on an overhead projector. Ask students to point out the key features they identified in the Opening Exercise on the displayed graph. If students do not mention it, emphasize that the long-term behavior of these functions is they are always increasing, although very slowly.

As representatives from each group make their presentations, record their findings on a chart. This chart can be used to help summarize the lesson and to later display in the classroom.

	$f(x) = \log(x)$	$g(x) = \log_2(x)$	$h(x) = \log_5(x)$
Domain of the Function	$(0, \infty)$	$(0, \infty)$	$(0, \infty)$
Range of the Function	$(-\infty, \infty)$	$(-\infty, \infty)$	$(-\infty, \infty)$
x-intercept	1	1	1
y-intercept	*None*	*None*	*None*
Point with y-value 1	$(10, 1)$	$(2, 1)$	$(5, 1)$
Behavior as $x \to 0$	$f(x) \to -\infty$	$g(x) \to -\infty$	$h(x) \to -\infty$
End Behavior as $x \to \infty$	$f(x) \to \infty$	$g(x) \to \infty$	$h(x) \to \infty$

Discussion (5 minutes)

Debrief the Opening Exercise by asking students to generalize the key features of the graphs $y = \log_b(x)$. If possible, display the graph of all three functions $f(x) = \log(x)$, $g(x) = \log_2(x)$, and $h(x) = \log_5(x)$ together on the same axes during this discussion.

We saw in Lesson 5 that the expression 2^x is defined for all real numbers x; therefore, the range of the function $g(x) = \log_2(x)$ is all real numbers. Likewise, the expressions 10^x and 5^x are defined for all real numbers x, so the range of the functions f and h is all real numbers. Notice that since the range is all real numbers in each case, there must be logarithms that are irrational. We saw examples of such logarithms in Lesson 16.

- What are the domain and range of the logarithm functions?
  - *The domain is the positive real numbers, and the range is all real numbers.*
- What do the three graphs of $f(x) = \log(x)$, $g(x) = \log_2(x)$, and $h(x) = \log_5(x)$ have in common?
  - *The graphs all cross the x-axis at $(1, 0)$.*
  - *None of the graphs intersect the y-axis.*
  - *They have the same end behavior as $x \to \infty$, and they have the same behavior as $x \to 0$.*
  - *The functions all increase quickly for $0 < x < 1$, then increase more and more slowly.*
- What do you expect the graph of $y = \log_3(x)$ to look like?
  - *It will look just like the other graphs, except that it will lie between the graphs of $y = \log_2(x)$ and $y = \log_5(x)$ because $2 < 3 < 5$.*
- What do you expect the graph of $y = \log_b(x)$ to look like for any number $b > 1$?
  - *It will have the same key features of the other graphs of logarithmic functions. As the value of b increases, the graph will flatten as $x \to \infty$.*

Exercise 1 (8 minutes)

Keep students in the same groups for this exercise. Students plot points and sketch the graph of $y = \log_{\frac{1}{b}}(x)$ for $b = 10$, $b = 2$, or $b = 5$, depending on whether they are on the 10-team, the 2-team, or the 5-team. Then, students observe the relationship between their two graphs, justify the relationship using properties of logarithms, and generalize the observed relationship to graphs of $y = \log_b(x)$ and $y = \log_{\frac{1}{b}}(x)$ for $b > 0$, $b \neq 1$.

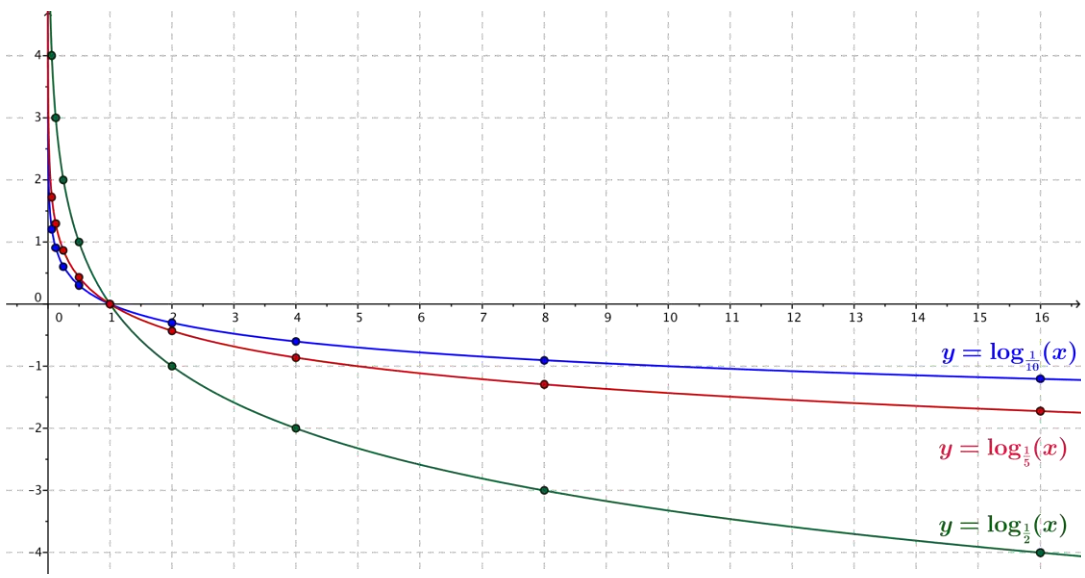
Exercises

1. Graph the points in the table for your assigned function $r(x) = \log_{\frac{1}{10}}(x)$, $s(x) = \log_{\frac{1}{2}}(x)$, or $t(x) = \log_{\frac{1}{5}}(x)$ for $0 < x \leq 16$. Then, sketch a smooth curve through those points, and answer the questions that follow.

10-team $r(x) = \log_{\frac{1}{10}}(x)$	
x	$r(x)$
0.0625	1.20
0.125	0.90
0.25	0.60
0.5	0.30
1	0
2	-0.30
4	-0.60
8	-0.90
16	-1.20

2-team $s(x) = \log_{\frac{1}{2}}(x)$	
x	$s(x)$
0.0625	4
0.125	3
0.25	2
0.5	1
1	0
2	-1
4	-2
8	-3
16	-4

e-team $t(x) = \log_{\frac{1}{5}}(x)$	
x	$t(x)$
0.0625	1.72
0.125	1.29
0.25	0.86
0.5	0.43
1	0
2	-0.43
4	-0.86
8	-1.29
16	-1.72



a. **What is the relationship between your graph in the Opening Exercise and your graph from this exercise?**

The second graph is the reflection of the graph in the Opening Exercise across the x-axis.

b. **Why does this happen? Use the change of base formula to justify what you have observed in part (a).**

Using the change of base formula, we have $\log_{\frac{1}{2}}(x) = \dfrac{\log_2(x)}{\log_2\left(\frac{1}{2}\right)}$.

Since $\log_2\left(\frac{1}{2}\right) = -1$, *we have* $\log_{\frac{1}{2}}(x) = \dfrac{\log_2(x)}{-1}$, *so* $\log_{\frac{1}{2}}(x) = -\log_2(x)$.

Thus, the graph of $y = \log_{\frac{1}{2}}(x)$ *is the reflection of the graph of* $y = \log_2(x)$ *across the x-axis.*

Scaffolding:

Students struggling with the comparison of graphs may find it easier to draw the graphs on transparent plastic sheets and compare them that way.

Discussion (4 minutes)

Ask students from each team to share their graphs results from part (a) of Exercise 1 with the class. During their presentations, complete the chart below.

	$r(x) = \log_{\frac{1}{10}}(x)$	$s(x) = \log_{\frac{1}{2}}(x)$	$t(x) = \log_{\frac{1}{5}}(x)$
Domain of the Function	$(0, \infty)$	$(0, \infty)$	$(0, \infty)$
Range of the Function	$(-\infty, \infty)$	$(-\infty, \infty)$	$(-\infty, \infty)$
x-intercept	1	1	1
y-intercept	None	None	None
Point with y-value -1	$(10, -1)$	$(2, -1)$	$(5, -1)$
Behavior as $x \to 0$	$r(x) \to \infty$	$s(x) \to \infty$	$t(x) \to \infty$
End Behavior as $x \to \infty$	$r(x) \to \infty$	$s(x) \to -\infty$	$t(x) \to -\infty$

Then proceed to hold the following discussion.

- From what we have seen of these three sets of graphs of functions, can we state the relationship between the graphs of $y = \log_b(x)$ and $y = \log_{\frac{1}{b}}(x)$, for $b \neq 1$?
  - *If $b \neq 1$, then the graphs of $y = \log_b(x)$ and $y = \log_{\frac{1}{b}}(x)$ are reflections of each other across the x-axis.*
- Describe the key features of the graph of $y = \log_b(x)$ for $0 < b < 1$.
  - *The graph crosses the x-axis at $(1, 0)$.*
  - *The graph does not intersect the y-axis.*
  - *The graph passes through the point $(b, -1)$.*
  - *As $x \to 0$, the function values increase quickly; that is, $f(x) \to \infty$.*
  - *As $x \to \infty$, the function values continue to decrease; that is, $f(x) \to -\infty$.*
  - *There are no relative maxima or relative minima.*

Exercises 2–3 (6 minutes)

Keep students in the same groups for this set of exercises. Students plot points and sketch the graph of $y = \log_b(bx)$ for $b = 10$, $b = 2$, or $b = 5$, depending on whether they are on the 10-team, the 2-team, or the 5-team. Then, students observe the relationship between their two graphs, justify the relationship using properties of logarithms, and generalize the observed relationship to graphs of $y = \log_b(x)$ and $y = \log_b(x)$ for $b > 0$, $b \neq 1$. If there is time at the end of these exercises, consider using GeoGebra or other dynamic geometry software to demonstrate the property illustrated in Exercise 3 below by graphing $y = \log_2(x)$, $y = \log_2(2x)$, and $y = 1 + \log_2(x)$ on the same axes.

Consider having students graph these functions on the same axes as used in the Opening Exercise.

2. In general, what is the relationship between the graph of a function $y = f(x)$ and the graph of $y = f(kx)$ for a constant k?

The graph of $y = f(kx)$ is a horizontal scaling of the graph of $y = f(x)$.

3. Graph the points in the table for your assigned function $u(x) = \log(10x)$, $v(x) = \log_2(2x)$, or $w(x) = \log_5(5x)$ for $0 < x \leq 16$. Then sketch a smooth curve through those points, and answer the questions that follow.

10-team $u(x) = \log(10x)$	
x	$u(x)$
0.0625	-0.20
0.125	0.10
0.25	0.40
0.5	0.70
1	1
2	1.30
4	1.60
8	1.90
16	2.20

2-team $v(x) = \log_2(2x)$	
x	$v(x)$
0.0625	-3
0.125	-2
0.25	-1
0.5	0
1	1
2	2
4	3
8	4
16	5

5-team $w(x) = \log_5(5x)$	
x	$w(x)$
0.0625	-0.72
0.125	-0.29
0.25	0.14
0.5	0.57
1	1
2	1.43
4	1.86
8	2.29
16	2.72

MP.7

a. Describe a transformation that takes the graph of your team's function in this exercise to the graph of your team's function in the Opening Exercise.

The graph produced in this exercise is a vertical translation of the graph from the Opening Exercise by one unit upward.

b. Do your answers to Exercise 2 and part (a) agree? If not, use properties of logarithms to justify your observations in part (a).

The answers to Exercise 2 and part (a) do not appear to agree. However, because $\log_b(bx) = \log_b(b) + \log_b(x) = 1 + \log_b(x)$***, the graph of*** $y = \log_b(bx)$ ***and the graph of*** $y = 1 + \log_b(x)$ ***coincide.***

Closing (3 minutes)

Ask students to respond to these questions in writing or orally to a partner.

- In which quadrants is the graph of the function $f(x) = \log_b(x)$ located?
  - *The first and fourth quadrants*
- When $b > 1$, for what values of x are the values of the function $f(x) = \log_b(x)$ negative?
  - *When* $b > 1$, $f(x) = \log_b(x)$ *is negative for* $0 < x < 1$.
- When $0 < b < 1$, for what values of x are the values of the function $f(x) = \log_b(x)$ negative?
  - *When* $0 < b < 1$, $f(x) = \log_b(x)$ *is negative for* $x > 1$.
- What are the key features of the graph of a logarithmic function $f(x) = \log_b(x)$ when $b > 1$?
  - *The domain of the function is all positive real numbers, and the range is all real numbers. The* x*-intercept is* 1*, the graph passes through* $(b, 1)$ *and there is no* y*-intercept. As* $x \to 0$, $f(x) \to -\infty$ *quickly, and as* $x \to \infty$, $f(x) \to \infty$ *slowly.*
- What are the key features of the graph of a logarithmic function $f(x) = \log_b(x)$ when $0 < b < 1$?
  - *The domain of the function is the positive real numbers, and the range is all real numbers. The* x*-intercept is* 1*, the graph passes through* $(b, -1)$*, and there is no* y*-intercept. As* $x \to 0$, $f(x) \to \infty$ *quickly, and as* $x \to \infty$, $f(x) \to -\infty$ *slowly.*

Lesson Summary

The function $f(x) = \log_b(x)$ **is defined for irrational and rational numbers. Its domain is all positive real numbers. Its range is all real numbers.**

The function $f(x) = \log_b(x)$ **goes to negative infinity as** x **goes to zero. It goes to positive infinity as** x **goes to positive infinity.**

The larger the base b**, the more slowly the function** $f(x) = \log_b(x)$ **increases.**

By the change of base formula, $\log_{\frac{1}{b}}(x) = -\log_b(x)$.

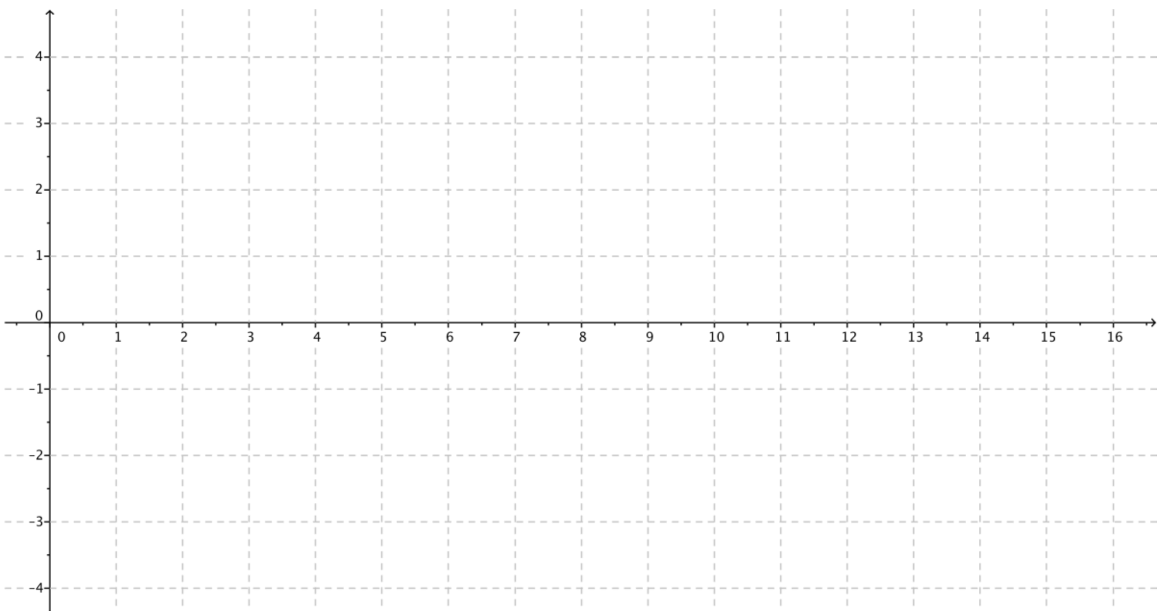
Exit Ticket (5 minutes)

Name ____ Date ____________________

Lesson 17: Graphing the Logarithm Function

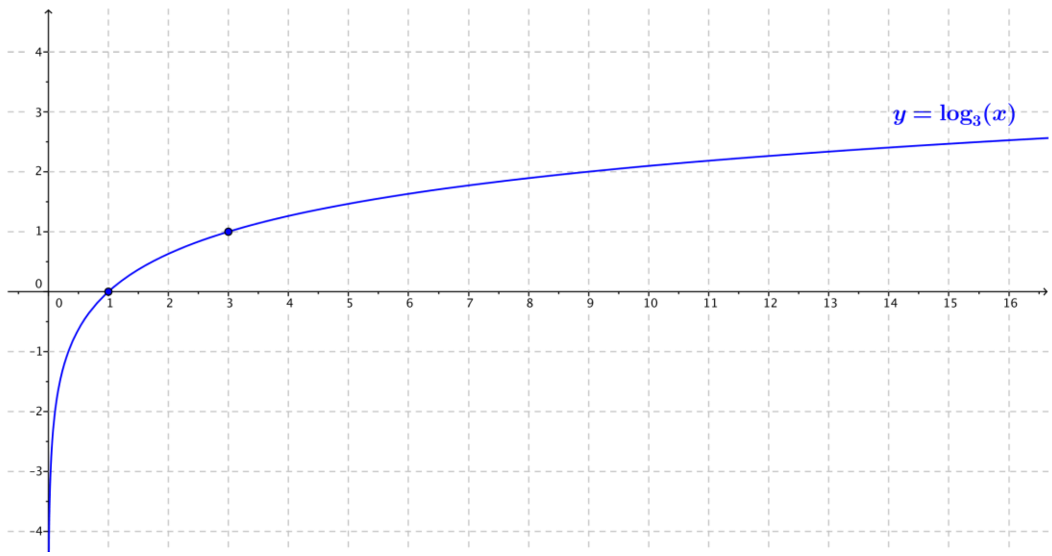
Exit Ticket

Graph the function $f(x) = \log_3(x)$ without using a calculator, and identify its key features.



Exit Ticket Sample Solutions

Graph the function $f(x) = \log_3(x)$ without using a calculator, and identify its key features.



Key features:

The domain is $(0, \infty)$.

The range is all real numbers.

End behavior:

As $x \to 0$, $f(x) \to -\infty$.

As $x \to \infty$, $f(x) \to \infty$.

Intercepts:

x-intercept: There is one x-intercept at 1.

y-intercept: The graph does not cross the y-axis.

The graph passes through $(3, 1)$.

Problem Set Sample Solutions

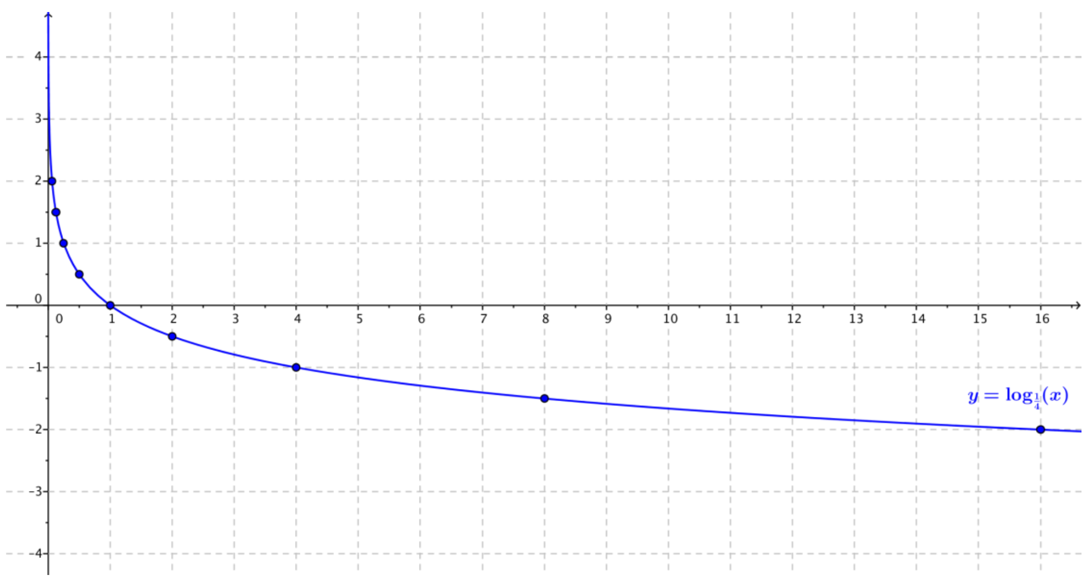
For the Problem Set, students need graph paper. They should not use calculators or other graphing technology, except where specified in extension Problems 11 and 12. In Problems 2 and 3, students compare different representations of logarithmic functions. Problems 4–6 continue the reasoning from the lesson in which students observed logarithmic properties through the transformations of logarithmic graphs.

Fluency problems 9–10 are a continuation of work done in Algebra I and have been placed in this lesson so that students recall concepts required in Lesson 19. Similar review problems occur in the next lesson.

1. The function $Q(x) = \log_b(x)$ has function values in the table at right.

a. Use the values in the table to sketch the graph of $y = Q(x)$.

x	$Q(x)$
0.1	1.66
0.3	0.87
0.5	0.50
1.00	0.00
2.00	-0.50
4.00	-1.00
6.00	-1.29
10.00	-1.66
12.00	-1.79



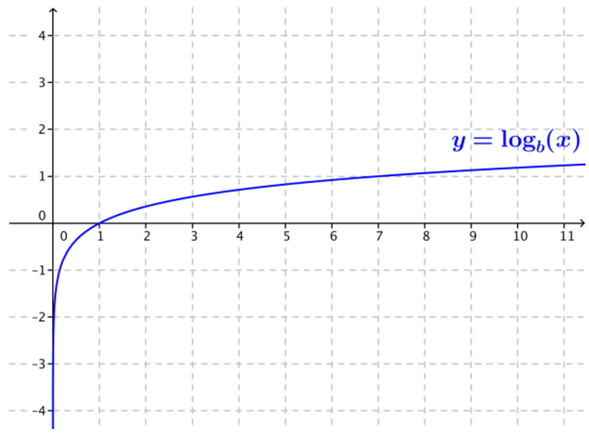
b. What is the value of b in $Q(x) = \log_b(x)$? Explain how you know.

Because the point $(4, -1)$ is on the graph of $y = Q(x)$, we know $\log_b(4) = -1$, so $b^{-1} = 4$. It follows that $b = \frac{1}{4}$.

c. Identify the key features in the graph of $y = Q(x)$.

Because $0 < b < 1$, the function values approach ∞ as $x \to 0$, and the function values approach $-\infty$ as $x \to \infty$. There is no y-intercept, and the x-intercept is 1. The domain of the function is $(0, \infty)$, the range is $(-\infty, \infty)$, and the graph passes through $(b, 1)$.

2. Consider the logarithmic functions $f(x) = \log_b(x)$, $g(x) = \log_5(x)$, where b is a positive real number, and $b \neq 1$. The graph of f is given at right.



   a. Is $b > 5$, or is $b < 5$? Explain how you know.

   Since $f(7) = 1$, and $g(7) \approx 1.21$, the graph of f lies below the graph of g for $x \geq 1$. This means that b is larger than 5, so we have $b > 5$. (Note: The actual value of b is 7.)

   b. Compare the domain and range of functions f and g.

   Functions f and g have the same domain, $(0, \infty)$, and the same range, $(-\infty, \infty)$.

   c. Compare the x-intercepts and y-intercepts of f and g.

   Both f and g have an x-intercept at 1 and no y-intercepts.

   d. Compare the end behavior of f and g.

   As $x \to \infty$, both $f(x) \to \infty$ and $g(x) \to \infty$.

3. Consider the logarithmic functions $f(x) = \log_b(x)$ and $g(x) = \log_{\frac{1}{2}}(x)$, where b is a positive real number and $b \neq 1$. A table of approximate values of f is given below.

x	$f(x)$
$\frac{1}{4}$	0.86
$\frac{1}{2}$	0.43
1	0
2	-0.43
4	-0.86

   a. Is $b > \frac{1}{2}$, or is $b < \frac{1}{2}$? Explain how you know.

   Since $g(2) = -1$, and $f(2) \approx -0.43$, the graph of f lies above the graph of g for $x \geq 1$. This means that b is closer to 0 than $\frac{1}{2}$ is, so we have $b < \frac{1}{2}$. (Note: The actual value of b is $\frac{1}{5}$.)

   b. Compare the domain and range of functions f and g.

   Functions f and g have the same domain, $(0, \infty)$, and the same range, $(-\infty, \infty)$.

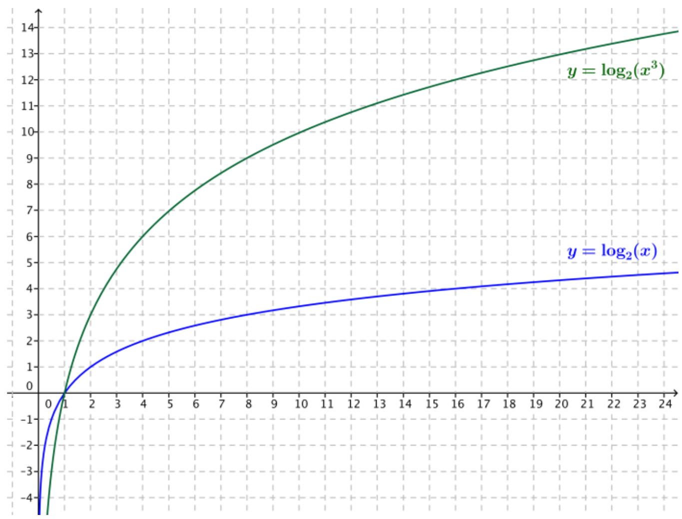
c. Compare the x-intercepts and y-intercepts of f and g.

Both f and g have an x-intercept at 1 and no y-intercepts.

d. Compare the end behavior of f and g.

As $x \to \infty$, both $f(x) \to -\infty$ and $g(x) \to -\infty$.

4. On the same set of axes, sketch the functions $f(x) = \log_2(x)$ and $g(x) = \log_2(x^3)$.



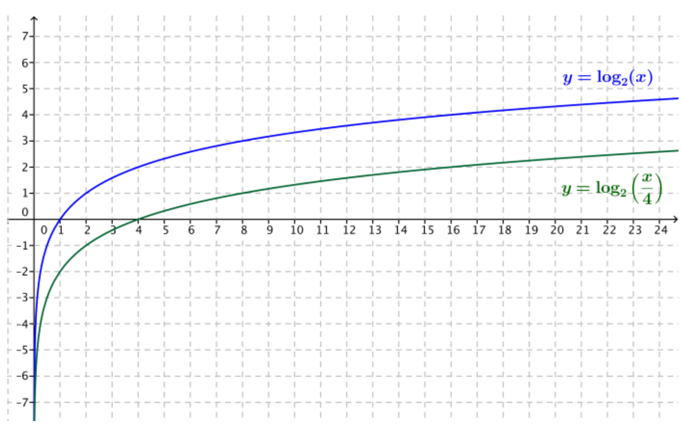
a. Describe a transformation that takes the graph of f to the graph of g.

The graph of g is a vertical scaling of the graph of f by a factor of 3.

b. Use properties of logarithms to justify your observations in part (a).

Using properties of logarithms, we know that $g(x) = \log_2(x^3) = 3\log_2(x) = 3\,f(x)$. Thus, the graph of f is a vertical scaling of the graph of g by a factor of 3.

5. On the same set of axes, sketch the functions $f(x) = \log_2(x)$ and $g(x) = \log_2\left(\frac{x}{4}\right)$.



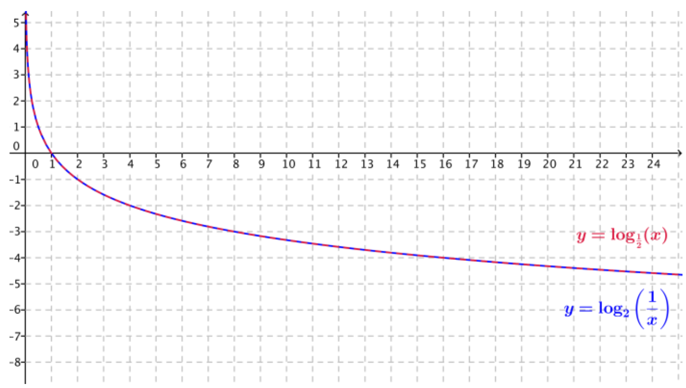
a. Describe a transformation that takes the graph of f to the graph of g.

The graph of g is the graph of f translated down by 2 units.

b. Use properties of logarithms to justify your observations in part (a).

Using properties of logarithms, $g(x) = \log_2\left(\frac{x}{4}\right) = \log_2(x) - \log_2(4) = f(x) - 2$. Thus, the graph of g is a translation of the graph of f down 2 units.

6. On the same set of axes, sketch the functions $f(x) = \log_{\frac{1}{2}}(x)$ and $g(x) = \log_2\left(\frac{1}{x}\right)$.



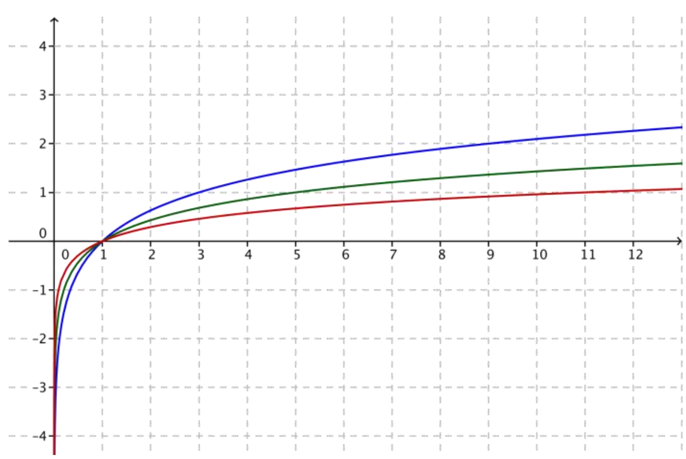
a. Describe a transformation that takes the graph of f to the graph of g.

These two graphs coincide, so the identity transformation takes the graph of f to the graph of g.

b. Use properties of logarithms to justify your observations in part (a).

If $\log_{\frac{1}{2}}(x) = y$, then $\left(\frac{1}{2}\right)^y = x$, so $\frac{1}{x} = 2^y$. Then, $y = \log_2$, so $\log_2\left(\frac{1}{x}\right) = \log_{\frac{1}{2}}(x)$; thus, $g(x) = f(x)$ for all $x > 0$.

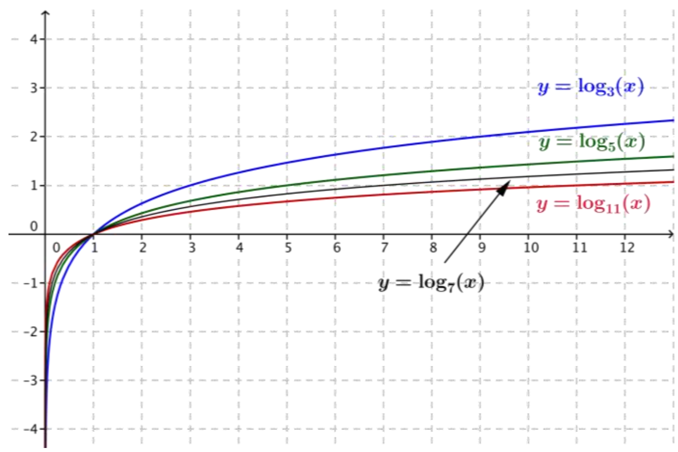
7. The figure below shows graphs of the functions $f(x) = \log_3(x)$, $g(x) = \log_5(x)$, and $h(x) = \log_{11}(x)$.



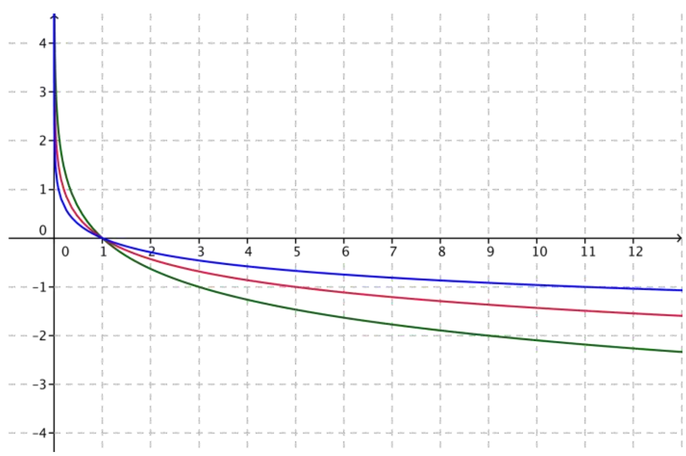
a. **Identify which graph corresponds to which function. Explain how you know.**

The top graph (in blue) is the graph of $f(x) = \log_3(x)$, the middle graph (in green) is the graph of $g(x) = \log_5(x)$, and the lower graph (in red) is the graph of $h(x) = \log_{11}(x)$. We know this because the blue graph passes through the point $(3, 1)$, the green graph passes through the point $(5, 1)$, and the red graph passes through the point $(11, 1)$. We also know that the higher the value of the base b, the flatter the graph, so the graph of the function with the largest base, 11, must be the red graph on the bottom, and the graph of the function with the smallest base, 3, must be the blue graph on the top.

b. **Sketch the graph of $k(x) = \log_7(x)$ on the same axes.**



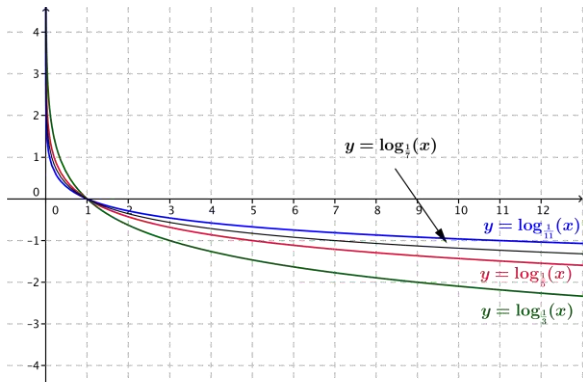
8. **The figure below shows graphs of the functions $f(x) = \log_{\frac{1}{3}}(x)$, $g(x) = \log_{\frac{1}{5}}(x)$, and $h(x) = \log_{\frac{1}{11}}(x)$.**



a. **Identify which graph corresponds to which function. Explain how you know.**

The top graph (in blue) is the graph of $h(x) = \log_{\frac{1}{11}}(x)$, the middle graph (in red) is the graph of $g(x) = \log_{\frac{1}{5}}(x)$, and the lower graph is the graph of $f(x) = \log_{\frac{1}{3}}(x)$. We know this because the blue graph passes through the point $(11, -1)$, the red graph passes through the point $(5, -1)$, and the green graph passes through the point $(3, -1)$.

b. Sketch the graph of $k(x) = \log_{\frac{1}{7}}(x)$ on the same axes.



9. For each function f, find a formula for the function h in terms of x. Part (a) has been done for you.

a. If $f(x) = x^2 + x$, find $h(x) = f(x+1)$.

$$\begin{aligned} h(x) &= f(x+1) \\ &= (x+1)^2 + (x+1) \\ &= x^2 + 3x + 2 \end{aligned}$$

b. If $f(x) = \sqrt{x^2 + \frac{1}{4}}$, find $h(x) = f\left(\frac{1}{2}x\right)$.

$$h(x) = \frac{1}{2}\sqrt{x^2 + 1}$$

c. If $f(x) = \log(x)$, find $h(x) = f\left(\sqrt[3]{10x}\right)$ when $x > 0$.

$$h(x) = \frac{1}{3} + \frac{1}{3}\log(x)$$

d. If $f(x) = 3^x$, find $h(x) = f(\log_3(x^2 + 3))$.

$$h(x) = x^2 + 3$$

e. If $f(x) = x^3$, find $h(x) = f\left(\frac{1}{x^3}\right)$ when $x \neq 0$.

$$h(x) = \frac{1}{x^6}$$

f. If $f(x) = x^3$, find $h(x) = f\left(\sqrt[3]{x}\right)$.

$$h(x) = x$$

g. If $f(x) = \sin(x)$, find $h(x) = f\left(x + \frac{\pi}{2}\right)$.

$$h(x) = \sin\left(x + \frac{\pi}{2}\right)$$

h. If $f(x) = x^2 + 2x + 2$, find $h(x) = f(\cos(x))$.

$$h(x) = (\cos(x))^2 + 2\cos(x) + 2$$

10. For each of the functions f and g below, write an expression for (i) $f(g(x))$, (ii) $g(f(x))$, and (iii) $f(f(x))$ in terms of x. Part (a) has been done for you.

a. $f(x) = x^2$, $g(x) = x + 1$

i. $f(g(x)) = f(x+1)$
$= (x+1)^2$

ii. $g(f(x)) = g(x^2)$
$= x^2 + 1$

iii. $f(f(x)) = f(x^2)$
$= (x^2)^2$
$= x^4$

b. $f(x) = \frac{1}{4}x - 8$, $g(x) = 4x + 1$

i. $f(g(x)) = x - \frac{31}{4}$

ii. $g(f(x)) = x - 31$

iii. $f(f(x)) = \frac{1}{16}x - 10$

c. $f(x) = \sqrt[3]{x+1}$, $g(x) = x^3 - 1$

i. $f(g(x)) = x$

ii. $g(f(x)) = x$

iii. $f(f(x)) = \sqrt[3]{\sqrt[3]{x+1} + 1}$

d. $f(x) = x^3$, $g(x) = \frac{1}{x}$

i. $f(g(x)) = \frac{1}{x^3}$

ii. $g(f(x)) = \frac{1}{x^3}$

iii. $f(f(x)) = x^9$

e. $f(x) = |x|$, $g(x) = x^2$

i. $f(g(x)) = |x^2| = x^2$

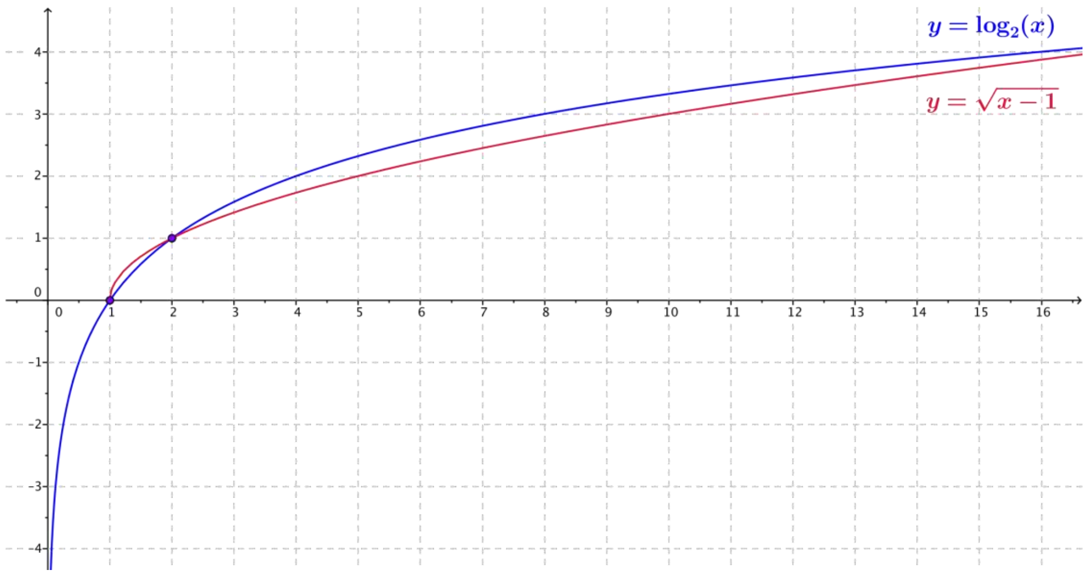
ii. $g(f(x)) = (|x|)^2 = x^2$

iii. $f(f(x)) = |x|$

Extension:

11. Consider the functions $f(x) = \log_2(x)$ and $(x) = \sqrt{x-1}$.

a. Use a calculator or other graphing utility to produce graphs of $f(x) = \log_2(x)$ and $g(x) = \sqrt{x-1}$ for $x \leq 17$.



b. Compare the graph of the function $f(x) = \log_2(x)$ with the graph of the function $g(x) = \sqrt{x-1}$. Describe the similarities and differences between the graphs.

They are not the same, but they have a similar shape when $x \geq 1$. Both graphs pass through the points $(1, 0)$ and $(2, 1)$. Both functions appear to approach infinity slowly as $x \to \infty$.

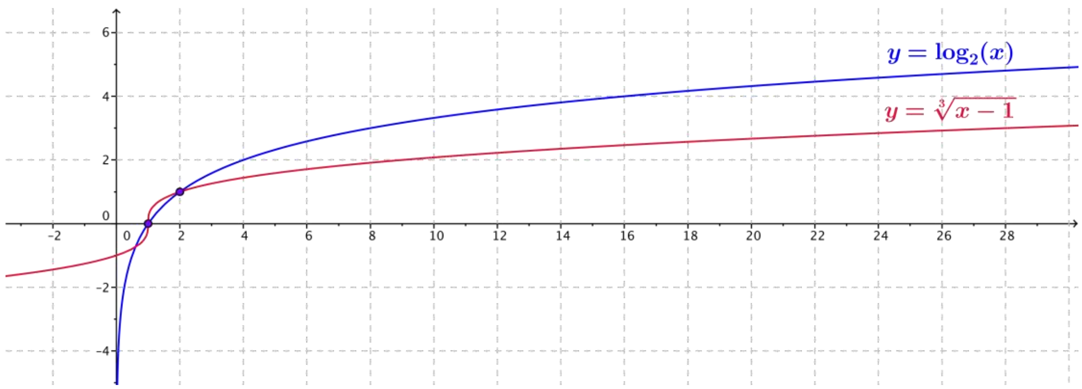
The graph of $f(x) = \log_2(x)$ lies below the graph of $g(x) = \sqrt{x-1}$ on the interval $(1, 2)$, and the graph of f appears to lie above the graph of g on the interval $(2, \infty)$. The logarithm function f is defined for $x > 0$, and the radical function g is defined for $x \geq 1$. Both functions appear to slowly approach infinity as $x \to \infty$.

c. Is it always the case that $\log_2(x) > \sqrt{x-1}$ for $x > 2$?

No, for $2 < x \leq 19$, $\log_2(x) > \sqrt{x-1}$. Between 19 and 20, the graphs cross again, and we have $\sqrt{x-1} > \log_2(x)$ for $x \geq 20$.

12. Consider the functions $f(x) = \log_2(x)$ and $(x) = \sqrt[3]{x-1}$.

   a. Use a calculator or other graphing utility to produce graphs of $f(x) = \log_2(x)$ and $h(x) = \sqrt[3]{x-1}$ for $x \leq 28$.



   b. Compare the graph of the function $f(x) = \log_2(x)$ with the graph of the function $h(x) = \sqrt[3]{x-1}$. Describe the similarities and differences between the graphs.

   They are not the same, but they have a similar shape when $x \geq 1$. Both graphs pass through the points $(1, 0)$ and $(2, 1)$. Both functions appear to approach infinity slowly as $x \to \infty$.

   The graph of $f(x) = \log_2(x)$ lies below the graph of $h(x) = \sqrt[3]{x-1}$ on the interval $(1, 2)$, and the graph of f appears to lie above the graph of h on the interval $(2, \infty)$. The logarithm function f is defined for $x > 0$, and the radical function h is defined for all real numbers x. Both functions appear to approach infinity slowly as $x \to \infty$.

   c. Is it always the case that $\log_2(x) > \sqrt[3]{x-1}$ for $x > 2$?

   No, if we extend the viewing window on the calculator, we see that the graphs cross again between 983 and 984. Thus, $\log_2(x) > \sqrt[3]{x-1}$ for $2 < x \leq 983$, and $\log_2(x) < \sqrt[3]{x-1}$ for $x \geq 984$.

Lesson 18: Graphs of Exponential Functions and Logarithmic Functions

Student Outcomes

- Students compare the graph of an exponential function to the graph of its corresponding logarithmic function.
- Students note the geometric relationship between the graph of an exponential function and the graph of its corresponding logarithmic function.

Lesson Notes

In the previous lesson, students practiced graphing transformed logarithmic functions and observed the effects of the logarithmic properties in the graphs. In this lesson, students graph the logarithmic functions along with their corresponding exponential functions. Be careful to ensure that the scale is the same on both axes so that the geometric relationship between the graph of the exponential function and the graph of the logarithmic function is apparent. Part of the focus of the lesson is for students to begin seeing that these functions are the inverses of each other—but without the teacher actually saying it yet. Encourage students to draw the graphs carefully so that they can see that the two graphs are reflections of each other about the diagonal. The asymptotic nature of the two graphs should be discussed. (**F-IF.B.4**, **F-IF.C.7e**) The teacher is encouraged to consider using graphing software such as GeoGebra.

Classwork

Opening Exercise (5 minutes)

Allow students to work in pairs or small groups on the following exercise in which they graph a few points on the curve $y = 2^x$, reflect these points over the diagonal line with the equation $y = x$, and analyze the result.

Scaffolding:

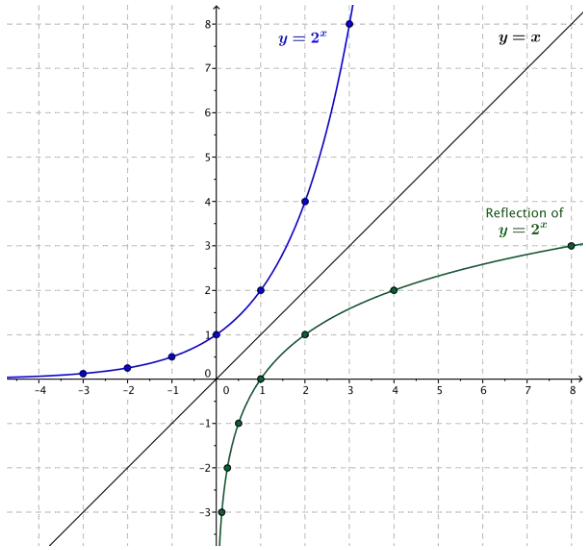
- Model the process of reflecting a set of points, such as $\triangle ABC$ with vertices $A(-3, 2)$, $B(-3, 7)$, and $C(2, 7)$, over the diagonal line $y = x$ before asking students to do the same.
- After the graph of $y = 2^x$ and its reflection are shown, ask advanced students, "If the first graph represents the points that satisfy $y = 2^x$, then what equation do the points on the reflected graph satisfy?"

Opening Exercise

Complete the following table of values of the function $f(x) = 2^x$. We want to sketch the graph of $y = f(x)$ and then reflect that graph across the diagonal line with equation $y = x$.

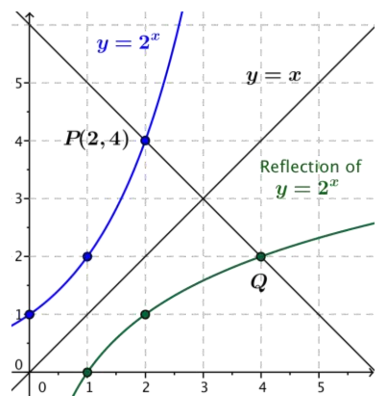
x	$y = 2^x$	Point (x, y) on the Graph of $y = 2^x$
-3	$\frac{1}{8}$	$\left(-3, \frac{1}{8}\right)$
-2	$\frac{1}{4}$	$\left(-2, \frac{1}{4}\right)$
-1	$\frac{1}{2}$	$\left(-1, \frac{1}{2}\right)$
0	1	$(0, 1)$
1	2	$(1, 2)$
2	4	$(2, 4)$
3	8	$(3, 8)$

On the set of axes below, plot the points from the table and sketch the graph of $y = 2^x$. Next, sketch the diagonal line with equation $y = x$, and then reflect the graph of $y = 2^x$ across the line.



Discussion (4 minutes)

Use the following discussion to reinforce the process by which a point is reflected across the diagonal line given by $y = x$ and the reasoning for why reflecting points on an exponential curve produces points on the corresponding logarithmic curve.



- How do we find the reflection of the point $P(2,4)$ across the line given by $y = x$?
  - *Point $P(2,4)$ is reflected to point Q on the line through $(2,4)$ that is perpendicular to the line given by $y = x$ so that points P and Q are equidistant from the diagonal line.*
- What is the slope of the line through P and Q? Explain how you know. (*Draw the figure to the right.)*
  - *The slope of $\overleftrightarrow{PQ}$ is -1 because this line is perpendicular to the diagonal line that has slope 1.*
- We know that P and Q are the same distance from the diagonal line. What are the coordinates of the point Q?
  - *Point Q has coordinates $(4,2)$.*
- What are the coordinates of the reflection of the point $(1,2)$ across the line given by $y = x$?
  - *The reflection of the point $(1,2)$ is the point $(2,1)$.*
- What are the coordinates of the reflection of the point (a,b) across the line given by $y = x$?
  - *When we reflect about the line with equation $y = x$, we actually switch the axes themselves by folding the plane along this line. Therefore, the reflection of the point (a,b) is the point (b,a).*

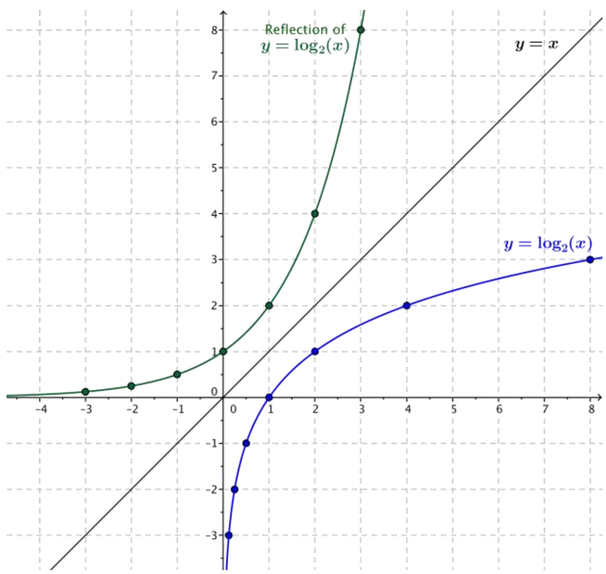
Exercise 1 (7 minutes)

Exercises

1. **Complete the following table of values of the function $g(x) = \log_2(x)$. We want to sketch the graph of $y = g(x)$ and then reflect that graph across the diagonal line with equation $y = x$.**

x	$y = \log_2(x)$	Point (x, y) on the graph of $y = \log_2(x)$
$-\frac{1}{8}$	-3	$\left(\frac{1}{8}, -3\right)$
$-\frac{1}{4}$	-2	$\left(\frac{1}{4}, -2\right)$
$-\frac{1}{2}$	-1	$\left(\frac{1}{2}, -1\right)$
1	0	$(1, 0)$
2	1	$(2, 1)$
4	2	$(4, 2)$
8	3	$(8, 3)$

On the set of axes below, plot the points from the table and sketch the graph of $y = \log_2(x)$. Next, sketch the diagonal line with equation $y = x$, and then reflect the graph of $y = \log_2(x)$ across the line.



Discussion (5 minutes)

This discussion makes clear that the reflection of the graph of an exponential function is the graph of a corresponding logarithmic function, and vice-versa.

- How do we find the reflection of the point $P(2, 4)$ across the line given by $y = x$?
- What similarities do you notice about this exercise and the Opening Exercise?

MP.7

  - *The points $(0, 1)$, $(2, 1)$, and $(4, 2)$ on the logarithmic graph are the reflections of the points we plotted on this first graph of $f(x) = 2^x$ across the diagonal line.*

- *The point $(2, 4)$ on the graph of the exponential function is the reflection across the diagonal line of the point $(4, 2)$ on the graph of the logarithm, and the point $(4, 2)$ on the graph of the logarithm function is the reflection across the diagonal line of the point $(2, 4)$ on the graph of the exponential function.*
- *The point (a, b) on the graph of the exponential function is the reflection across the diagonal line of the point (b, a) on the graph of the logarithm, and the point (b, a) on the graph of the logarithm function is the reflection across the diagonal line of the point (a, b) on the graph of the exponential function.*
- *The graphs of the functions $f(x) = 2^x$ and $g(x) = \log_2(x)$ are reflections of each other across the diagonal line given by $y = x$.*

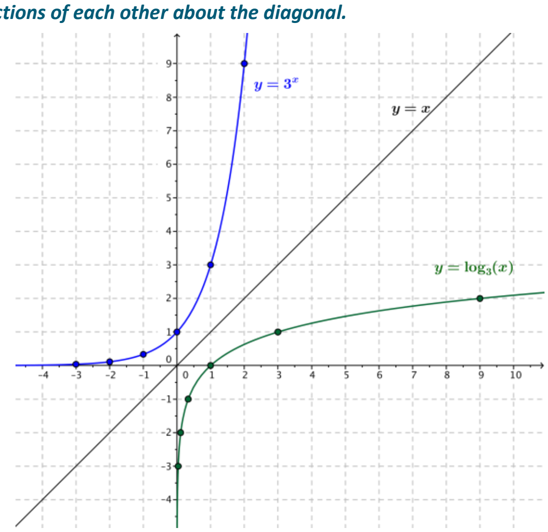
- Why does this happen? How does the definition of the logarithm tell us that if (a, b) is a point on the exponential graph, then (b, a) is a point on the logarithmic graph? How does the definition of the logarithm tell us that if (b, a) is a point on the logarithmic graph, then (a, b) is a point on the exponential graph?
  - *If (a, b) is a point on the graph of the exponential function $f(x) = 2^x$, then*
$$f(a) = 2^a$$
$$b = 2^a$$
$$\log_2(b) = a.$$
  - *So, the point (b, a) is on the graph of the logarithmic function $g(x) = \log_2(x)$.*
    Likewise, if (b, a) is a point on the graph of the logarithmic function $g(x) = \log_2(x)$, then
$$g(b) = \log_2(b)$$
$$\log_2(b) = a$$
$$2^a = b.$$
- So, the point (a, b) is on the graph of the exponential function $f(x) = 2^x$.

Exercise 2 (5 minutes)

MP.3

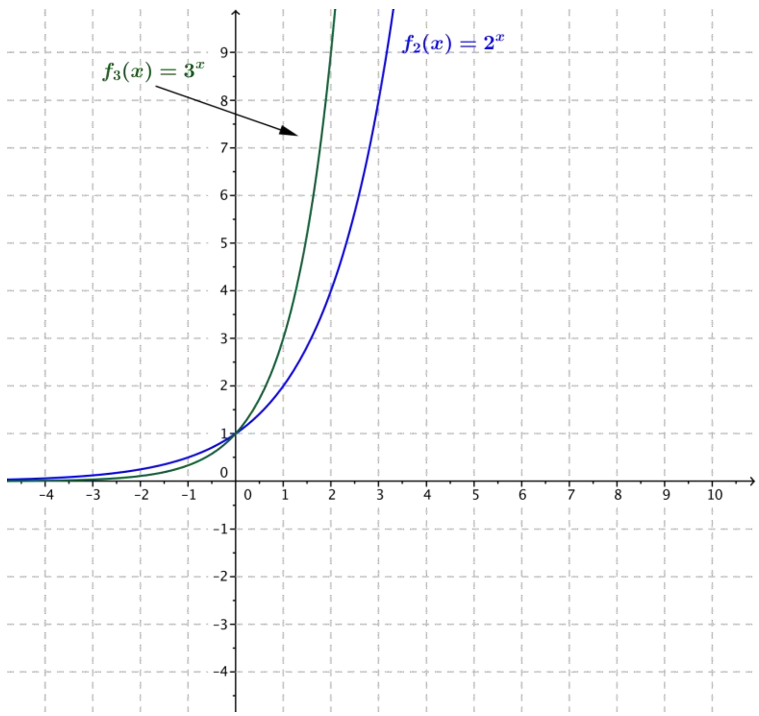
2. Working independently, predict the relation between the graphs of the functions $f(x) = 3^x$ and $g(x) = \log_3(x)$. Test your predictions by sketching the graphs of these two functions. Write your prediction in your notebook, provide justification for your prediction, and compare your prediction with that of your neighbor.

The graphs will be reflections of each other about the diagonal.



Exercises 3–4 (10 minutes)

3. Now let's compare the graphs of the functions $f_2(x) = 2^x$ and $f_3(x) = 3^x$. Sketch the graphs of the two exponential functions on the same set of axes; then, answer the questions below.



a. **Where do the two graphs intersect?**

The two graphs intersect at the point $(0, 1)$.

b. **For which values of x is $2^x < 3^x$?**

If $x > 0$, then $2^x < 3^x$.

c. **For which values of x is $2^x > 3^x$?**

If $x < 0$, then $2^x > 3^x$.

d. **What happens to the values of the functions f_2 and f_3 as $x \to \infty$?**

As $x \to \infty$, both $f_2(x) \to \infty$ and $f_3(x) \to \infty$.

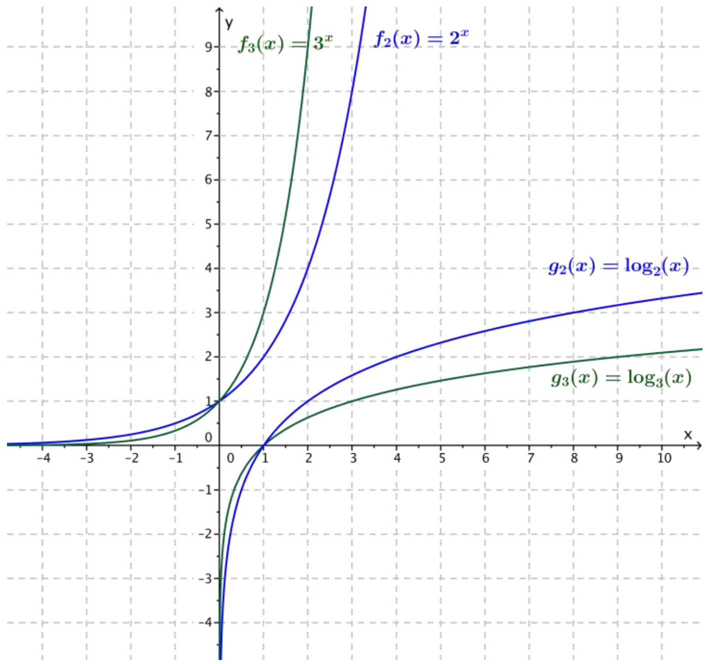
e. **What happens to the values of the functions f_2 and f_3 as $x \to -\infty$?**

As $x \to -\infty$, both $f_2(x) \to 0$ and $f_3(x) \to 0$.

f. **Does either graph ever intersect the x-axis? Explain how you know.**

No. For every value of x, we know $2^x \neq 0$ and $3^x \neq 0$.

4. Add sketches of the two logarithmic functions $g_2(x) = \log_2(x)$ and $g_3(x) = \log_3(x)$ to the axes with the graphs of the exponential functions from Exercise 3; then, answer the questions below.



a. Where do the two logarithmic graphs intersect?

The two graphs intersect at the point $(1, 0)$.

b. For which values of x is $\log_2(x) < \log_3(x)$?

If $x < 1$, then $\log_2(x) < \log_3(x)$.

c. For which values of x is $\log_2(x) > \log_3(x)$?

If $x > 1$, then $\log_2(x) > \log_3(x)$.

d. What happens to the values of the functions g_2 and g_3 as $x \to \infty$?

As $x \to \infty$, both $g_2(x) \to \infty$ and $g_3(x) \to \infty$.

e. What happens to the values of the functions g_2 and g_3 as $x \to 0$?

As $x \to 0$, both $g_2(x) \to -\infty$ and $g_3(x) \to -\infty$.

f. Does either graph ever intersect the y-axis? Explain how you know.

No. Logarithms are only defined for positive values of x.

g. Describe the similarities and differences in the behavior of $f_2(x)$ and $g_2(x)$ as $x \to \infty$.

As $x \to \infty$, both $f_2(x) \to \infty$ and $g_2(x) \to \infty$; however, the exponential function gets very large very quickly, and the logarithmic function gets large rather slowly.

Closing (4 minutes)

Ask students to summarize the key points of the lesson with a partner or in writing. Make sure that students have used the specific examples from the lesson to create some generalizations about the graphs of exponential and logarithmic functions.

MP.8

- Graphical analysis was done for the functions $f_2(x) = 2^x$ and $f_3(x) = 3^x$. What generalizations can we make about functions of the form $f(x) = a^x$ for $a > 1$?
    - *The function values increase to infinity as $x \to \infty$. The function values get closer to 0 as $x \to -\infty$.*
- Graphical analysis was done for functions $g_2(x) = \log_2(x)$ and $g_3(x) = \log_3(x)$. What generalizations can we make about functions of the form $g(x) = \log_b(x)$ for $b > 1$?
    - *The function values increase to infinity as $x \to \infty$. The function values approach $-\infty$ as $x \to 0$.*
- How are the graphs of the functions $f(x) = 2^x$ and $g(x) = \log_2(x)$ related?
    - *They are reflections of each other across the diagonal line given by $y = x$.*
- What can we say, in general, about the graphs of $f(x) = b^x$ and $g(x) = \log_b(x)$ where $b > 1$?
    - *They are reflections of each other about the diagonal line with equation $y = x$.*

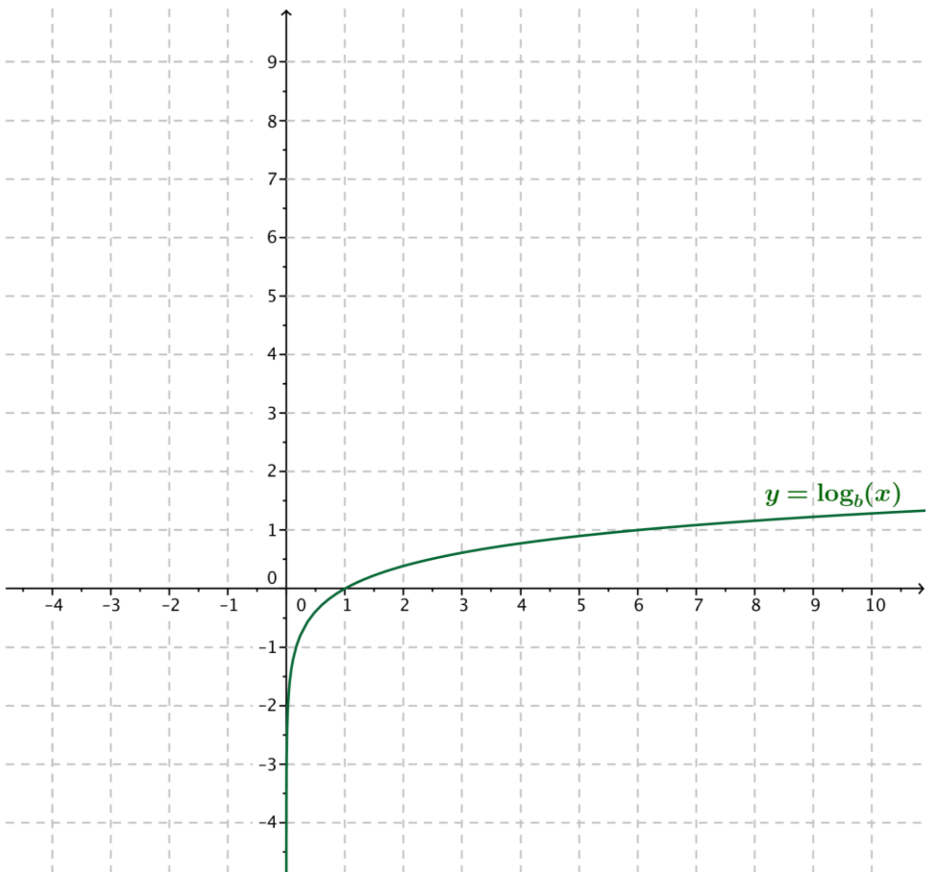
Exit Ticket (5 minutes)

Name ____ Date ____________________

Lesson 18: Graphs of Exponential Functions and Logarithmic Functions

Exit Ticket

The graph of a logarithmic function $g(x) = \log_b(x)$ is shown below.

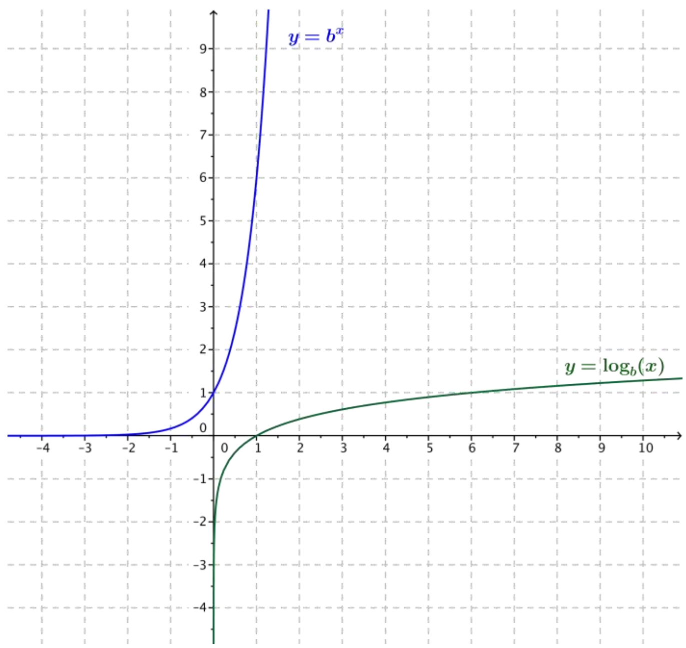


a. Explain how to find points on the graph of the function $f(x) = b^x$.

b. Sketch the graph of the function $f(x) = b^x$ on the same axes.

Exit Ticket Sample Solutions

The graph of a logarithmic function $g(x) = \log_b(x)$ is shown below.



a. **Explain how to find points on the graph of the function $f(x) = b^x$.**

A point (x, y) is on the graph of f if the corresponding point (y, x) is on the graph of g.

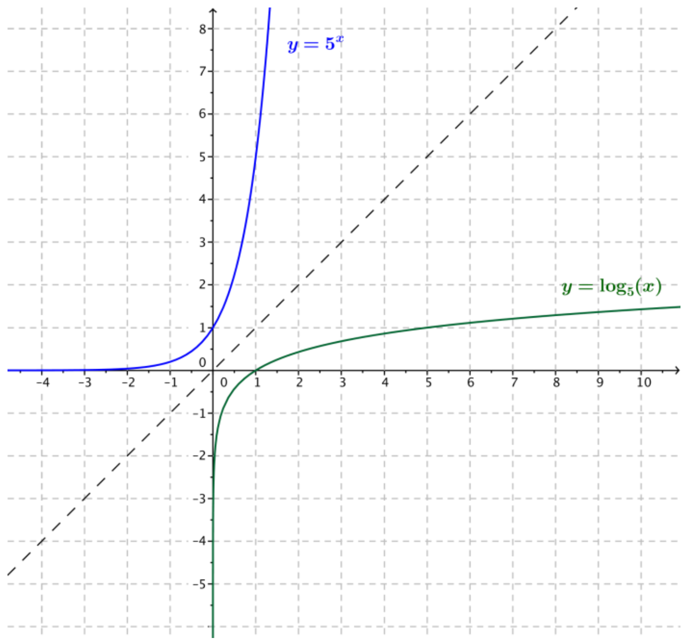
b. **Sketch the graph of the function $f(x) = b^x$ on the same axes.**

See graph above.

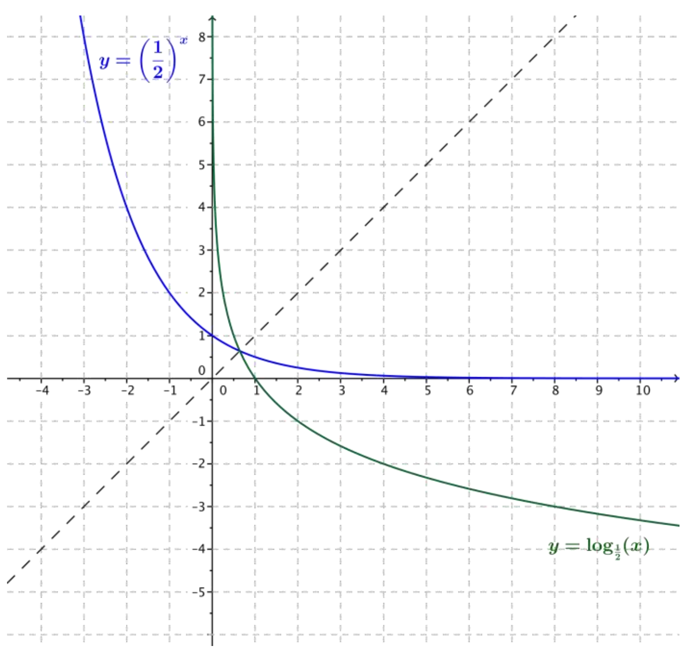
Problem Set Sample Solutions

Problems 5–7 serve to review the process of computing $f(g(x))$ for given functions f and g in preparation for work with inverses of functions in Lesson 19.

1. **Sketch the graphs of the functions $f(x) = 5^x$ and $g(x) = \log_5(x)$.**



2. **Sketch the graphs of the functions $f(x) = \left(\frac{1}{2}\right)^x$ and $g(x) = \log_{\frac{1}{2}}(x)$.**



3. Sketch the graphs of the functions $f_1(x) = \left(\frac{1}{2}\right)^x$ and $f_2(x) = \left(\frac{3}{4}\right)^x$ on the same sheet of graph paper, and answer the following questions.

a. Where do the two exponential graphs intersect?

The graphs intersect at the point $(0, 1)$.

b. For which values of x is $\left(\frac{1}{2}\right)^x < \left(\frac{3}{4}\right)^x$?

If $x > 0$, *then* $\left(\frac{1}{2}\right)^x < \left(\frac{3}{4}\right)^x$.

c. For which values of x is $\left(\frac{1}{2}\right)^x > \left(\frac{3}{4}\right)^x$?

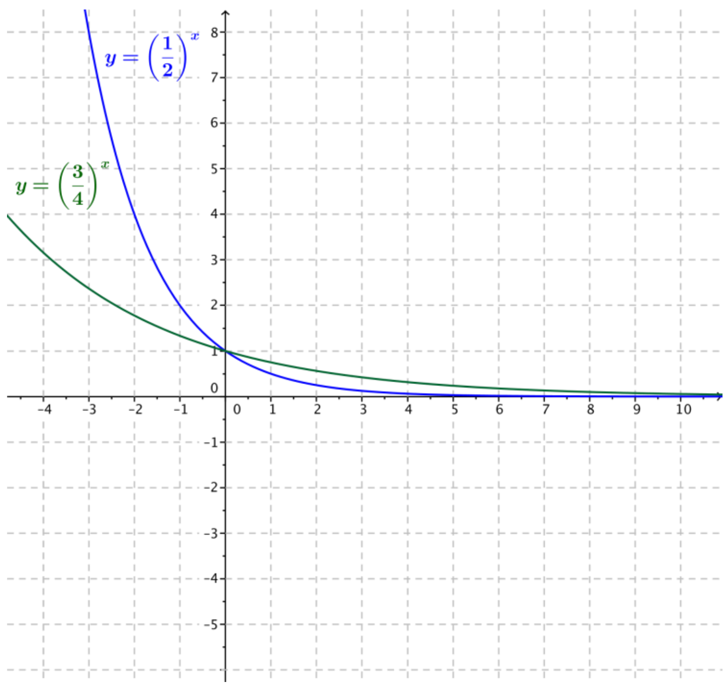
If $x < 0$, *then* $\left(\frac{1}{2}\right)^x > \left(\frac{3}{4}\right)^x$.

d. What happens to the values of the functions f_1 and f_2 as $x \to \infty$?

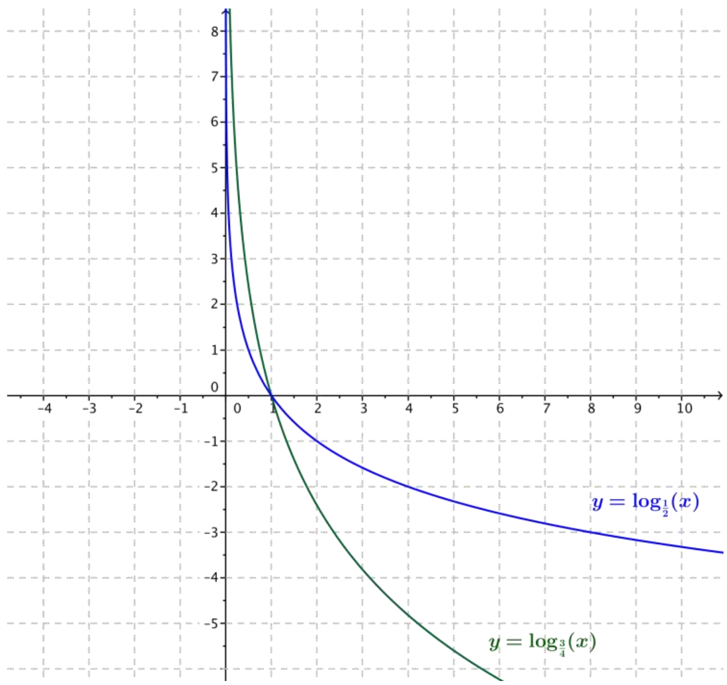
As $x \to \infty$, *both* $f_1(x) \to 0$ *and* $f_2(x) \to 0$.

e. What are the domains of the two functions f_1 and f_2?

Both functions have domain $(-\infty, \infty)$.



4. Use the information from Problem 3 together with the relationship between graphs of exponential and logarithmic functions to sketch the graphs of the functions $g_1(x) = \log_{\frac{1}{2}}(x)$ and $g_2(x) = \log_{\frac{3}{4}}(x)$ on the same sheet of graph paper. Then, answer the following questions.



a. Where do the two logarithmic graphs intersect?

The graphs intersect at the point $(1, 0)$.

b. For which values of x is $\log_{\frac{1}{2}}(x) < \log_{\frac{3}{4}}(x)$?

When $x < 1$, we have $\log_{\frac{1}{2}}(x) < \log_{\frac{3}{4}}(x)$.

c. For which values of x is $\log_{\frac{1}{2}}(x) > \log_{\frac{3}{4}}(x)$?

When $x > 1$, we have $\log_{\frac{1}{2}}(x) > \log_{\frac{3}{4}}(x)$.

d. What happens to the values of the functions g_1 and g_2 as $x \to \infty$?

As $x \to \infty$, both $g_1(x) \to -\infty$ and $g_2(x) \to -\infty$.

e. What are the domains of the two functions g_1 and g_2?

Both functions have domain $(0, \infty)$.

5. For each function f, find a formula for the function h in terms of x.

a. If $f(x) = x^3$, find $h(x) = 128f\left(\frac{1}{4}x\right) + f(2x)$.

$h(x) = 10x^3$

b. If $(x) = x^2 + 1$, find $h(x) = f(x+2) - f(2)$.

$h(x) = x^2 + 4x$

c. If $f(x) = x^3 + 2x^2 + 5x + 1$, find $h(x) = \frac{f(x) + f(-x)}{2}$.

$h(x) = 2x^2 + 1$

d. If $f(x) = x^3 + 2x^2 + 5x + 1$, find $h(x) = \frac{f(x) - f(-x)}{2}$.

$h(x) = x^3 + 5x$

6. In Problem 5, parts (c) and (d), list at least two aspects about the formulas you found as they relate to the function $f(x) = x^3 + 2x^2 + 5x + 1$.

The formula for 1(c) is all of the even power terms of f. The formula for 1(d) is all of the odd power terms of f. The sum of the two functions gives f back again; that is, $\frac{f(x)+f(-x)}{2} + \frac{f(x)-f(-x)}{2} = f$.

7. For each of the functions f and g below, write an expression for (i) $f(g(x))$, (ii) $g(f(x))$, and (iii) $f(f(x))$ in terms of x.

a. $f(x) = x^{\frac{2}{3}}$, $g(x) = x^{12}$

i. $f(g(x)) = x^8$

ii. $g(f(x)) = x^8$

iii. $f(f(x)) = x^{\frac{4}{9}}$

b. $f(x) = \frac{b}{x-a}$, $g(x) = \frac{b}{x} + a$ for two numbers a and b, when x is not 0 or a

i. $f(g(x)) = x$

ii. $g(f(x)) = x$

iii. $f(f(x)) = \frac{b}{\frac{b}{x-a} - a}$, *which is equivalent to* $f(f(x)) = \frac{b(x-a)}{b+a^2-ax}$

c. $f(x) = \frac{x+1}{x-1}$, $g(x) = \frac{x+1}{x-1}$, when x is not 1 or -1

i. $f(g(x)) = x$

ii. $g(f(x)) = x$

iii. $f(f(x)) = x$

d. $f(x) = 2^x$, $g(x) = \log_2(x)$

i. $f(g(x)) = x$

ii. $g(f(x)) = x$

iii. $f(f(x)) = x$

e. $f(x) = \ln(x),\ g(x) = e^x$

i. $f(g(x)) = x$

ii. $g(f(x)) = x$

iii. $f(f(x)) = \ln(\ln(x))$

f. $f(x) = 2 \cdot 100^x,\ g(x) = \frac{1}{2}\log\left(\frac{1}{2}x\right)$

i. $f(g(x)) = x$

ii. $g(f(x)) = x$

iii. $f(f(x)) = 2 \cdot 10000^{100^x}$

Lesson 19: The Inverse Relationship Between Logarithmic and Exponential Functions

Student Outcomes

- Students understand that the logarithmic function base b and the exponential function base b are inverse functions.

Lesson Notes

In the previous lesson, students learned that if they reflected the graph of a logarithmic function across the diagonal line with equation $y = x$, then the reflection is the graph of the corresponding exponential function, and vice-versa. In this lesson, we formalize this graphical observation with the idea of inverse functions. Students have not yet been exposed to the idea of an inverse function, but it is natural for us to have that discussion in this module. In particular, this lesson attends to these standards:

F-BF.B.4a: Solve an equation of the form $f(x) = c$ for a simple function f that has an inverse and write an expression for the inverse.

F-LE.A.4: For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology.

In order to clarify the procedure for finding an inverse function, we start with algebraic functions before returning to transcendental logarithms and exponential functions.

Note: You might want to consider splitting this lesson over two days.

Classwork

Opening Exercise (8 minutes)

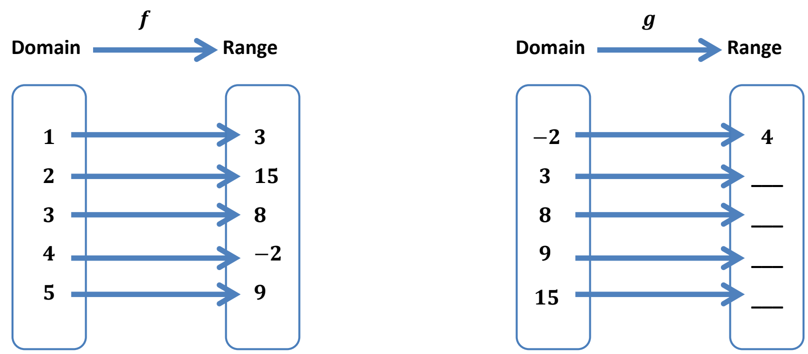
Before talking about inverse functions, review the idea of inverse operations. At this point, students have had a lot of practice thinking of division as undoing multiplication (in other words, multiplying by 5 and then dividing by 5 gives back the original number) and thinking of subtraction as undoing addition.

You may also want to remind your students about the composition of two transformations. For example, in geometry, the image of a *counterclockwise* rotation of a triangle $\triangle ABC$ by $30°$ around a point P is a new triangle congruent to the original. If we apply a $30°$ *clockwise* rotation to this new triangle around P (rotation by $-30°$), the image is the original triangle again. That is, the rotation $R_{P,-30°}$ undoes the rotation $R_{P,30°}$:
$R_{P,-30°}\left(R_{P,30°}(\triangle ABC)\right) = \triangle ABC$.

In this lesson, we study functions that undo other functions. That is, given a function f, sometimes there is another function g, so that if $y = f(x)$, then $g(y) = x$. Then, if we apply these functions in succession, doing f and then g on x, the composition returns the original value x. Such a function g is called the inverse of the function f, just as division is the inverse operation for multiplication.

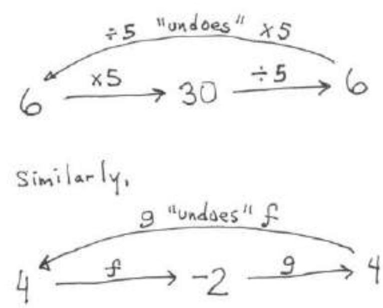
Opening Exercise

a. Consider the mapping diagram of the function f below. Fill in the blanks of the mapping diagram of g to construct a function that undoes each output value of f by returning the original input value of f. (The first one is done for you.)



1, 3, 5, 2

As you walk around the room, help struggling students by drawing the analogy of multiplication and division of what students are asked to do above.



b. Write the set of input-output pairs for the functions f and g by filling in the blanks below. (The set F for the function f has been done for you.)

$F = \{(1,3), (2,15), (3,8), (4,-2), (5,9)\}$

$G = \{(-2,4),$ $(3,1),$ $(8,3),$ $(9,5),$ $(15,2)\}$

c. How can the points in the set G be obtained from the points in F?

The points in G can be obtained from the points in F by switching the first entry (first coordinate) with the second entry (second coordinate), that is, if (a,b) is a point of F, then (b,a) is a point of G.

d. Peter studied the mapping diagrams of the functions f and g above and exclaimed, "I can get the mapping diagram for g by simply taking the mapping diagram for f and reversing all of the arrows!" Is he correct?

He is almost correct. It is true that he can reverse the arrows, but he would also need to switch the domain and range labels to reflect that the range of f is the domain of g, and the domain of f is the range of g.

We explore questions like those asked in parts (a), (c), and (d) of the Opening Exercise in more detail in the examples that follow.

Discussion (8 minutes)

You may need to point out to students the meaning of "Let $y = f(x)$" in this context. Usually, the equation $y = f(x)$ is an equation to be solved for solutions of the form (x, y). However, when we state "Let $y = f(x)$," we are using the equal symbol to assign the value $f(x)$ to y.

Complete this table either on the board or on an overhead projector.

- Consider the two functions $f(x) = 3x$ and $g(x) = \frac{x}{3}$. What happens if we compose these two functions in sequence? Let's make a table of values by letting y be the value of f when evaluated at x and then evaluating g on the result.

x	Let $y = f(x)$	$g(y)$
-2	-6	-2
-1	-3	-1
0	0	0
1	3	1
2	6	2
3	9	3

- What happens when we evaluate the function f on a value of x and then the function g on the result?
  - *We get back the original value of x.*
- Now, let's make a table of values by letting y be the value of g when evaluated at x and then evaluating f on the result.

x	Let $y = g(x)$	$f(y)$
-2	$-\frac{2}{3}$	-2
-1	$-\frac{1}{3}$	-1
0	0	0
1	$\frac{1}{3}$	1
2	$\frac{2}{3}$	2
3	1	3

- What happens when we evaluate the function g on a value of x and then the function f on the result?
  - *We get back the original value of x.*
- Does this happen with any two functions? What is special about the functions f and g?

  The formula for the function f multiplies its input by 3, and the formula for g divides its input by 3. If we first evaluate f on an input and then evaluate g on the result, we are multiplying by 3 and then dividing by 3, which has a net effect of multiplying by $\frac{3}{3} = 1$, so the result of the composition of f followed by g is the original input.

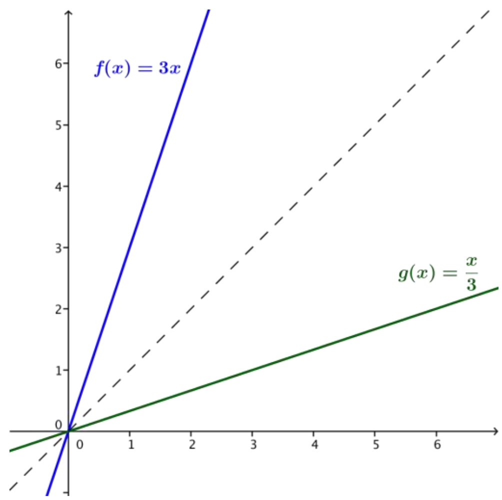
Likewise, if we first evaluate g on an input and then evaluate f on the result, we are dividing a number by 3 and then multiplying the result by 3 so that the net effect is again multiplication by $\frac{3}{3} = 1$, and the result of the composition is the original input.

This does not happen with two arbitrarily chosen functions. It is special when it does, and the functions have a special name:

- Functions $f(x) = 3x$ and $g(x) = \frac{x}{3}$ are examples of inverse functions—functions that, if you take the output of one function for a given input and put the output into the other function, you get the original input back.

Inverse functions have a special relationship between their graphs. Let's explore that now and tie it back to what we learned earlier.

- Graph $f(x) = 3x$ and $g(x) = \frac{x}{3}$. (Also, sketch in the graph of the diagonal line $y = x$.)



MP.7

- What do we notice about these two graphs?
  - *They are reflections of each other across the diagonal line given by $y = x$.*
- What is the rule for the transformation that reflects the Cartesian plane across the line given by $y = x$?
  - $r_{y=x}(x, y) = (y, x)$
- Where have we seen this switching of first and second coordinates before in this lesson? How is that situation similar to this one?
  - *In the Opening Exercise, to obtain the set G from the set F, we took each ordered pair of F and switched the first and second coordinates to get a point of G. Since plotting the points of F and G produce the graphs of those functions, we see that the graphs are reflections of each other across the diagonal given by $y = x$, that is, $r_{y=x}(F) = G$.*
    Similarly, for $f(x) = 3x$ and $g(x) = \frac{x}{3}$, we find $r_{y=x}$(Graph of f) = Graph of g.

Finally, let's tie what we learned about graphs of inverse functions to what we learned in the previous lesson.

- What other two functions have we seen whose graphs are reflections of each other across the diagonal line?
  - *The graph of a logarithmic and an exponential function with the same base b are reflections of each other across the diagonal line.*
- Make a conjecture about logarithmic and exponential functions.
  - *A logarithmic and an exponential function with the same base are inverses of each other.*
- Can we verify that using the properties of logarithms and exponents? What happens if we compose the functions by evaluating them one right after another? Let $f(x) = \log(x)$ and $g(x) = 10^x$. What do we know about the result of evaluating f for a number x and then evaluating g on the resulting output? What about evaluating g and then f?
  - *Let $y = f(x)$. Then $y = \log(x)$, so $g(y) = 10^y = 10^{\log(x)}$. By logarithmic property 4, $10^{\log(x)} = x$, so evaluating f at x, and then g on the results gives us the original input x back.*
  - *Let $y = g(x)$. Then $y = 10^x$, so $f(y) = \log(10^x)$. By logarithmic property 3, $\log(10^x) = x$, so evaluating g at x, and then f on the results gives us the original input x back.*

> *Scaffolding:*
>
> Remind students of the logarithmic properties:
>
> 3. $\log_b(b^x) = x$,
> 4. $b^{\log_b(x)} = x$.
>
> It may also be helpful to include an example next to each property, such as $\log_3(3^x) = x$ and $10^{\log(5)} = 5$.

- So, yes, a logarithmic function and its corresponding exponential function are inverse functions.

Discussion (8 minutes)

What if we have the formula of a function f, and we want to know the formula for its inverse function g? At this point, all we know is that if we have the graph of f and reflect it across the diagonal line we get the graph of its inverse g. We can use this fact to derive the formula for the inverse function g from the formula of f.

Above, we saw that

$$r_{x=y}(\text{Graph of } f) = \text{Graph of } g.$$

Let's write out what those sets look like. For $f(x) = 3x$, the graph of f is the same as the graph of the equation $y = f(x)$, that is, $y = 3x$:

$$\text{Graph of } f = \{(x, y) \mid y = 3x\}.$$

For $g(x) = \frac{x}{3}$, the graph of g is the same as the graph of the equation $y = \frac{x}{3}$, which is the same as the graph of the equation $x = 3y$ (why are they same?):

$$\text{Graph of } g = \{(x, y) \mid x = 3y\}.$$

Thus, the reflection across the diagonal line of the graph of f can be written as follows:

$$\{(x, y) \mid y = 3x\} \xrightarrow{r_{x=y}} \{(x, y) \mid x = 3y\}.$$

MP.7

- What relationship do you see between the set $\{(x, y) \mid y = 3x\}$ and the set $\{(x, y) \mid x = 3y\}$? How does this relate to the reflection map $r_{x=y}(x, y) = (y, x)$?
  - *To get the second set, we interchange x and y in the equation that defines the first set. This is exactly what the reflection map is telling us to do.*

- Let's see if the same relationship holds for $f(x) = \log(x)$ and its inverse $g(x) = 10^x$. Focusing on g, we see that the graph of g is the same as the graph of the equation $y = 10^x$. We can rewrite the equation $y = 10^x$ using logarithms as $x = \log(y)$. (Why are they the same?) Thus, the reflection across the diagonal line of the graph of f can be written as follows:

$$\{(x,y) \mid y = \log(x)\} \xrightarrow{r_{x=y}} \{(x,y) \mid x = \log(y)\}.$$

This pair of sets also has the same relationship.

Scaffolding:

Use $f(x) = 3x$ to help students discover the steps:

$$f(x) = 3x$$
$$y = 3x$$
$$x = 3y$$
$$\frac{x}{3} = y$$
$$y = \frac{x}{3}$$
$$g(x) = \frac{x}{3}.$$

MP.7

- How can we use that relationship to obtain the equation for the graph of g from the graph of f?
  - *Write the equation $y = f(x)$, and then interchange the symbols to get $x = f(y)$.*
- How can we use the equation $x = f(y)$ to find the formula for the function g?
  - *Solve the equation for y to write y as an expression in x. The formula for g is the expression in x.*
- In general, to find the formula for an inverse function g of a given function f:
  - i. Write $y = f(x)$ using the formula for f.
  - ii. Interchange the symbols x and y to get $x = f(y)$.
  - iii. Solve the equation for y to write y as an expression in x.
  - iv. Then, the formula for g is the expression in x found in step (iii).

Exercises 1–7 (8 minutes)

Give students a couple of minutes to work in pairs to use the above procedure in Exercise 1. Allow them time to think through the procedure on their own and generate questions that they can ask either their partner or you. After giving students a few minutes, work through Exercise 1 as a whole class, and move on to a selection of the remaining problems.

Scaffolding:

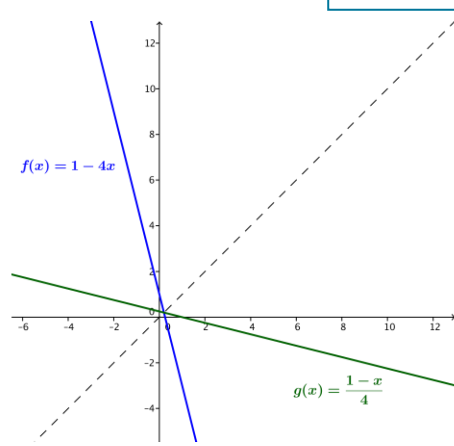
For Exercises 1–5, it may be useful to ask what axes setting is required on a calculator to check whether the graphs of the two functions are reflections of each other. (The x-axis and y-axis must have the same scale.)

Exercises

For each function f in Exercises 1–5, find the formula for the corresponding inverse function g. Graph both functions on a calculator to check your work.

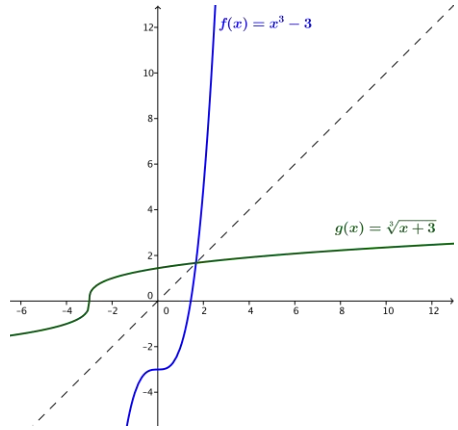
1. $f(x) = 1 - 4x$

$$y = 1 - 4x$$
$$x = 1 - 4y$$
$$4y = 1 - x$$
$$y = \frac{(1-x)}{4}$$
$$g(x) = \frac{(1-x)}{4}$$



2. $f(x) = x^3 - 3$

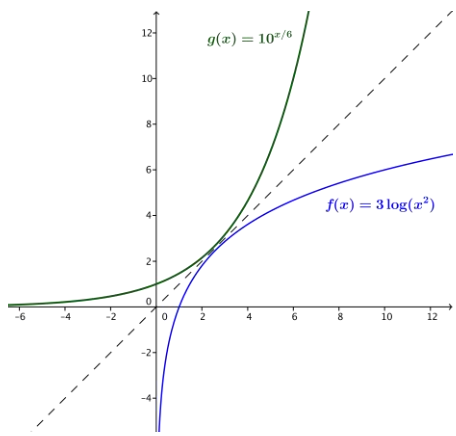
$$y = x^3 - 3$$
$$x = y^3 - 3$$
$$y^3 = x + 3$$
$$y = \sqrt[3]{x+3}$$
$$g(x) = \sqrt[3]{x+3}$$



For Exercise 2, you may need to mention that, unlike principal square roots, there are real principal cube roots for negative numbers. This leads to the following identities that hold for all real numbers: $\sqrt[3]{x^3} = x$ and $\left(\sqrt[3]{x}\right)^3 = x$ for any real number x. Problems such as these are practiced further in the Problem Set.

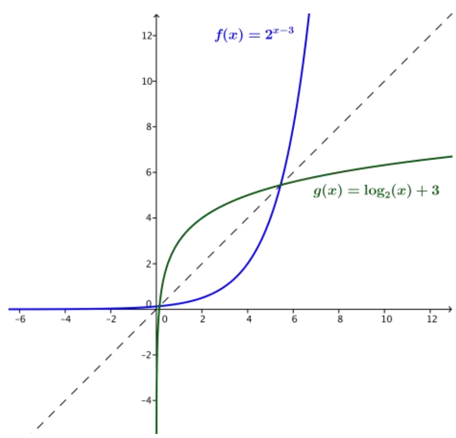
3. $f(x) = 3\log(x^2)$ for $x > 0$

$$y = 3\log(x^2)$$
$$x = 3\log(y^2)$$
$$x = 3 \cdot 2\log(y)$$
$$\log(y) = \frac{x}{6}$$
$$y = 10^{\frac{x}{6}}$$
$$g(x) = 10^{\frac{x}{6}}$$



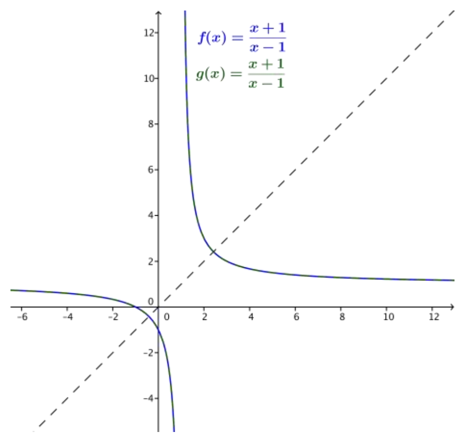
4. $f(x) = 2^{x-3}$

$$y = 2^{x-3}$$
$$x = 2^{y-3}$$
$$\log_2(x) = y - 3$$
$$y = \log_2(x) + 3$$
$$g(x) = \log_2(x) + 3$$



5. $f(x) = \frac{x+1}{x-1}$ for $x \neq 1$

$$y = \frac{x+1}{x-1}$$
$$x = \frac{y+1}{y-1}$$
$$x(y-1) = y+1$$
$$xy - x = y+1$$
$$xy - y = x+1$$
$$y(x-1) = x+1$$
$$y = \frac{x+1}{x-1}$$
$$g(x) = \frac{x+1}{x-1} \text{ for } x \neq 1$$



Exercise 5 is quite interesting. You might have students note that the functions f and g are the same. What do students notice about the graph of f? (This issue is further explored in the Problem Set.)

6. **Cindy thinks that the inverse of $f(x) = x - 2$ is $g(x) = 2 - x$. To justify her answer, she calculates $f(2) = 0$ and then substitutes the output 0 into g to get $g(0) = 2$, which gives back the original input. Show that Cindy is incorrect by using other examples from the domain and range of f.**

   Answers will vary, but any point other than 2 works. For example, $f(3) = 1$, but $g(1) = 1$, not 3 as needed.

7. **After finding the inverse for several functions, Henry claims that *every* function must have an inverse. Rihanna says that his statement is not true and came up with the following example: If $f(x) = |x|$ has an inverse, then because $f(3)$ *and* $f(-3)$ both have the same output 3, the inverse function g would have to map 3 to *both* 3 *and* -3 simultaneously, which violates the definition of a function. What is another example of a function without an inverse?**

   Answers will vary. Any even degree polynomial function, such as $f(x) = x^2$, does not have an inverse.

You might consider showing students graphs of functions without inverses and discuss what the graphs look like after reflecting them along the diagonal line (where it becomes obvious that the reflected figure cannot be a graph of a function).

Example (5 minutes)

Now we need to address the question of how the domain and range of the function f and its inverse function g relate. You may need to review domain and range of a function with your students first.

- In all exercises we did above, what numbers were in the domain of g? Why?
  - *The domain of g contains the same numbers that were in the range of f. This is because, as the inverse of f, the function g takes the output of f (the range) as its input.*

> *Scaffolding:*
> For students who are struggling, use concrete examples from the Opening Exercise or from the exercises they just did. For example, "If $f(x) = 2^{x-3}$, what is an example of a number that is in the domain? The range?"

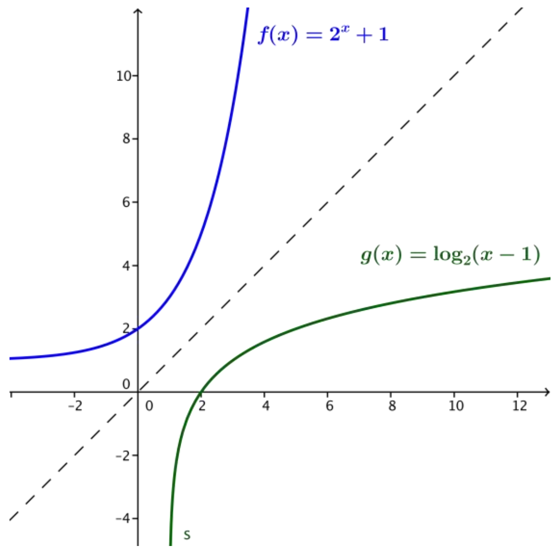
- What numbers were in the range of g? Why?
  - *The range of g contains the same numbers that were in the domain of f. When the function g is evaluated on an output value of f, its output is the original input of f (the domain of f).*

Example

Consider the function $f(x) = 2^x + 1$, whose graph is shown to the right.

a. **What are the domain and range of f?**

Since the function $h(x) = 2^x$ has domain all real numbers and range $(0, \infty)$, we know that the translated function $f(x) = 2^x + 1$ has domain all real numbers and range $(1, \infty)$.



b. **Sketch the graph of the inverse function g on the graph. What type of function do you expect g to be?**

Since logarithmic and exponential functions are inverses of each other, g should be some form of a logarithmic function (shown in green).

c. **What are the domain and range of g? How does that relate to your answer in part (a)?**

The range of g is all real numbers, and the domain of g is $(1, \infty)$, which makes sense since the range of g is the domain of f, and the domain of g is the range of f.

d. **Find the formula for g.**

$$y = 2^x + 1$$
$$x = 2^y + 1$$
$$2^y = x - 1$$
$$y = \log_2(x - 1)$$
$$g(x) = \log_2(x - 1), \text{ for } x > 1$$

Closing (3 minutes)

Ask students to summarize the important points of the lesson either in writing, orally with a partner, or as a class. Use this as an opportunity to informally assess understanding of the lesson. In particular, ask students to articulate the process for both graphing and finding the formula for the inverse of a given function. Some important summary elements are contained in the box below.

Lesson Summary

- **INVERTIBLE FUNCTION:** Let f be a function whose domain is the set X and whose image is the set Y. Then f is *invertible* if there exists a function g with domain Y and image X such that f and g satisfy the property:

  For all x in X and y in Y, $f(x) = y$ if and only if $g(y) = x$.

  The function g is called the *inverse* of f.
- If two functions whose domain and range are a subset of the real numbers are inverses, then their graphs are reflections of each other across the diagonal line given by $y = x$ in the Cartesian plane.
- If f and g are inverses of each other, then
  - The domain of f is the same set as the range of g.
  - The range of f is the same set as the domain of g.
- In general, to find the formula for an inverse function g of a given function f:
  i. Write $y = f(x)$ using the formula for f.
  ii. Interchange the symbols x and y to get $x = f(y)$.
  iii. Solve the equation for y to write y as an expression in x.
  iv. Then, the formula for g is the expression in x found in step (iii).
- The functions $f(x) = \log_b(x)$ and $g(x) = b^x$ are inverses of each other.

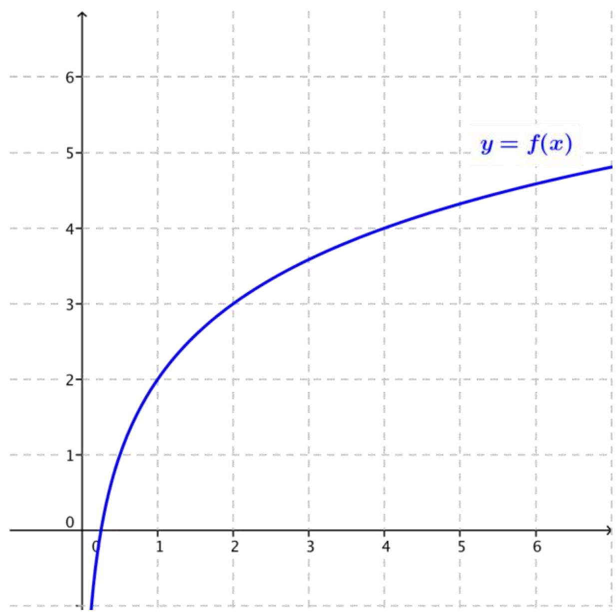
Exit Ticket (5 minutes)

Name ____ Date ____________________

Lesson 19: The Inverse Relationship Between Logarithmic and Exponential Functions

Exit Ticket

1. The graph of a function f is shown below. Sketch the graph of its inverse function g on the same axes.

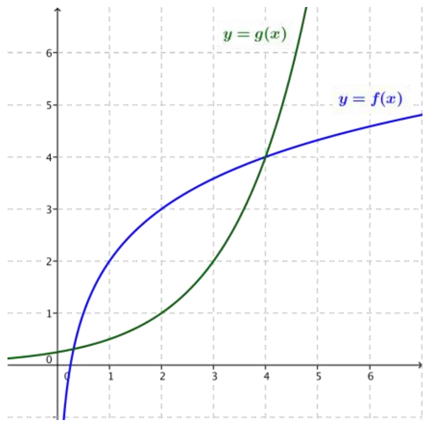


2. Explain how you made your sketch.

3. The function f graphed above is the function $f(x) = \log_2(x) + 2$ for $x > 0$. Find a formula for the inverse of this function.

Exit Ticket Sample Solutions

1. The graph of a function f is shown below. Sketch the graph of its inverse function g on the same axes.



2. Explain how you made your sketch.

   Answers will vary. Example: I drew the line given by $y = x$ and reflected the graph of f across it.

3. The graph of the function f above is the function $f(x) = \log_2(x) + 2$ for $x > 0$. Find a formula for the inverse of this function.

$$y = \log_2(x) + 2$$
$$x = \log_2(y) + 2$$
$$x - 2 = \log_2(y)$$
$$\log_2(y) = x - 2$$
$$y = 2^{x-2}$$
$$g(x) = 2^{x-2}$$

Problem Set Sample Solutions

1. For each function h below, find two functions f and g such that $h(x) = f(g(x))$. (There are many correct answers.)

   a. $h(x) = (3x + 7)^2$

   Possible answer: $f(x) = x^2$, $g(x) = 3x + 7$

   b. $h(x) = \sqrt[3]{x^2 - 8}$

   Possible answer: $f(x) = \sqrt[3]{x}$, $g(x) = x^2 - 8$

   c. $h(x) = \frac{1}{2x-3}$

   Possible answer: $f(x) = \frac{1}{x}$, $g(x) = 2x - 3$

d. $h(x) = \frac{4}{(2x-3)^3}$

Possible answer: $f(x) = \frac{4}{x^3}$, $g(x) = 2x - 3$

e. $h(x) = (x+1)^2 + 2(x+1)$

Possible answer: $f(x) = x^2 + 2x$, $g(x) = x + 1$

f. $h(x) = (x+4)^{\frac{4}{5}}$

Possible answer: $f(x) = x^{\frac{4}{5}}$, $g(x) = x + 4$

g. $h(x) = \sqrt[3]{\log(x^2+1)}$

Possible answer: $f(x) = \sqrt[3]{\log(x)}$, $g(x) = x^2 + 1$

h. $h(x) = \sin(x^2 + 2)$

Possible answer: $f(x) = \sin(x)$, $g(x) = x^2 + 2$

i. $h(x) = \ln(\sin(x))$

Possible answer: $f(x) = \ln(x)$, $g(x) = \sin(x)$

2. Let f be the function that assigns to each student in your class his or her biological mother.

a. Use the definition of function to explain why f is a function.

The function has a well-defined domain (students in the class) and range (their mothers), and each student is assigned one and only one biological mother.

b. In order for f to have an inverse, what condition must be true about the students in your class?

If a mother has several children in the same classroom, then there would be no way to define an inverse function that picks one and only one student for each mother. The condition that must be true is that there are no siblings in the class.

c. If we enlarged the domain to include all students in your school, would this larger domain function have an inverse?

Probably not. Most schools have several students who are siblings.

3. The table below shows a partially filled-out set of input-output pairs for two functions f and h that have the same finite domain of $\{0, 5, 10, 15, 20, 25, 30, 35, 40\}$.

x	0	5	10	15	20	25	30	35	40
$f(x)$	0	0.3	1.4		2.1		2.7	6	
$h(x)$	0	0.3	1.4		2.1		2.7	6	

a. Complete the table so that f is invertible but h is definitely not invertible.

Answers will vary. For f, all output values should be different. For h, at least two output values for two different inputs should be the same number.

b. **Graph both functions and use their graphs to explain why f is invertible and h is not.**

Answers will vary. The graph of f has one unique output for every input, so it is possible to undo f and map each of its outputs to a unique input. The graph of h has at least two input values that map to the same output value. Hence, there is no way to map that output value back to a unique multiple of 5. Hence, h cannot have an inverse function because such a correspondence is not a function.

4. **Find the inverse of each of the following functions. In each case, indicate the domain and range of both the original function and its inverse.**

a. $f(x) = \frac{3x-7}{5}$

$$x = \frac{3y-7}{5}$$
$$5x = 3y - 7$$
$$\frac{5x+7}{3} = y$$

The inverse function is $g(x) = \frac{5x+7}{3}$. Both functions f and g have a domain and range of all real numbers.

b. $f(x) = \frac{5+x}{6-2x}$

$$x = \frac{5+y}{6-2y}$$
$$6x - 2yx = 5 + y$$
$$6x - 5 = 2yx + y$$
$$6x - 5 = (2x+1)y$$
$$\frac{6x-5}{2x+1} = y$$

The inverse function is $g(x) = \frac{6x-5}{2x+1}$.

Domain of f and range of g: all real numbers x with $x \neq 3$

Range of f and domain of g: all real numbers x with $x \neq -\frac{1}{2}$

c. $f(x) = e^{x-5}$

$$x = e^{y-5}$$
$$\ln(x) = y - 5$$
$$\ln(x) + 5 = y$$

The inverse function is $g(x) = \ln(x) + 5$.

Domain of f and range of g: all real numbers x

Range of f and domain of g: all real numbers x with $x > 0$

d. $f(x) = 2^{5-8x}$

$$x = 2^{5-8y}$$
$$\log_2(x) = 5 - 8y$$
$$8y = 5 - \log_2(x)$$
$$y = \frac{1}{8}(5 - \log_2(x))$$

The inverse function is $g(x) = \frac{1}{8}\left(5 - \log_2(x)\right)$.

Domain of f and range of g: all real numbers x

Range of f and domain of g: all real numbers x with $x > 0$

e. $f(x) = 7\log(1 + 9x)$

$$x = 7\log(1 + 9y)$$
$$\frac{x}{7} = \log(1 + 9y)$$
$$10^{\frac{x}{7}} = 1 + 9y$$
$$\frac{1}{9}\left(10^{\frac{x}{7}} - 1\right) = y$$

The inverse function is $g(x) = \frac{1}{9}\left(10^{\frac{x}{7}} - 1\right)$.

Domain of f and range of g: all real numbers x with $x > -\frac{1}{9}$

Range of f and domain of g: all real numbers x

f. $f(x) = 8 + \ln(5 + \sqrt[3]{x})$

$$x = 8 + \ln(5 + \sqrt[3]{y})$$
$$x - 8 = \ln(5 + \sqrt[3]{y})$$
$$e^{x-8} = 5 + \sqrt[3]{y}$$
$$e^{x-8} - 5 = \sqrt[3]{y}$$
$$(e^{x-8} - 5)^3 = y$$

The inverse function is $g(x) = (e^{x-8} - 5)^3$.

Domain of f and range of g: all real numbers x with $x > -125$

Range of f and domain of g: all real numbers x

g. $f(x) = \log\left(\frac{100}{3x+2}\right)$

$$x = \log\left(\frac{100}{3y+2}\right)$$
$$x = \log(100) - \log(3y+2)$$
$$x = 2 - \log(3y+2)$$
$$2 - x = \log(3y+2)$$
$$10^{2-x} = 3y+2$$
$$\frac{1}{3}(10^{2-x} - 2) = y$$

The inverse function is $g(x) = \frac{1}{3}(10^{2-x} - 2)$.

Domain of f *and range of* g*: all real numbers* x *with* $x > -\frac{2}{3}$

Range of f *and domain of* g*: all real numbers* x

h. $f(x) = \ln(x) - \ln(x+1)$

$$x = \ln(y) - \ln(y+1)$$
$$x = \ln\left(\frac{y}{y+1}\right)$$
$$e^x = \frac{y}{y+1}$$
$$ye^x + e^x = y$$
$$ye^x - y = -e^x$$
$$y(e^x - 1) = -e^x$$
$$y = \frac{e^x}{1-e^x}$$

The inverse function is $g(x) = \frac{e^x}{1-e^x}$.

Domain of f *and range of* g*: all real numbers* x *with* $x > 0$

Range of f *and domain of* g*: all real numbers* $x < 0$

i. $f(x) = \frac{2^x}{2^x+1}$

$$x = \frac{2^y}{2^y+1}$$
$$x2^y + x = 2^y$$
$$x2^y - 2^y = -x$$
$$2^y(x-1) = -x$$
$$2^y = \frac{-x}{x-1} = \frac{x}{1-x}$$
$$y\ln(2) = \ln\left(\frac{x}{1-x}\right)$$
$$y = \ln\frac{\left(\frac{x}{1-x}\right)}{\ln(2)}$$

The inverse function is $g(x) = \frac{\ln\left(\frac{x}{1-x}\right)}{\ln(2)}$.

Domain of f *and range of* g*: all real numbers* x

Range of f *and domain of* g*: all real numbers* x, $0 < x < 1$

5. Even though there are no real principal square roots for negative numbers, principal cube roots do exist for negative numbers: $\sqrt[3]{-8}$ is the real number -2 since $-2 \cdot -2 \cdot -2 = -8$. Use the identities $\sqrt[3]{x^3} = x$ and $\left(\sqrt[3]{x}\right)^3 = x$ for any real number x to find the inverse of each of the functions below. In each case, indicate the domain and range of both the original function and its inverse.

a. $f(x) = \sqrt[3]{2x}$ for any real number x.

$$y = \sqrt[3]{2x}$$
$$x = \sqrt[3]{2y}$$
$$x^3 = 2y$$
$$2y = x^3$$
$$y = \frac{1}{2}(x^3)$$
$$g(x) = \frac{1}{2}(x^3)$$

Domain of f and range of g: all real numbers x
Range of f and domain of g: all real numbers x

b. $f(x) = \sqrt[3]{2x-3}$ for any real number x.

$$y = \sqrt[3]{2x-3}$$
$$x = \sqrt[3]{2y-3}$$
$$x^3 = 2y - 3$$
$$2y = x^3 + 3$$
$$y = \frac{1}{2}(x^3 + 3)$$
$$g(x) = \frac{1}{2}(x^3 + 3)$$

Domain of f and range of g: all real numbers x
Range of f and domain of g: all real numbers x

c. $f(x) = (x-1)^3 + 3$ for any real number x.

$$y = (x-1)^3 + 3$$
$$x = (y-1)^3 + 3$$
$$x - 3 = (y-1)^3$$
$$\sqrt[3]{x-3} = y - 1$$
$$y - 1 = \sqrt[3]{x-3}$$
$$y = \sqrt[3]{x-3} + 1$$
$$g(x) = \sqrt[3]{x-3} + 1$$

Domain of f and range of g: all real numbers x

Range of f and domain of g: all real numbers x

6. Suppose that the inverse of a function is the function itself. For example, the inverse of the function $f(x) = \frac{1}{x}$ (for $x \neq 0$) is just itself again, $g(x) = \frac{1}{x}$ (for $x \neq 0$). What symmetry must the graphs of all such functions have? (Hint: Study the graph of Exercise 5 in the lesson.)

All graphs of functions that are self-inverses are symmetric with respect to the diagonal line given by the equation $y = x$. That is, a reflection across the line given by $y = x$ takes the graph back to itself.

7. There are two primary scales for measuring daily temperature: degrees Celsius and degrees Fahrenheit. The United States uses the Fahrenheit scale, and many other countries use the Celsius scale. When traveling abroad you often need to convert between these two temperature scales.

Let f be the function that inputs a temperature measure in degrees Celsius, denoted by °C, and outputs the corresponding temperature measure in degrees Fahrenheit, denoted by °F.

a. Assuming that f is linear, we can use two points on the graph of f to determine a formula for f. In degrees Celsius, the freezing point of water is 0, and its boiling point is 100. In degrees Fahrenheit, the freezing point of water is 32, and its boiling point is 212. Use this information to find a formula for the function f. (Hint: Plot the points and draw the graph of f first, keeping careful track of the meaning of values on the x-axis and y-axis.)

$f(t) = \frac{9}{5}t + 32$

b. If the temperature in Paris is 25°C, what is the temperature in degrees Fahrenheit?

Since $f(25) = 77$, it is 77°F in Paris.

c. Find the inverse of the function f and explain its meaning in terms of degrees Fahrenheit and degrees Celsius.

The inverse of f is $g(t) = \frac{5}{9}(t - 32)$. Given the measure of a temperature reported in degrees Fahrenheit, the function converts that measure to degrees Celsius.

d. The graphs of f and its inverse g are two lines that intersect in one point. What is that point? What is its significance in terms of degrees Celsius and degrees Fahrenheit?

The point is $(-40, -40)$. This means that -40°C is the same temperature as -40°F.

Extension: Use the fact that, for $b > 1$, the functions $f(x) = b^x$ and $g(x) = \log_b(x)$ are increasing to solve the following problems. Recall that an increasing function f has the property that if both a and b are in the domain of f and $a < b$, then $f(a) < f(b)$.

8. For which values of x is $2^x < \frac{1}{1{,}000{,}000}$?

$$2^x < \frac{1}{1,000,000}$$
$$x < \log_2\left(\frac{1}{1,000,000}\right) = -\log_2(1,000,000)$$

9. For which values of x is $\log_2(x) < -1,000,000$?

$$\log_2(x) < -1,000,000$$
$$x < 2^{-1,000,000}$$

Lesson 20: Transformations of the Graphs of Logarithmic and Exponential Functions

Student Outcomes

- Students study transformations of the graphs of logarithmic functions and learn the standard form of generalized logarithmic and exponential functions.
- Students use the properties of logarithms and exponents to produce equivalent forms of exponential and logarithmic expressions. In particular, they notice that different types of transformations can produce the same graph due to these properties.

Lesson Notes

Students revisit the use of transformations to produce graphs of exponential and logarithmic functions (**F-BF.B.3**, **F-IF.B.4**, **F-IF.C.7e**). They make and verify conjectures about why certain transformations of the graphs of functions produce the same graph by applying the properties to produce equivalent expressions (MP.3). This work leads to a general form of both logarithmic and exponential functions where the given parameters can be quickly analyzed to determine key features and to sketch graphs of logarithmic and exponential functions (MP.7, MP.8). This lesson reinforces knowledge of transformations and the properties of logarithms as students sketch graphs of transformed logarithmic and exponential functions.

Classwork

Opening Exercise (8 minutes)

Since much of the work on this lesson involves the connections between scaling and translating graphs of functions, this Opening Exercise presents students with an opportunity to reflect on what they already know about transformations of graphs of functions using a simple polynomial function and the sine function. Observe students carefully as they work on these exercises to gauge how much re-teaching or additional support may be needed in the Exploratory Challenge that follows. If students struggle to recall their knowledge of transformations, you may need to provide additional guidance and practice throughout the lesson. A grid is provided for students to use when sketching the graphs in Opening Exercise, part (a), but students could also complete this exercise using graphing technology.

Opening Exercise

a. **Sketch the graphs of the three functions $f(x) = x^2$, $g(x) = (2x)^2 + 1$, and $h(x) = 4x^2 + 1$.**

  i. **Describe the sequence of transformations that takes the graph of $f(x) = x^2$ to the graph of $g(x) = (2x)^2 + 1$.**

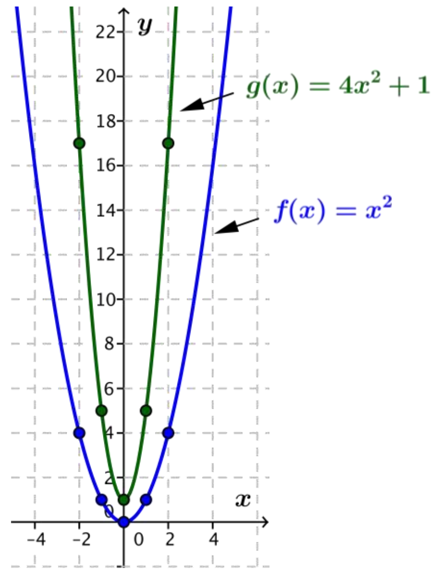
  The graph of g is a horizontal scaling by a factor of $\frac{1}{2}$ and a vertical translation up 1 unit of the graph of f.

ii. **Describe the sequence of transformations that takes the graph of $f(x) = x^2$ to the graph of $h(x) = 4x^2 + 1$.**

The graph of h is a vertical scaling by a factor of 4 and a vertical translation up 1 unit of the graph of f.

iii. **Explain why g and h from parts (i) and (ii) are equivalent functions.**

These functions are equivalent and have the same graph because the expressions $(2x)^2 + 1$ and $4x^2 + 1$ are equivalent. The blue graph shown is the graph of f, and the green graph is the graph of g and h.



b. **Describe the sequence of transformations that takes the graph of $f(x) = \sin(x)$ to the graph of $g(x) = \sin(2x) - 3$.**

The graph of g is a horizontal scaling by a factor of $\frac{1}{2}$ and a vertical translation down 3 units of the graph of f.

c. **Describe the sequence of transformations that takes the graph of $f(x) = \sin(x)$ to the graph of $h(x) = 4\sin(x) - 3$.**

The graph of h is a vertical scaling by a factor of 4 and a vertical translation down 3 units of the graph of f.

d. **Explain why g and h from parts (b)–(c) are *not* equivalent functions.**

These functions are not equivalent because they do not have the same graphs, and the two expressions are not equivalent.

Scaffolding:

- For students who struggle with visual processing, provide larger graph paper and/or colored pencils to color code their graphs.
- For struggling students, prominently display the properties of logarithms and exponents on the board or on chart paper in your room for visual reference. Refer students back to these charts during the exploration with questions such as: "Which property could you use to rewrite the expression?"

Have students share responses and revise their work in small groups. Be sure to emphasize that students need to label their graphs since many of them appear on the same set of axes. Lead a brief whole-group discussion that focuses on the responses to Opening Exercise part (a-iii) and part (d). Perhaps have one or two students share their written response with the whole class, using either the board or the document camera. Check to make sure students are actually writing responses to these questions.

Conclude this discussion by helping students to understand the following ideas that, as we demonstrate later, are also true with graphs of logarithmic and exponential functions.

- Certain transformations of the graph of a function can be identical to other transformations depending on the properties of the given function.
- For example, the function $g(x) = x + 1$ is either a horizontal translation of 1 unit to the left or a vertical translation of 1 unit up of the graph of $f(x) = x$. Similarly, $g(x) = |-2x + 2|$ has the same graph as $h(x) = 2|x - 1|$, but could be described using different transformations of the graph of $f(x) = |x|$.

Announce to students that in this lesson they explore the properties of logarithms and exponents to understand graphing transformations of those types of functions, and they explore when two different functions have the same graph in order to reinforce those properties.

Exploratory Challenge (15 minutes)

Students should work in small groups to complete this sequence of questions. Provide support to individual groups or students as you move around the classroom. As you circulate, keep questioning students as to the meaning of a logarithm with questions like those below.

MP.1

- What does $\log_2(4)$ mean?
  - *The exponent when the number 4 is written as a power of 2.*
- Why is $\log_2\left(\frac{1}{4}\right)$ negative?
  - *It is negative because $\frac{1}{4} = 2^{-2}$, and the logarithm is the exponent when the number $\frac{1}{4}$ is written as a power of 2.*
- What is the domain and range of the function f? Why does this make sense given the definition of a logarithm?
  - *The domain is all real numbers greater than 0. The range is all real numbers. This makes sense because the range of the exponential function $f(x) = 2^x$ is all real numbers greater than 0, the domain is all real numbers, and the logarithmic function base 2 is the inverse of the exponential function base 2.*

> *Scaffolding:*
>
> Use technology to support learners who are still struggling with arithmetic and need visual reinforcement. Students can investigate using graphing calculators or online graphing programs. Newer calculators and graphing programs have a $\log_b(x)$ function built in. On older models, you may need to coach students to use the change of base property to enter these functions (i.e., to graph $f(x) = \log_2(x)$, you need to enter the expression $\frac{\log(x)}{\log(2)}$ into the graphing calculator).

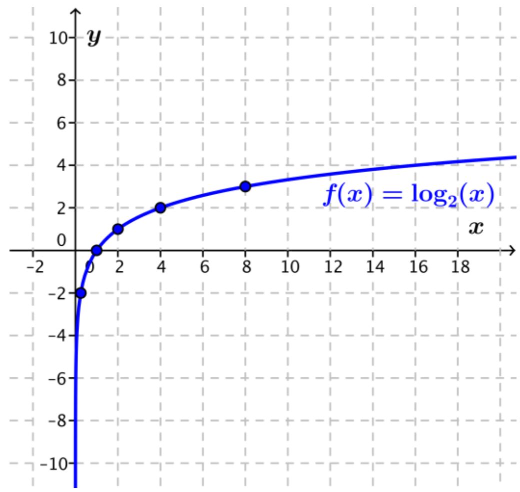
You can extend this lesson by using graphing software such as GeoGebra to create parameterized graphs with sliders (variables a and b that can be dynamically changed while viewing graphs). By manipulating the values of a and k in the functions $g(x) = \log_2(ax)$ and $h(x) = k + \log_2(x)$, you can verify that the graphs of $g(x) = \log_2(8x)$ and $h(x) = 3 + \log_2(x)$ are the same. This result reinforces properties of logarithms since $\log_2(8x) = \log_2(8) + \log_2(x)$. Students and teachers can similarly confirm the other examples in this lesson as well.

Exploratory Challenge

a. Sketch the graph of $f(x) = \log_2(x)$ by identifying and plotting at least five key points. Use the table below to get started.

The graph of f is shown below.

x	$\log_2(x)$
$\frac{1}{4}$	-2
$\frac{1}{2}$	-1
1	0
2	1
4	2
8	3



b. Describe a sequence of transformations that takes the graph of f to the graph of $g(x) = \log_2(8x)$.

The graph of g is a horizontal scaling by a factor of $\frac{1}{8}$ of the graph of f.

c. Describe a sequence of transformations that takes the graph of f to the graph of $h(x) = 3 + \log_2(x)$.

The graph of h is a vertical translation up 3 units of the graph of f.

MP.3 & MP.8

d. Complete the table below for f, g, and h and describe any noticeable patterns.

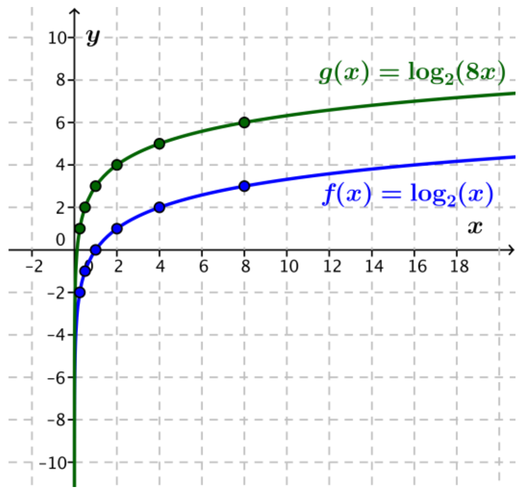
x	$f(x)$	$g(x)$	$h(x)$
$\frac{1}{4}$	-2	0	0
$\frac{1}{2}$	-1	2	2
1	0	3	3
2	1	4	4
4	2	5	5
8	3	6	6

The functions g and h have the same range values at each domain value in the table.

MP.3 & MP.8

e. **Graph the three functions on the same coordinate axes and describe any noticeable patterns.**

The graphs of g and h are identical.



f. **Use a property of logarithms to show that g and h are equivalent.**

By the product property and the definition of logarithm, $\log_2(8x) = \log_2(8) + \log_2(x) = 3 + \log_2(x)$, so $g(x) = h(x)$ for all positive real numbers x and g and h are equivalent functions.

Call the entire class together at this point to debrief their work so far. Make sure students understand that by applying the product or quotient property of logarithms, they can rewrite a single logarithmic expression as a sum or difference. In this way, a horizontal scaling of the graph of a logarithmic function can produce the same graph as a vertical translation. The next three parts of the Exploratory Challenge can be used to informally assess student understanding of the idea that two different transformations can produce the same graph because of the properties of logarithms.

MP.3

g. **Describe the graph of $p(x) = \log_2\left(\frac{x}{4}\right)$ as a vertical translation of the graph of $f(x) = \log_2(x)$. Justify your response.**

The graph of p is a vertical translation down 2 units of the graph of f because $\log_2\left(\frac{x}{4}\right) = \log_2(x) - 2$.

h. **Describe the graph of $h(x) = 3 + \log_2(x)$ as a horizontal scaling of the graph of $f(x) = \log_2(x)$. Justify your response.**

The graph of h is a horizontal scaling by a factor of $\frac{1}{8}$ of the graph of f because

$3 + \log_2(x) = \log_2(8) + \log_2(x) = \log_2(8x)$.

i. **Do the functions $h(x) = \log_2(8) + \log_2(x)$ and $k(x) = \log_2(x + 8)$ have the same graphs? Justify your reasoning.**

No, they do not. By substituting 1 for x in both f and g, you can see that the graphs of the two functions do not have the same y-coordinate at this point. Therefore, the graphs cannot be the same if at least one point is different.

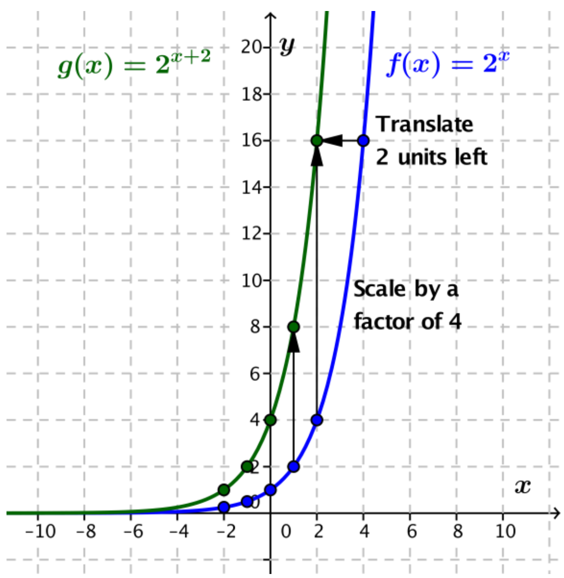
Carefully review the answers to the preceding questions to check for student understanding. Students may struggle with expressing 3 as $\log_2(8)$ in part (h). In the last portion of the Exploratory Challenge, students turn their attention to exponential functions and apply the properties of exponents to explain why graphs of certain exponential functions are identical. Circulate around the room while students are working and encourage them to create the graph of the parent function by plotting key points on the graph of the function and then transforming those key points according to the transformation they described.

j. **Use properties of exponents to explain why graphs of $f(x) = 4^x$ and $g(x) = 2^{2x}$ are identical.**

Using the power property of exponents, $2^{2x} = (2^2)^x = 4^x$. Since the expressions are equal, the graphs of the functions would be the same.

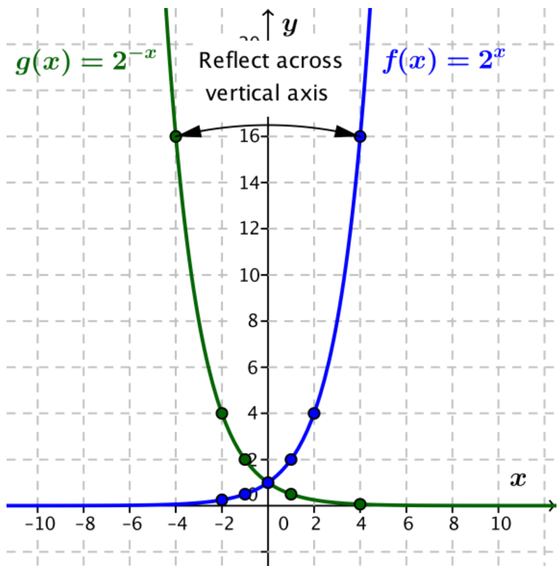
MP.7

k. **Use the properties of exponents to predict what the graphs of $g(x) = 4 \cdot 2^x$ and $h(x) = 2^{x+2}$ look like compared to one another. Describe the graphs of g and h as transformations of the graph of $f(x) = 2^x$. Confirm your prediction by graphing f, g, and h on the same coordinate axes.**



The graphs of the two functions g and h are the same since $2^{x+2} = 2^x \cdot 2^2 = 4 \cdot 2^x$ by the multiplication property of exponents and the commutative property. The graph of g is the graph of $f(x) = 2^x$ scaled vertically by a factor of 4. The graph of h is the graph of $f(x) = 2^x$ translated horizontally 2 units to the left.

l. Graph $f(x) = 2^x$, $g(x) = 2^{-x}$, and $h(x) = \left(\frac{1}{2}\right)^x$ on the same coordinate axes. Describe the graphs of g and h as transformations of the graph of f. Use the properties of exponents to explain why g and h are equivalent.



The graph of g and the graph of h are both reflections about the vertical axis of the graph of f. They are equivalent because $\left(\frac{1}{2}\right)^x = (2^{-1})^x = 2^{-x}$ by the definition of a negative exponent and the power property of exponents.

Have groups volunteer to present their findings on the last three parts of the Exploratory Challenge. When debriefing, model both transformations for students by marking the sketch as shown in the solutions above.

Discuss these transformations.

- In part (k), how do you see the transformations that produce the graphs of f and g from the graph of $y = 2^x$?
  - *I see the horizontal translation of 2 units to the left, but others might see the vertical scaling that takes each y-value and multiplies it by 4.*
- In part (l), how do the transformations validate the definition of a negative exponent?
  - *Since the graphs of g and h were identical, we have visual confirmation that $\left(\frac{1}{2}\right)^x = 2^{-x}$, which can only be true if $\frac{1}{2} = 2^{-1}$.*

Then, have students respond to the reflection question below in writing or with a partner.

- How do the properties of logarithms and exponents justify the fact that different transformations of the graph of a function can sometimes produce the same graph?
  - *We can use the properties to rewrite logarithmic and exponential expressions in equivalent forms that then represent different transformations of the same original function.*

If time permits, you can also tie these transformations to a simple real-world context. For example, suppose that we can model a simple population by $P(t) = 2^{t+3}$. If we rewrite the function as $P(t) = 8 \cdot 2^t$, students can see that evaluating P at $t + 3$ would be like looking three years forward in time. This means the population doubled three times, which is why we are multiplying by 8.

Example 1 (4 minutes): Graphing Transformations of the Logarithm Functions

MP.7 & MP.8

MP.2

Introduce the general form of a logarithm function, noting that we do not need a horizontal scaling parameter since a horizontal scaling can always be rewritten as a vertical translation. Continue to reinforce learning from the previous lessons by asking students why the restrictions on b and $x - h$ are necessary. Students should be able to work through part (a) without your assistance, but monitor their work to make sure that all students have the correct answer to refer to when they work the Problem Set. Model your expectations for sketching the graphs of logarithm functions in part (b) so students are able to produce accurate and precise graphs. Demonstrate how to plot the key points, and then transform the individual points to produce the graph of g.

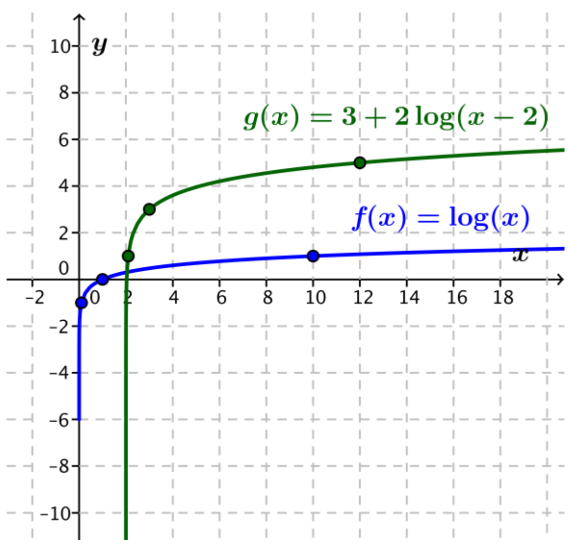
Example 1: Graphing Transformations of the Logarithm Functions

The general form of a logarithm function is given by $f(x) = k + a\log_b(x - h)$, where a, b, k, and h are real numbers such that b is a positive number not equal to 1, and $x - h > 0$.

a. **Given $g(x) = 3 + 2\log(x - 2)$, describe the graph of g as a transformation of the common logarithm function.**

The graph of g is a horizontal translation 2 units to the right, a vertical scaling by a factor of 2, and a vertical translation up 3 units of the graph of the common logarithm function.

b. **Graph the common logarithm function and g on the same coordinate axes.**



The common logarithm function is shown in blue, and the graph of g is shown in green. Notice the key points that students should include on their hand-drawn sketches.

Example 2 (4 minutes): Graphing Transformations of Exponential Functions

Introduce the general form of the exponential function, noting that we do not need a horizontal scaling or a horizontal translation since these can always be rewritten using the properties of exponents. Demonstrate, or let students work with a partner on part (a), and make sure students have correct work to refer to when they work on the Problem Set. Since earlier lessons applied transformations to graphing exponential functions, parts (b), (c), and (d) should move along rather quickly. Continue to reinforce your expectations for sketching graphs of functions using transformations.

MP.7 & MP.8

Example 2: Graphing Transformations of Exponential Functions

The general form of the exponential function is given by $f(x) = a \cdot b^x + k$, where a, b, and k are real numbers such that b is a positive number not equal to 1.

a. **Use the properties of exponents to transform the function $g(x) = 3^{2x+1} - 2$ to the general form, and then graph it. What are the values of a, b, and k?**

Using the properties of exponents, $3^{2x+1} - 2 = 3^{2x} \cdot 3^1 - 2 = 3 \cdot 9^x - 2$. Thus, $g(x) = 3(9)^x - 2$, so $a = 3$, $b = 9$, and $k = -2$.

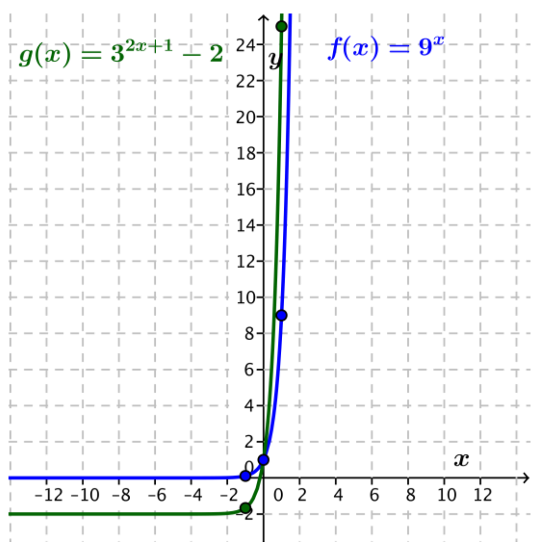
b. **Describe the graph of g as a transformation of the graph of $f(x) = 9^x$.**

The graph of g is a vertical scaling by a factor of 3 and a vertical translation down 2 units of the graph of f.

c. **Describe the graph of g as a transformation of the graph of $f(x) = 3^x$.**

The graph of g is a horizontal scaling by a factor of $\frac{1}{2}$, a vertical scaling by a factor of 3, and a vertical translation down 2 units of the graph of f.

d. **Sketch the graph of g using transformations.**



The graph of $f(x) = 9^x$ is shown in blue, and the graph of g is shown in green.

Exercises (4 minutes)

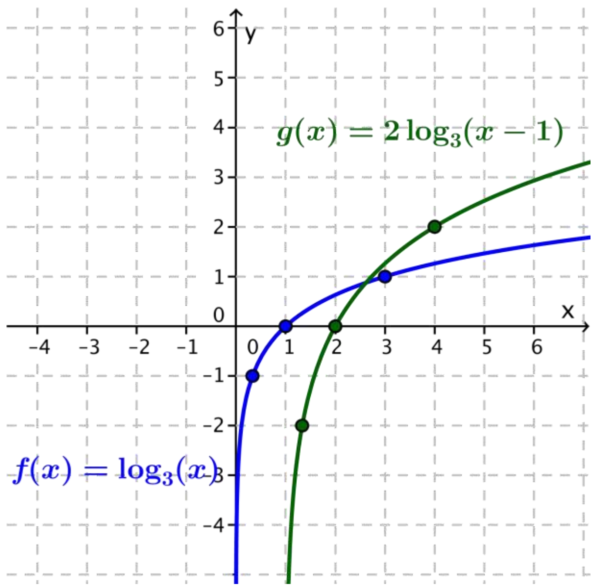
Students can work on these exercises independently or with a partner. Monitor their work by circulating around the classroom and checking for accuracy. Encourage students to describe the graph of g as a transformation of the graph of f in more than one way and to justify their answer analytically. In particular, emphasize how rewriting the expression using the properties of logarithms can make sketching the graphs easier because a horizontal scaling is revealed to have the same effect as a vertical translation when graphing logarithm functions, and a vertical translation is easier to sketch.

Exercises

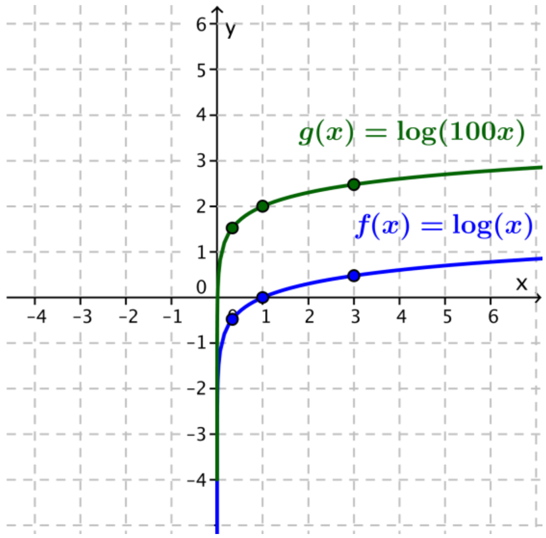
Graph each pair of functions by first graphing f and then graphing g by applying transformations of the graph of f. Describe the graph of g as a transformation of the graph of f.

1. $f(x) = \log_3(x)$ and $g(x) = 2\log_3(x-1)$

   The graph of g is the graph of f translated 1 unit to the right and stretched vertically by a factor of 2.

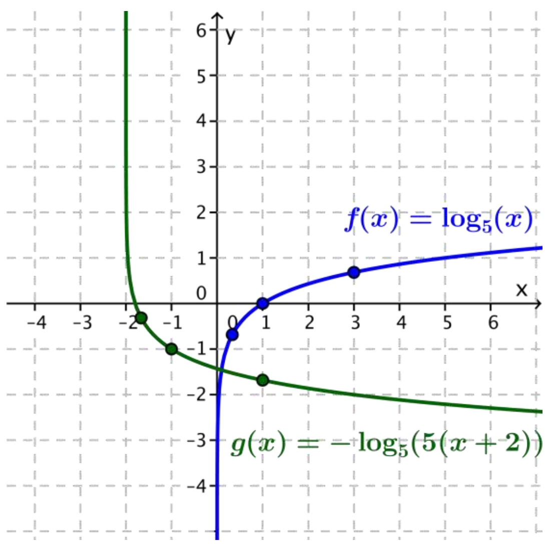


2. $f(x) = \log(x)$ and $g(x) = \log(100x)$



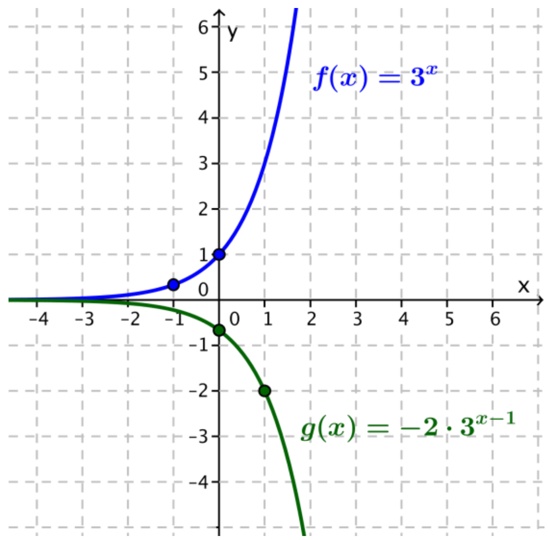
Because of the product property of logarithms, $g(x) = 2 + \log(x)$. The graph of g is the graph of f translated vertically 2 units.

3. $f(x) = \log_5 x$ and $g(x) = -\log_5(5(x+2))$



Since $-\log_5\big(5(x+2)\big) = -1 - \log_5(x+2)$ by the product property of logarithms and the distributive property, the graph of g is the graph of f translated 2 units to the left, reflected across the horizontal axis, and translated down 1 unit.

4. $f(x) = 3^x$ and $g(x) = -2 \cdot 3^{x-1}$



Since $-2 \cdot 3^{x-1} = -2 \cdot 3^x \cdot 3^{-1} = -\frac{2}{3} \cdot 3^x$ by the properties of exponents and the commutative property, the graph of g is the graph of f reflected across the horizontal axis and compressed by a factor of $\frac{2}{3}$.

There are multiple ways to obtain the graph of g from the graph of f through a sequence of transformations. One choice is to use the structure of $g(x) = -2 \cdot 3^{x-1}$ to translate the graph of f horizontally one unit left, reflect across the horizontal axis, and then scale vertically by a factor of 2. Another choice is to rewrite $g(x)$ as $g(x) = -\frac{2}{3}(3^x)$. Then the graph of g is obtained from the graph of f by reflecting the graph of f across the horizontal axis and then vertically scaling by a factor of $\frac{2}{3}$.

After a few minutes, have different groups share how they saw the transformations and discuss when it is advantageous to rewrite an expression before graphing and when it is not. For example, it might be easier in Exercise 4 to simply translate the graph 1 unit to the right rather than scale it by a factor of $\frac{2}{3}$. Also, make sure students are including a sketch of the end behavior of the functions.

Closing (5 minutes)

Provide students with an opportunity to summarize their learning with a partner by responding to the questions below. Their summaries should provide you with additional evidence of their understanding of this lesson.

- How do you apply properties of logarithms or exponents to rewrite $f(x) = \log_2(5x)$ and $g(x) = 3^{x+2} + 2$ in general form?
  - *Using the product properties:* $\log_2(5x) = \log_2(5) + \log_2(x)$, *so* $f(x) = \log_2(5) + \log_2(x)$ *in general form where* $k = \log_2(5)$, $a = 1$, *and* $h = 0$ *in the general form.*
  - *Using the product properties:* $3^{x+2} = 3^x \cdot 3^2$, *so* $g(x) = 9 \cdot 3^x + 2$ *in general form where* $a = 9$ *and* $k = 2$.
- How do transformations help you to sketch quick and accurate graphs of functions?
  - *Once you have a sketch of a basic logarithmic or exponential function, you can use transformations to quickly sketch the graph of a new logarithmic or exponential function.*

A summary of the key points of this lesson is provided. Review them with the class before beginning the Exit Ticket.

> **Lesson Summary**
>
> **GENERAL FORM OF A LOGARITHMIC FUNCTION: $f(x) = k + a\log_b(x - h)$ such that a, h, and k are real numbers, b is any positive number not equal to 1, and $x - h > 0$.**
>
> **GENERAL FORM OF AN EXPONENTIAL FUNCTION: $f(x) = a \cdot b^x + k$ such that a and k are real numbers, and b is any positive number not equal to 1.**
>
> **The properties of logarithms and exponents can be used to rewrite expressions for functions in equivalent forms that can then be graphed by applying transformations.**

Exit Ticket (5 minutes)

Name ______________________________ Date ______________

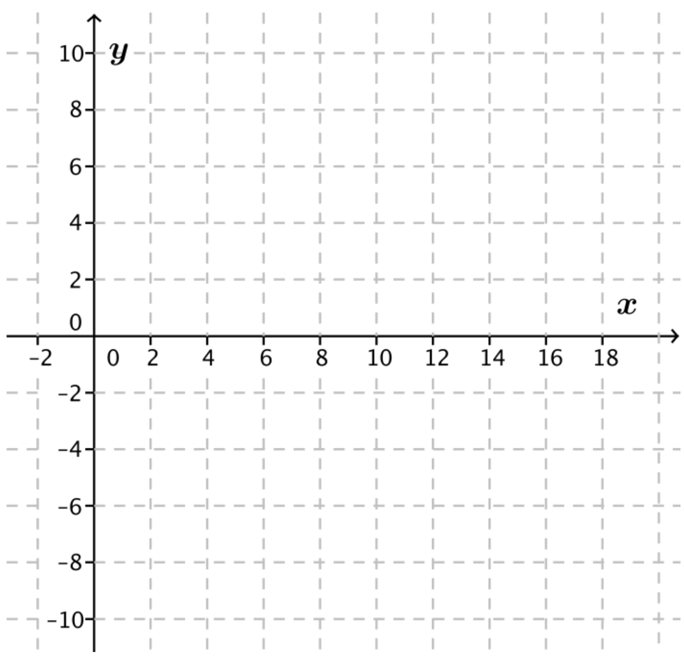
Lesson 20: Transformations of the Graphs of Logarithmic and Exponential Functions

Exit Ticket

1. Express $g(x) = -\log_4(2x)$ in the general form of a logarithmic function, $f(x) = k + a\log_b(x - h)$. Identify a, b, h, and k.

2. Use the structure of g when written in general form to describe the graph of g as a transformation of the graph of $h(x) = \log_4(x)$.

3. Graph g and h on the same coordinate axes.



Exit Ticket Sample Solutions

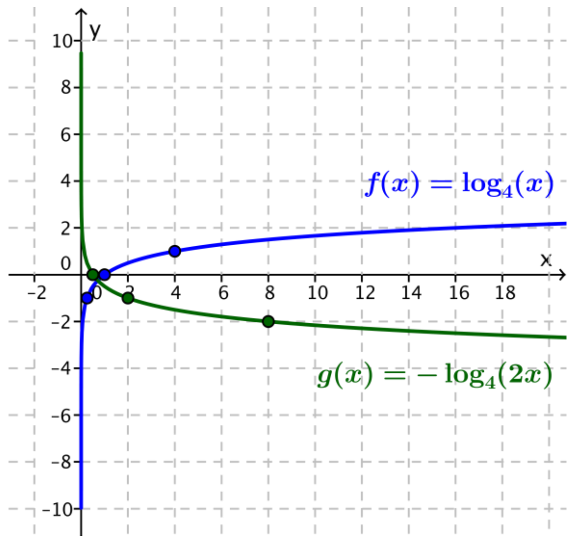
1. Express $g(x) = -\log_4(2x)$ in the general form of a logarithmic function, $f(x) = k + a\log_b(x - h)$. Identify a, b, h, and k.

   Since $-\log_4(2x) = -\log_4(2) + \log_4(x) = -\frac{1}{2} - \log_4(x)$, *the function is* $g(x) = -\frac{1}{2} - \log_4(x)$, *and* $a = -1$, $b = 4$, $h = 0$, *and* $k = -\frac{1}{2}$.

2. Use the structure of g when written in general form to describe the graph of g as a transformation of the graph of $h(x) = \log_4(x)$.

   The graph of g is the graph of h reflected about the horizontal axis and translated down $\frac{1}{2}$ unit.

3. Graph g and h on the same coordinate axes.



   The graph of h is shown in blue, and the graph of g is shown in green.

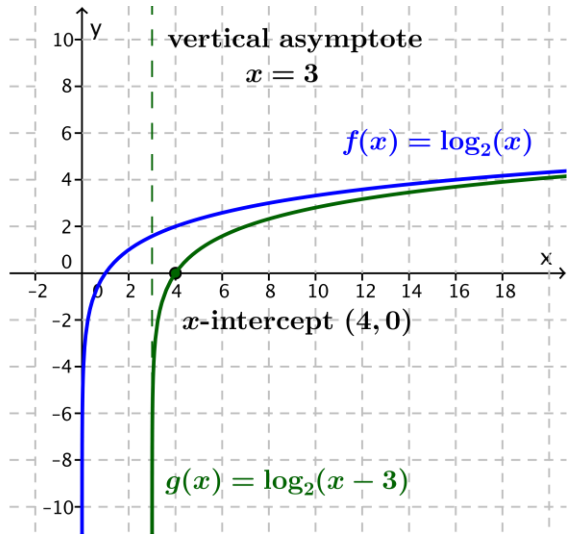
Problem Set Sample Solutions

Students should complete these problems without the use of a calculator.

1. **Describe each function as a transformation of the graph of a function in the form $f(x) = \log_b(x)$. Sketch the graph of f and the graph of g by hand. Label key features such as intercepts, intervals where g is increasing or decreasing, and the equation of the vertical asymptote.**

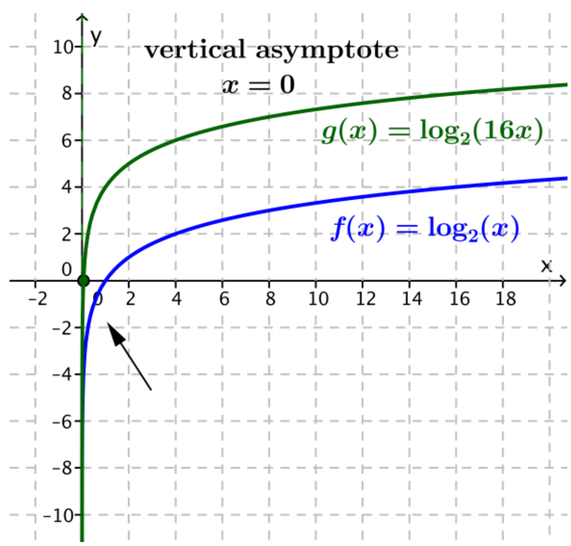
   a. $g(x) = \log_2(x - 3)$

   The graph of g is the graph of $f(x) = \log_2(x)$ translated horizontally 3 units to the right. The function g is increasing on $(3, \infty)$. The x-intercept is 4, and the vertical asymptote is $x = 3$.



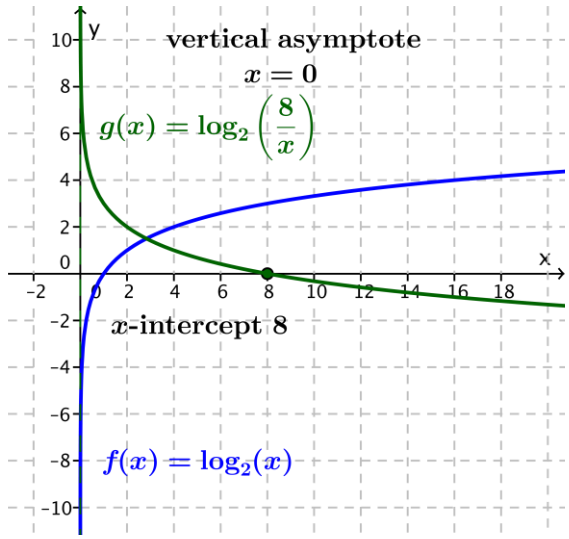
   b. $g(x) = \log_2(16x)$

   The graph of g is the graph of $f(x) = \log_2(x)$ translated vertically up 4 units. The function g is increasing on $(0, \infty)$. The x-intercept is 2^{-4}. The vertical asymptote is $x = 0$.



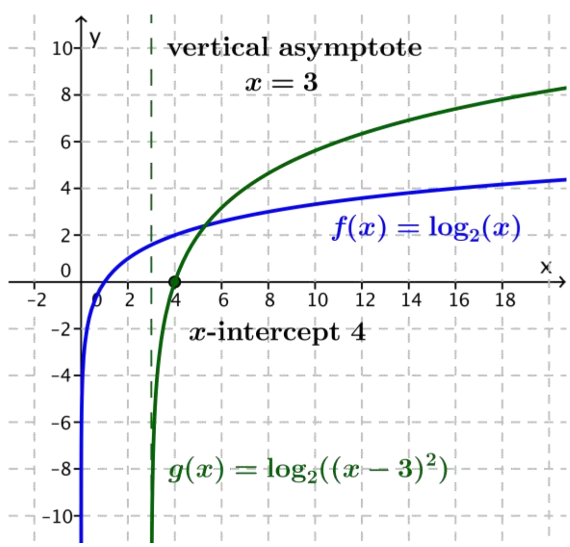
c. $g(x) = \log_2\left(\frac{8}{x}\right)$

The graph of g is the graph of $f(x) = \log_2(x)$ reflected about the horizontal axis and translated vertically up 3 units. The function g is decreasing on $(0, \infty)$. The x-intercept is 2^3. The vertical asymptote is $x = 0$. The reflected graph and the final graph are both shown in the solution.



d. $g(x) = \log_2((x-3)^2)$ for $x > 3$

The graph of g is the graph of $f(x) = \log_2(x)$ stretched vertically by a factor of 2 and translated horizontally 3 units to the right. The function g is increasing on $(3, \infty)$. The x-intercept is 4, and the vertical asymptote is $x = 3$.



2. Each function graphed below can be expressed as a transformation of the graph of $f(x) = \log(x)$. Write an algebraic function for g and h, and state the domain and range.

In Figure 1, $g(x) = -\log(x - 2)$ for $x > 2$. The domain of g is $x > 2$, and the range of g is all real numbers.

In Figure 2, $h(x) = 2 + \log(x)$ for $x > 0$. The domain of h is $x > 0$, and the range of h is all real numbers.

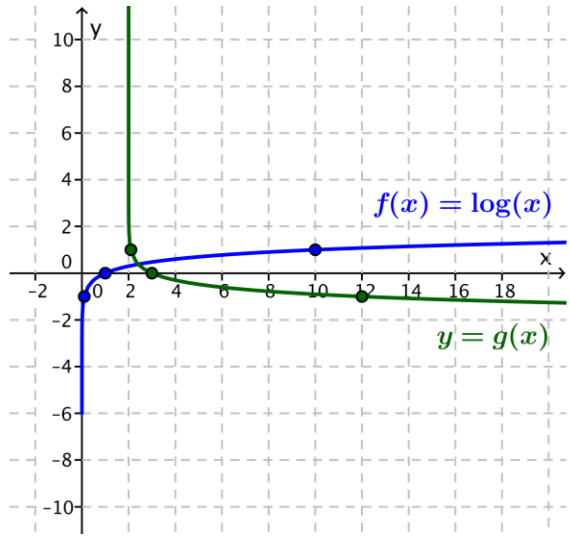


Figure 1: Graphs of $f(x) = \log(x)$ and the function g

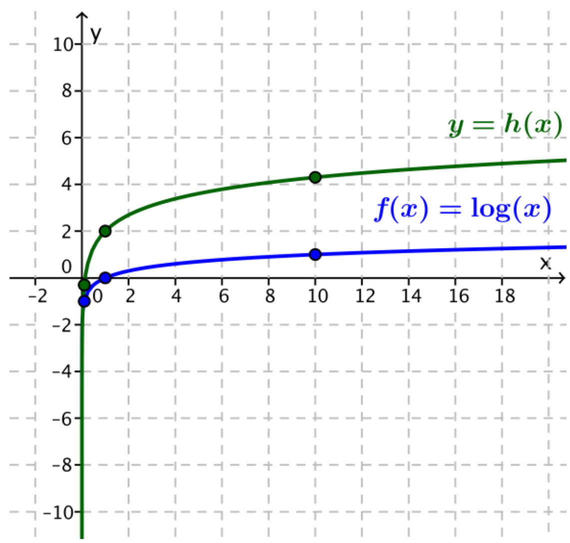
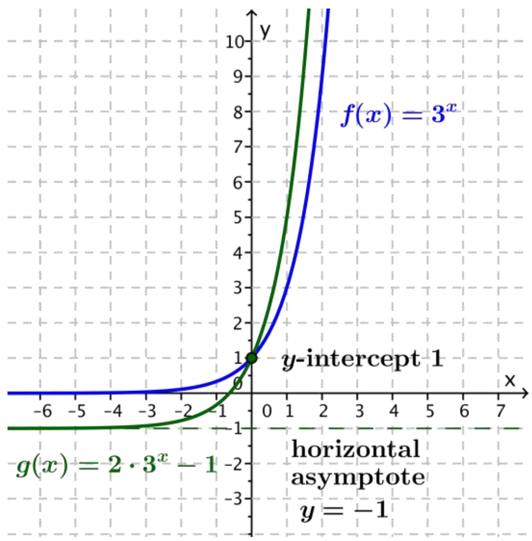


Figure 2: Graphs of $f(x) = \log(x)$ and the function h

3. Describe each function as a transformation of the graph of a function in the form $f(x) = b^x$. Sketch the graph of f and the graph of g by hand. Label key features such as intercepts, intervals where g is increasing or decreasing, and the horizontal asymptote.

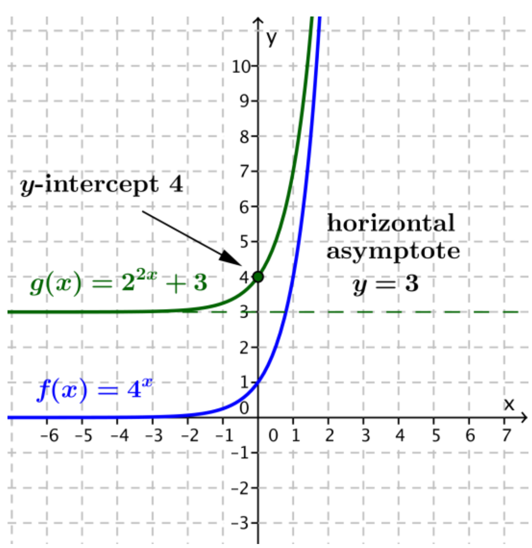
a. $g(x) = 2 \cdot 3^x - 1$

The graph of g is the graph of $f(x) = 3^x$ scaled vertically by a factor of 2 and translated vertically down 1 unit. The equation of the horizontal asymptote is $y = -1$. The y-intercept is 1, and the x-intercept is approximately -0.631. The function g is increasing for all real numbers.



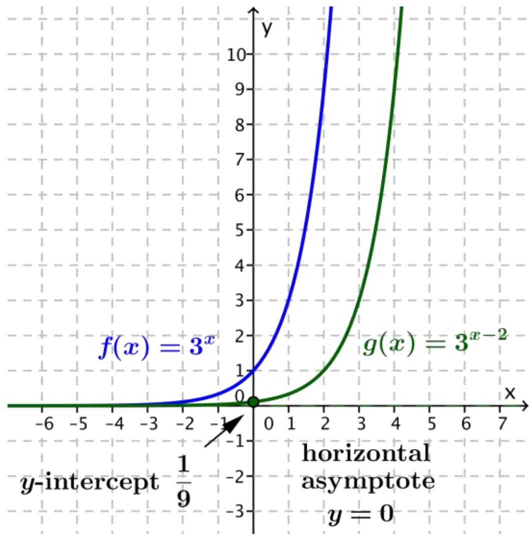
b. $g(x) = 2^{2x} + 3$

The graph of g is the graph of $f(x) = 4^x$ translated vertically up 3 units. The equation of the horizontal asymptote is $y = 3$. The y-intercept is 4. There is no x-intercept. The function g is increasing for all real numbers.



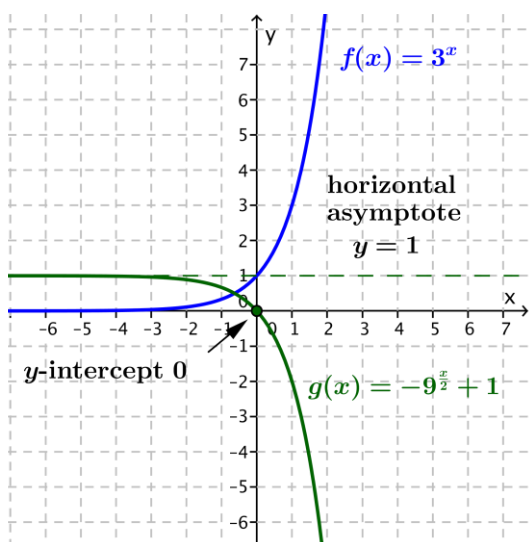
c. $g(x) = 3^{x-2}$

The graph of g is the graph of $f(x) = 3^x$ translated horizontally 2 units to the right OR the graph of f scaled vertically by a factor of $\frac{1}{9}$. The equation of the horizontal asymptote is $y = 0$. The y-intercept is $\frac{1}{9}$. There is no x-intercept, and the function g is increasing for all real numbers.



d. $g(x) = -9^{\frac{x}{2}} + 1$

The graph of g is the graph of $f(x) = 3^x$ reflected about the horizontal axis and then translated vertically up 1 unit. The equation of the horizontal asymptote is $y = 1$. The y-intercept is 0, and the x-intercept is also 0. The function g is decreasing for all real numbers.



4. Using the function $f(x) = 2^x$, create a new function g whose graph is a series of transformations of the graph of f with the following characteristics:
   - The function g is decreasing for all real numbers.
   - The equation for the horizontal asymptote is $y = 5$.
   - The y-intercept is 7.

   One possible solution is $g(x) = 2 \cdot 2^{-x} + 5$.

5. Using the function $f(x) = 2^x$, create a new function g whose graph is a series of transformations of the graph of f with the following characteristics:
   - The function g is increasing for all real numbers.
   - The equation for the horizontal asymptote is $y = 5$.
   - The y-intercept is 4.

   One possible solution is $g(x) = -(2^{-x}) + 5$.

6. Consider the function $g(x) = \left(\frac{1}{4}\right)^{x-3}$:

   a. Write the function g as an exponential function with base 4. Describe the transformations that would take the graph of $f(x) = 4^x$ to the graph of g.

   $$\left(\frac{1}{4}\right)^{x-3} = (4^{-1})^{x-3} = 4^{-x+3} = 4^3 \cdot 4^{-x}$$

   Thus, $g(x) = 64 \cdot 4^{-x}$. The graph of g is the graph of f reflected about the vertical axis and scaled vertically by a factor of 64.

   b. Write the function g as an exponential function with base 2. Describe two different series of transformations that would take the graph of $f(x) = 2^x$ to the graph of g.

   $$\left(\frac{1}{4}\right)^{x-3} = (2^{-2})^{x-3} = 2^{-2(x-3)} = 2^{-2x+6} = 64 \cdot 2^{-2x}$$

   Thus, $g(x) = 64 \cdot 2^{-2x}$, or $g(x) = 2^{-2(x-3)}$. To obtain the graph of g from the graph of f, you can scale the graph horizontally by a factor of $\frac{1}{2}$, reflect the graph about the vertical axis, and scale it vertically by a factor of 64. Or, you can scale the graph horizontally by a factor of $\frac{1}{2}$, reflect the graph about the vertical axis, and translate the resulting graph horizontally 3 units to the right.

MP.3

7. Explore the graphs of functions in the form $f(x) = \log(x^n)$ for $n > 1$. Explain how the graphs of these functions change as the values of n increase. Use a property of logarithms to support your reasoning.

   The graphs appear to be a vertical scaling of the common logarithm function by a factor of n. This is true because of the property of logarithms that states $\log(x^n) = n\log(x)$.

8. Use a graphical approach to solve each equation. If the equation has no solution, explain why.

   a. $\log(x) = \log(x - 2)$

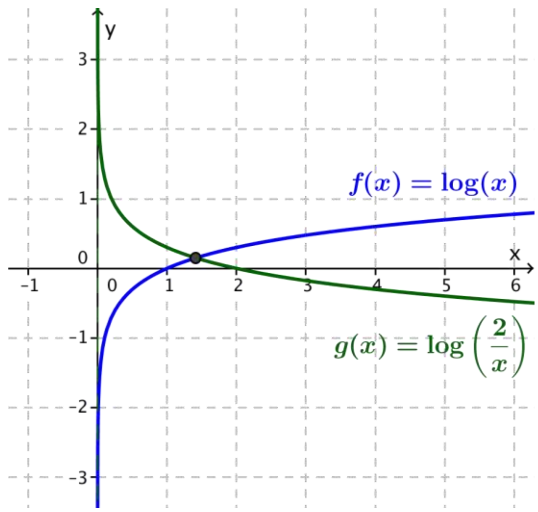
   This equation has no solution because the graphs of $y = \log(x)$ and $y = \log(x - 2)$ are horizontal translations of each other. Thus, their graphs do not intersect, and the corresponding equation has no solution.

b. $\log(x) = \log(2x)$

This equation has no solution because $\log(2x) = \log(2) + \log(x)$, which means that the graphs of $y = \log(x)$ and $y = \log(2x)$ are a vertical translation of each other. Thus, their graphs do not intersect, and the corresponding equation has no solution.

c. $\log(x) = \log\left(\frac{2}{x}\right)$

The solution is the x-coordinate of the intersection point of the graphs of $y = \log(x)$ and $y = \log(2) - \log(x)$. Since the graph of the function defined by the right side of the equation is a reflection across the horizontal axis and a vertical translation of the graph of the function defined by the left side of the equation, the graphs of these functions intersect in exactly one point with an x-coordinate approximately 1.5.



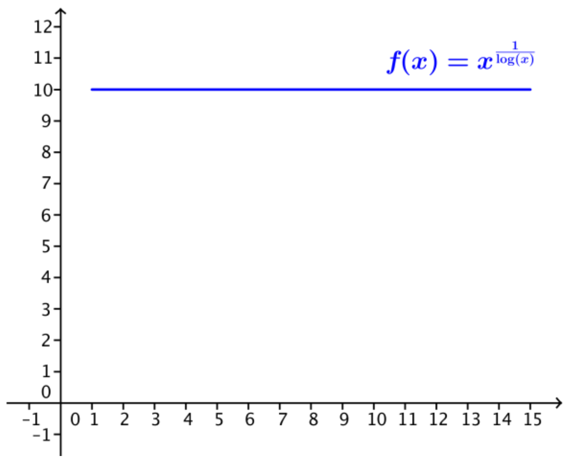
d. Show algebraically that the exact solution to the equation in part (c) is $\sqrt{2}$.

$$\log(x) = \log(2) - \log(x)$$
$$2\log(x) = \log(2)$$
$$\log(x) = \frac{1}{2}\log(2)$$
$$\log(x) = \log\left(2^{\frac{1}{2}}\right)$$
$$x = 2^{\frac{1}{2}}$$

Since $2^{\frac{1}{2}} = \sqrt{2}$, the exact solution is $\sqrt{2}$.

9. **Make a table of values for $f(x) = x^{\frac{1}{\log(x)}}$ for $x > 1$. *Graph the function f for $x > 1$.* Use properties of logarithms to explain what you see in the graph and the table of values.**

x	$f(x)$
10	10
100	$100^{\frac{1}{2}} = \sqrt{100} = \sqrt{10^2} = 10$
1000	$1\,000^{\frac{1}{3}} = \sqrt[3]{1000} = \sqrt[3]{10^3} = 10$
$10,000$	$10,000^{\frac{1}{4}} = \sqrt[4]{10000} = \sqrt[4]{10^4} = 10$
$100,000$	$100,000^{\frac{1}{5}} = \sqrt[5]{100000} = \sqrt[5]{10^5} = 10$



We see that $x^{\frac{1}{\log(x)}} = 10$ for all $x > 1$ because $\frac{1}{\log(x)} = \frac{\log(10)}{\log(x)} = \log_x(10)$. Therefore, when we substitute $\log_x(10)$ into the expression $x^{\frac{1}{\log(x)}}$, we get $x^{\log_x(10)}$, which is equal to 10 according to the definition of a logarithm.

Lesson 21: The Graph of the Natural Logarithm Function

Student Outcomes

- Students understand that the change of base property allows us to write every logarithm function as a vertical scaling of a natural logarithm function.
- Students graph the natural logarithm function and understand its relationship to other base b logarithm functions. They apply transformations to sketch the graph of natural logarithm functions by hand.

Lesson Notes

The focus of this lesson is developing fluency with sketching graphs of the natural logarithm function by hand and understanding that because of the change of base property of logarithms, every logarithm function can be expressed as a vertical scaling of the natural logarithm function (or any other base logarithm we choose, for that matter). This helps to explain why calculators typically feature a common logarithm button and a natural logarithm button. Students may question why we care so much about natural logarithms in mathematics. The importance of the particular base e will become apparent when they study calculus and learn that $f(x) = e^x$ is the only nonzero function that is its own derivative. For this reason, the exponential functions base e arise naturally as solutions to many differential equations, and are thus used to model many population scenarios. Additionally, $\ln(x)$ is equal to the area under the reciprocal function $f(t) = \frac{1}{t}$ from 1 to x; that is, $\ln(x) = \int_1^x \frac{1}{t} dt$.

This lesson begins by challenging students to compare and contrast logarithm functions with different bases in a group exploration. Students complete a graphic organizer to help focus their learning at the end of the exploration. Encourage students to use technology during this exploration. The focus should be on observing the patterns and making generalizations (MP.7, MP.8). A quick set of exercises primes students to explain their observations using the change of base property of logarithms. Once we have established that this property guarantees that graphs of logarithmic functions of one base are a vertical scaling of a graph of a logarithmic function of any other base, we tie the lesson back to the natural logarithm function. The lesson closes with demonstrations and practice with graphing natural logarithm functions to build fluency with creating sketches by hand (**F-IF.B.4**).

Classwork

Opening (3 minutes)

MP.1

Ask students to predict how the graphs of logarithmic functions are alike and how they are different when we consider different bases. Post this question on the board, give students a minute or two to think about their responses, and then have them share with a partner. Take a few responses from the entire class, but do not provide any concrete answers at this point. Student responses and the quality of their conversations will help you gauge their understanding of graphs of logarithmic functions up to this point and help you decide how to support student learning during the rest of this lesson.

- How are the graphs of $f(x) = \log_2(x)$, $g(x) = \log_3(x)$, and $h(x) = \log(x)$ similar? How are they different?
  - *They are always increasing for $b > 1$ and have one x-intercept at 1. As the base changes, they appear to increase more or less rapidly. They all have the same domain and range.*

Exploratory Challenge (15 minutes)

Have students work in groups of 4–5. Each group will need the student materials for this lesson, chart paper or personal white boards, and markers, and each student or pair of students needs access to graphing technology such as a graphing calculator. Have each student select at least one base value from the following list: $b = \left\{\frac{1}{10}, \frac{1}{2}, 2, 5, 20, 100\right\}$. Using a graphing calculator or other graphing technology, students should independently explore how their selected base-b logarithm function's graph compares to the graph of the common logarithm function $f(x) = \log(x)$. Next, have them describe what they observed in writing and report it to their group members. As a group, students then categorize their findings based on the value of b and record their observations on chart paper. Have each group present their findings to the entire class. As you debrief this exploration as a whole class, focus on clarifying student language in their descriptions and encourage students to revise their written descriptions to further clarify what they wrote. Student work should be similar to the sample responses shown below.

Scaffolding:

- For English language learners, the graphic organizer provides some scaffolding, but sentence frames may also help students respond to part (c) in this exploration. For example,

  "Compared to the graph of f, the graph of my function was ______."

  OR

  "My function's graph was a ____________ transformation of the graph of f."
- For advanced learners, rather than provide the explicit steps listed in the Exploratory Challenge, present the problem on the board, give them technology and chart paper, and start them on the group presentation.

MP.8

Exploratory Challenge

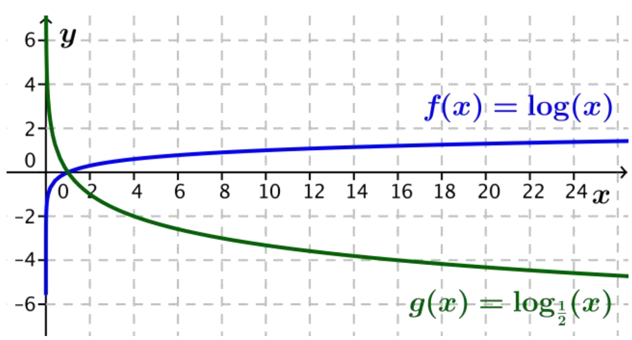
Your task is to compare graphs of base b logarithm functions to the graph of the common logarithm function $f(x) = \log(x)$ and summarize your results with your group. Recall that the base of the common logarithm function is 10. A graph of f is provided below.

a. **Select at least one base value from this list: $\frac{1}{10}, \frac{1}{2}, 2, 5, 20, 100$. Write a function in the form $g(x) = \log_b(x)$ for your selected base value, b.**

Students should use one of the numbers from the list to write their function. The sample solutions will use $g(x) = \log_{\frac{1}{2}}(x)$.

b. **Graph the functions f and g in the same viewing window using a graphing calculator or other graphing application, and then add a sketch of the graph of g to the graph of f shown below.**

The graph of $f(x) = \log(x)$ is shown in blue, and the graph of $g(x) = \log_{\frac{1}{2}}(x)$ is shown in green.



c. **Describe how the graph of g for the base you selected compares to the graph of $f(x) = \log(x)$.**

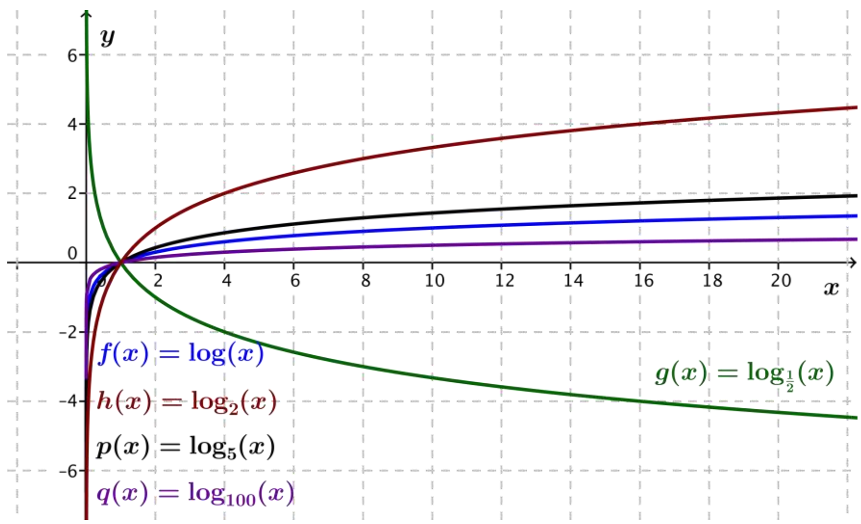
Answers will vary depending on the base selected. For example, when the base is 20, the graph of g appears to be a vertical scaling of the common logarithm function by a factor less than 1.

MP.8

d. **Share your results with your group and record observations on the graphic organizer below. Prepare a group presentation that summarizes the group's findings.**

How does the graph of $g(x) = \log_b(x)$ compare to the graph of $f(x) = \log(x)$ for various values of b?	
$0 < b < 1$	***The function g is decreasing. Its graph is a reflection about the horizontal axis of the graph of a logarithmic function whose base is the reciprocal of b.***
$1 < b < 10$	***When b is between 1 and 10, the graph of g appears to be a vertical scaling of the graph of f by a factor greater than 1. As b gets closer to 10, the graph of g gets closer to the graph of f and appears less steep.***
$b > 10$	***When b is greater than 10, the graph of g appears to be a vertical scaling of the graph of f by a factor between 0 and 1. As b grows, the graph of g grows at a slower rate and appears to move closer to the horizontal axis than the graphs of logarithmic functions whose bases are closest to 10.***

The graphs of logarithmic functions with several of the bases listed is shown below.



MP.3 & MP.7

As the groups work through this exploration, be sure to guide them to appropriate conclusions without explicitly telling them answers. After or during the group presentations, ask questions such as the ones listed below to clarify student understanding.

- Why are the functions decreasing when b is between 0 and 1?
  - *Consider $b = \frac{1}{2}$. Then, $y = \log_b(x) = \log_{\frac{1}{2}}(x)$, and $\left(\frac{1}{2}\right)^y = x$. If $x > 1$, then $y < 0$. As x increases, y becomes larger in magnitude while staying negative, so y decreases. Thus, the function is decreasing.*
  - *The exponential function for bases between 0 and 1 is a decreasing function, so when we have a logarithmic function with these bases, the range values will also decrease as the domain values increase.*

MP.3 & MP.7

- How does the graph of $y = \log_b(x)$ relate to the graph of $y = \log_{\frac{1}{b}}(x)$? Explain why this relationship exists.
  - *The graphs appear to be reflections of one another about the horizontal axis. The graphs of $y = b^x$ and $y = \left(\frac{1}{b}\right)^x$ are reflections about the vertical axis because $\left(\frac{1}{b}\right)^x = b^{-x}$. Thus, when we exchange the domain and range values to form the related logarithmic functions, they will also be reflections of one another but about the horizontal axis.*
- Why do smaller bases $b > 1$ produce steeper graphs and larger bases produce flatter graphs?
  - *Logarithmic functions with a smaller base grow at a faster rate, making the graph steeper. For example, $\log_2(64) = 6$, $\log_4(64) = 3$, and $\log_8(64) = 2$. The same input of 64 produces a smaller output as the size of the base increases.*
- Where would the graph of $y = \ln(x)$ sit in relation to these graphs? How do you know?
  - *The graph of $y = \ln(x)$ would be in between the two graphs of $y = \log_2(x)$ and $y = \log_5(x)$ because e is a number between 2 and 5.*
- The graphs of these functions appear to be vertical scalings of each other. How could we prove that this is true?
  - *We would have to show that we can rewrite each function as a constant multiple of another logarithmic function.*

Check to make sure each student has recorded appropriate information in the graphic organizer in part (d) before moving on. Post the group presentations on the board for reference during the rest of this lesson.

Exercise 1 (5 minutes)

Announce that now we will explore how all these graphs are related using a property of logarithms. Students should be able to complete this exercise quickly. Some students may already start to understand why the graphs appeared the way they did in the Exploratory Challenge as they work through these exercises.

Exercise 1

Use the change of base property to rewrite each logarithmic function in terms of the common logarithm function.

Base b	**Base 10 (Common logarithm)**
$g_1(x) = \log_{\frac{1}{4}}(x)$	$g_1(x) = \dfrac{\log(x)}{\log\left(\frac{1}{4}\right)}$
$g_2(x) = \log_{\frac{1}{2}}(x)$	$g_2(x) = \dfrac{\log(x)}{\log\left(\frac{1}{2}\right)}$
$g_3(x) = \log_2(x)$	$g_3(x) = \dfrac{\log(x)}{\log(2)}$
$g_4(x) = \log_5(x)$	$g_4(x) = \dfrac{\log(x)}{\log(5)}$
$g_5(x) = \log_{20}(x)$	$g_5(x) = \dfrac{\log(x)}{\log(20)}$
$g_6(x) = \log_{100}(x)$	$g_6(x) = \dfrac{\log(x)}{\log(100)}$

Discussion (5 minutes)

Lead a discussion to help students observe that each function in base 10 is divided by a constant (which is the same as multiplying by the reciprocal of that number). Have students explore the values of the constants using their calculators, and have them make sense of why the graphs appear the way they do compared to the graph of the common logarithm function. For example, $\log(2) \approx 0.69$. When dividing by a number between 0 and 1, you get the same result as multiplying by its reciprocal, which is a number greater than 1. The values of $\log\left(\frac{1}{2}\right)$ and $\log\left(\frac{1}{4}\right)$ are negative, which explains why the graphs of functions g_1 and g_2 are vertical scalings and reflections of the graph of the common logarithm function. When the base is greater than 10, as is the case with functions g_5 and g_6, we are dividing by a number greater than 1, which is the same as multiplying by a number between 0 and 1, which compresses the graph vertically.

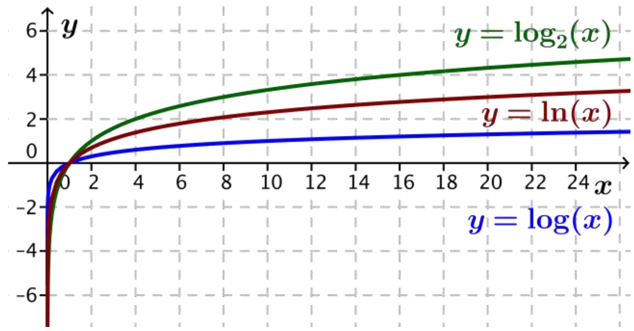
- How do the functions from Exercise 1 that you wrote in base 10 compare to the function $f(x) = \log(x)$?
  - *They are a constant multiple of the function f. For example, $\log(100) = 2$, so the function $g_6(x) = \frac{\log(x)}{\log(100)}$ could also be written as $g(x) = \frac{1}{2}\log(x)$.*
- Approximate the values of the constants in the functions from Exercise 1. How do those values help to explain why the graphs are a vertical stretch of the common logarithm function when the base is between 1 and 10, and a vertical compression when the base is greater than 10? Why are the functions decreasing when the base is between 0 and 1?
  - *When the base is between 1 and 10, the common logarithms are between 0 and 1. Dividing by a number between 0 and 1 is the same as multiplying by a number larger than 1, which will scale the graph vertically by a factor greater than 1. For bases greater than 10, the common logarithm function is multiplied by a number between 0 and 1. The functions decrease when the base is between 0 and 1 because the common logarithms of those numbers are less than 0.*

Next, revisit the question posed earlier regarding the graph of $y = \ln(x)$, the natural logarithm function, as a way to transition into the last portion of this lesson.

- Where would the graph of $y = \ln(x)$ sit in relation to these graphs? How do you know?
  - *The graph of $y = \ln(x)$ would lie in between the graphs of $y = \log_2(x)$ and $y = \log_5(x)$ because e is a number between 2 and 5.*

Example 1 (5 minutes): The Graph of the Natural Logarithm Function $f(x) = \ln(x)$

The example that follows demonstrates how to sketch the graph of the natural logarithm function by hand and shows more precisely where it sits in relation to base-2 and base-10 logarithm functions.

> **Example 1: The Graph of the Natural Logarithm Function $f(x) = \ln(x)$**
>
> **Graph the natural logarithm function below to demonstrate where it sits in relation to the graphs of the base-2 and base-10 logarithm functions.**
>
> 
>

The graphs are not labeled in the student file. You can question students about this to informally assess their understanding at this point.

- Which graph is $y = \log_2(x)$, and which one is $y = \log(x)$? How can you tell?
  - *Since the base 2 is smaller, the logarithm function base 2 grows more quickly than the base-10 logarithm function, so the green graph is the graph of $y = \log_2(x)$. You can also verify which graph is which by identifying a few points and substituting them into the equations to see which is true. For example, the blue graph appears to contain the point $(1, 10)$. Since $1 = \log(10)$, the blue graph represents the common logarithm function.*

Remind students that $e \approx 2.718$. Create a table of values like the one shown below and then plot these points. Connect the points with a smooth curve. When students are sketching by hand in the next example, have them plot fewer points, perhaps where the y-values are integers only.

x	$f(x) = \ln(x)$
$\frac{1}{e} \approx 0.369$	-1
1	0
$e^{0.5} \approx 1.649$	0.5
$e^{1} \approx 2.718$	1
$e^{1.5} \approx 4.482$	1.5
$e^{2} \approx 7.389$	2

Example 2 (5 minutes)

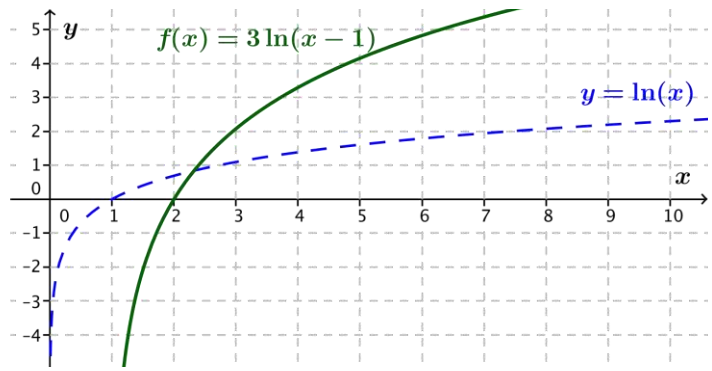
In this example, part (a) models how to sketch graphs by applying transformations. Then, show students in part (b) how to rewrite the function as a natural logarithm function, and sketch the graph by applying transformations of the graph of $f(x) = \ln(x)$. Model the transformations in stages. First, sketch the graph of $y = \ln(x)$; next, sketch a second graph applying the first transformation; finally, sketch a graph applying the last transformation to the second graph you made.

Example 2

Graph each function by applying transformations of the graphs of the natural logarithm function.

a. $f(x) = 3\ln(x-1)$

The graph of f is the graph of $y = \ln(x)$ shifted horizontally 1 unit to the right, stretched vertically by a factor of 3.

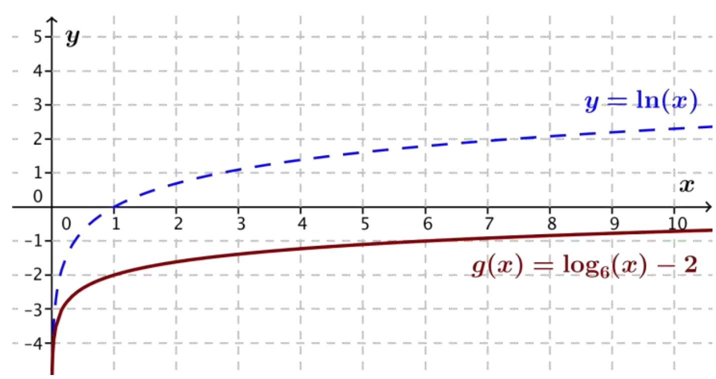


b. $g(x) = \log_6(x) - 2$

First, write g as a natural logarithm function.

$$g(x) = \frac{\ln(x)}{\ln(6)} - 2$$

Since $\frac{1}{\ln(6)} \approx 0.558$, the graph of g will be the graph of $y = \ln(x)$ scaled vertically by a factor of approximately 0.56 and translated down 2 units.



Closing (2 minutes)

Have students summarize what they have learned in this lesson by revisiting the question from the Opening. Students should revise their initial responses and either discuss their answers with a partner or write a brief individual reflection. The responses should be similar to what is listed in the Lesson Summary.

- How are the graphs of logarithmic functions with different bases alike? How are they different?
  - *They have the same x-intercept 1, and when the base is greater than 1, the functions are increasing. They all have the same domain and range. They are different because as the base changes, the steepness of the graph of the function changes. Logarithmic functions with larger bases grow at slower rates.*
- How does the change of base property guarantee that every logarithmic function could be expressed in the form $f(x) = k + a\ln(x - h)$?
  - *The change of base property guarantees that we can convert any logarithmic expression in base b to a natural logarithmic expression where the denominator of the expression is constant.*

Exit Ticket (5 minutes)

Name ___________________________________ Date ____________________

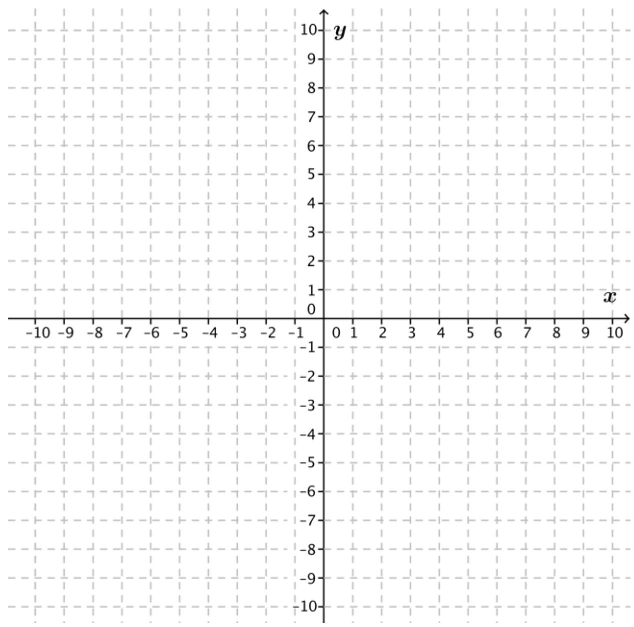
Lesson 21: The Graph of the Natural Logarithm Function

Exit Ticket

1.

a. Describe the graph of $g(x) = 2 - \ln(x+3)$ as a transformation of the graph of $f(x) = \ln(x)$.

b. Sketch the graphs of f and g by hand.



2. Explain where the graph of $g(x) = \log_3(2x)$ would sit in relation to the graph of $f(x) = \ln(x)$. Justify your answer using properties of logarithms and your knowledge of transformations of graph of functions.

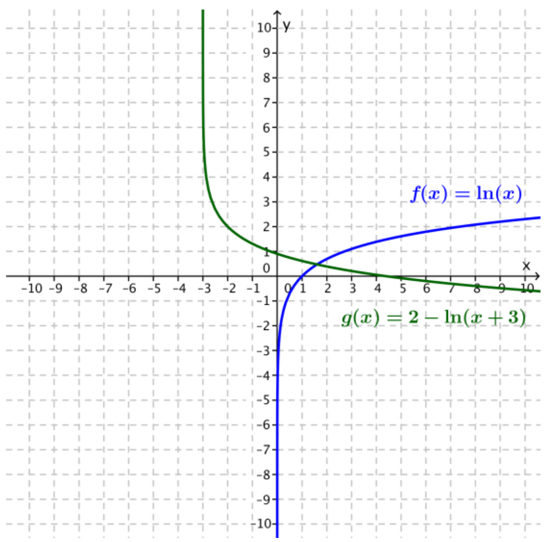
Exit Ticket Sample Solutions

1.

a. **Describe the graph of $g(x) = 2 - \ln(x+3)$ as a transformation of the graph of $f(x) = \ln(x)$.**

The graph of g is the graph of f translated 3 units to the left, reflected about the horizontal axis, and translated up 2 units.

b. **Sketch the graphs of f and g by hand.**



2. **Explain where the graph of $g(x) = \log_3(2x)$ would sit in relation to the graph of $f(x) = \ln(x)$. Justify your answer using properties of logarithms and your knowledge of transformations of graph of functions.**

Since $\log_3(2x) = \frac{\ln(2x)}{\ln(3)} = \frac{\ln(2)}{\ln(3)} + \frac{\ln(x)}{\ln(3)}$, the graph of g would be a vertical shift and a vertical scaling by a factor greater than 1 of the graph of f. The graph of g will lie vertically above the graph of f.

Problem Set Sample Solutions

1. **Rewrite each logarithmic function as a natural logarithm function.**

a. $f(x) = \log_5(x)$

$$f(x) = \frac{\ln(x)}{\ln(5)}$$

b. $f(x) = \log_2(x-3)$

$$f(x) = \frac{\ln(x-3)}{\ln(2)}$$

c. $f(x) = \log_2\left(\frac{x}{3}\right)$

$f(x) = \frac{\ln(x)}{\ln(2)} - \frac{\ln(3)}{\ln(2)}$

d. $f(x) = 3 - \log(x)$

$f(x) = 3 - \frac{\ln(x)}{\ln(10)}$

e. $f(x) = 2\log(x+3)$

$f(x) = \frac{2}{\ln(10)}\ln(x+3)$

f. $f(x) = \log_5(25x)$

$f(x) = 2 + \frac{\ln(x)}{\ln(5)}$

2. Describe each function as a transformation of the natural logarithm function $f(x) = \ln(x)$.

a. $g(x) = 3\ln(x+2)$

The graph of g is the graph of f translated 2 units to the left and scaled vertically by a factor of 3.

b. $g(x) = -\ln(1-x)$

The graph of g is the graph of f translated 1 unit to the right, reflected about $x = 1$, and then reflected about the horizontal axis.

c. $g(x) = 2 + \ln(e^2 x)$

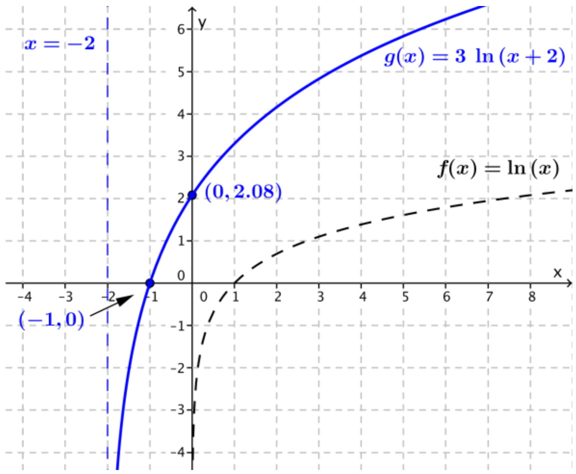
The graph of g is the graph of f translated up 4 units.

d. $g(x) = \log_5(25x)$

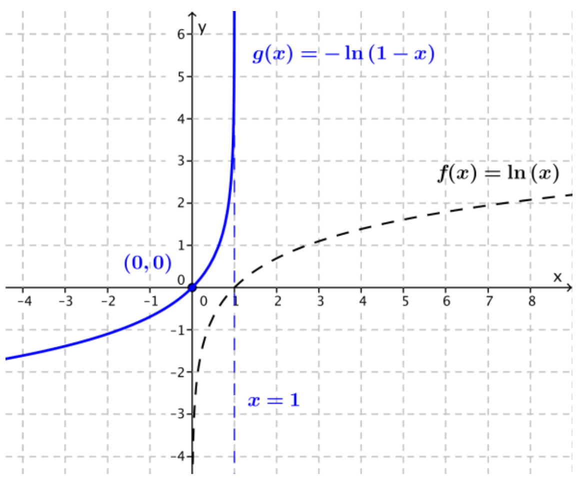
The graph of g is the graph of f translated up 2 units and scaled vertically by a factor of $\frac{1}{\ln(5)}$.

3. Sketch the graphs of each function in Problem 2, and identify the key features including intercepts, intervals where the function is increasing or decreasing, and the vertical asymptote.

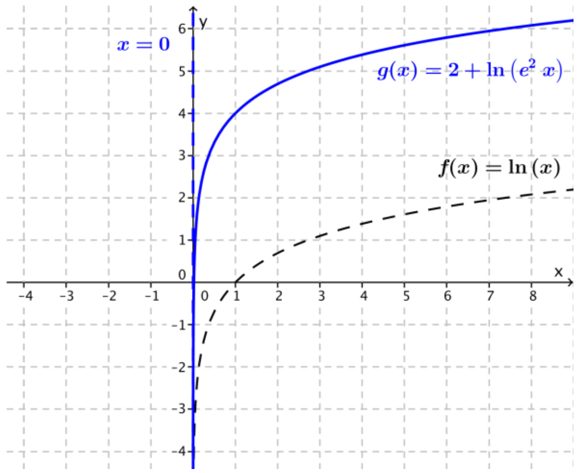
a. *The equation of the vertical asymptote is $x = -2$. The x-intercept is -1. The function is increasing for all $x > -2$. The y-intercept is approximately 2.079.*



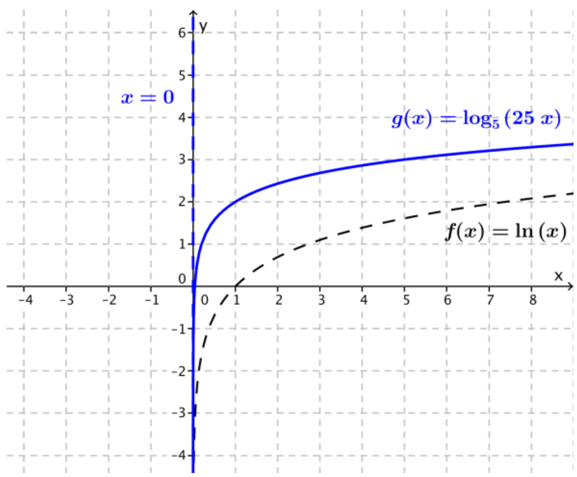
b. *The equation of the vertical asymptote is $x = 1$. The x-intercept is 0. The function is increasing for all $x < 1$. The y-intercept is 0.*



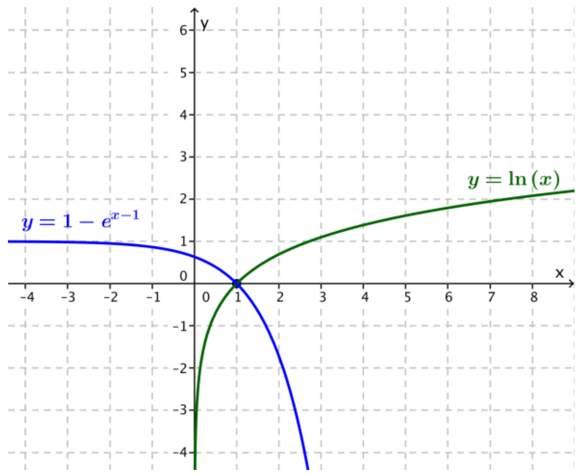
c. ***The equation of the vertical asymptote is $x = 0$. The x-intercept is approximately 0.018. The function is increasing for all $x > 0$.***



d. ***The equation of the vertical asymptote is $x = 0$. The x-intercept is 0.04. The function is increasing for all $x > 0$.***

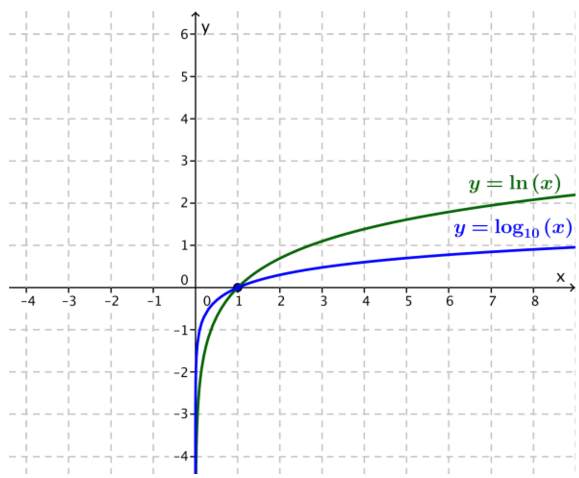


4. **Solve the equation $1 - e^{x-1} = \ln(x)$ graphically, without using a calculator.**



It appears that the two graphs intersect at the point $(1, 0)$. Checking, we see that $1 - e^{1-1} = 1 - 1 = 0$, and we know that $\ln(1) = 0$, so when $x = 0$, we have $1 - e^{x-1} = \ln(x)$. Thus 0 is a solution to this equation. From the graph, it is the only solution.

5. **Use a graphical approach to explain why the equation $\log(x) = \ln(x)$ has only one solution.**



The graphs intersect in only one point $(1, 0)$, so the equation has only one solution.

MP.3

6. Juliet tried to solve this equation as shown below using the change of base property and concluded there is no solution because $\ln(10) \neq 1$. Construct an argument to support or refute her reasoning.

$$\log(x) = \ln(x)$$
$$\frac{\ln(x)}{\ln(10)} = \ln(x)$$
$$\left(\frac{\ln(x)}{\ln(10)}\right)\frac{1}{\ln(x)} = (\ln(x))\frac{1}{\ln(x)}$$
$$\frac{1}{\ln(10)} = 1$$

Juliet's approach works as long as $\ln(x) \neq 0$, which occurs when $x = 1$. The solution to this equation is 1. When you divide both sides of an equation by an algebraic expression, you need to impose restrictions so that you are not dividing by 0. In this case, Juliet divided by $\ln(x)$, which is not valid if $x = 1$. This division caused the equation in the third and final lines of her solution to have no solution; however, the original equation is true when x is 1.

7. Consider the function f given by $f(x) = \log_x(100)$ for $x > 0$ and $x \neq 1$.

a. What are the values of $f(100)$, $f(10)$, and $f(\sqrt{10})$?

$f(100) = 1,\ f(10) = 2,\ f(\sqrt{10}) = 4$

b. Why is the value 1 excluded from the domain of this function?

The value 1 is excluded from the domain because 1 is not a base of an exponential function since it would produce the graph of a constant function. Since logarithmic functions by definition are related to exponential functions, we cannot have a logarithm with base 1.

c. Find a value x so that $f(x) = 0.5$.

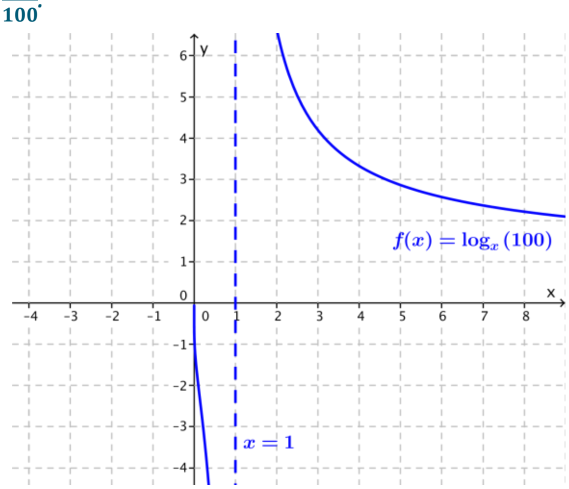
$$\log_x(100) = 0.5$$
$$x^{0.5} = 100$$
$$x = 10{,}000$$

The value of x that satisfies this equation is $10{,}000$.

d. Find a value w so that $f(w) = -1$.

The value of w that satisfies this equation is $\frac{1}{100}$.

e. Sketch a graph of $y = \log_x(100)$ for $x > 0$ and $x \neq 1$.



Lesson 22: Choosing a Model

Student Outcomes

- Students analyze data and real-world situations and find a function to use as a model.
- Students study properties of linear, quadratic, sinusoidal, and exponential functions.

Lesson Notes

When modeling authentic data, a mathematician or scientist must apply knowledge of conditions under which the data were gathered before fitting a function to the data. This lesson addresses focus standards **F-BF.A.1a** and **F-LE.A.2**, which ask students to determine an explicit expression for a function from a real-world context. Additionally, the entire lesson focuses on MP.4, as students use and analyze mathematical models for a variety of real-world situations. Students have already studied linear, quadratic, sinusoidal, and exponential functions, so the principal question being asked in this lesson is how to use what we know about the context to choose an appropriate function to model the data. We begin by fitting a curve to existing data points for which the data taken out of context do not clearly suggest the model. We then begin choosing a function type to model various scenarios. This is primarily a summative lesson that begins with a review of properties of linear, polynomial, exponential, and sinusoidal functions.

Classwork

Opening Exercise (8 minutes)

In this example, students are given points to plot and a real-world context. The point of this exercise is that either a quadratic model or a sinusoidal model can fit the data, but the real-world context is necessary to determine how to appropriately model the data. Encourage students to use calculators or another graphing utility to produce graphs of the functions in this exercise and to then copy the graph to the axes provided on the student pages.

Scaffolding:

Students who struggle with sketching the sinusoidal curve may need to be reminded that its zeros are at $\frac{2\pi n}{3}$ for real n. Those who struggle with sketching the parabola may need the hint that it has vertex $(2.25, 186)$ and passes through $(0, 105)$.

Opening Exercise

a. **You are working on a team analyzing the following data gathered by your colleagues:**

$$(-1.1, 5), (0, 105), (1.5, 178), (4.3, 120).$$

Your coworker Alexandra says that the model you should use to fit the data is

$$k(t) = 100 \cdot \sin(1.5t) + 105.$$

Sketch Alexandra's model on the axes at left on the next page.

b. **How does the graph of Alexandra's model $k(t) = 100 \cdot \sin(1.5t) + 105$ relate to the four points? Is her model a good fit to this data?**

The curve passes through or close to all four of those points, so this model fits the data well.

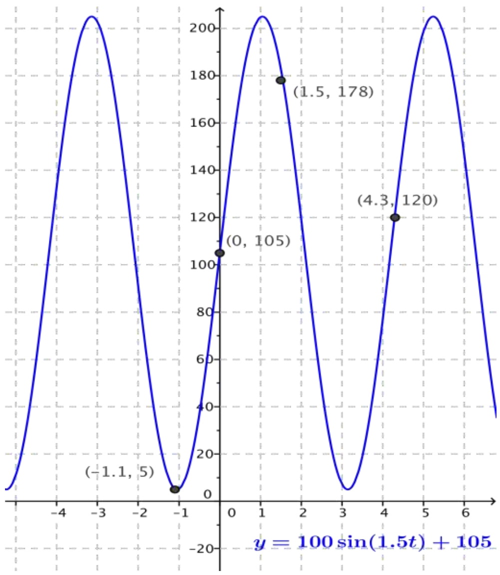
c. **Another teammate Randall says that the model you should use to fit the data is**

$$g(t) = -16t^2 + 72t + 105.$$

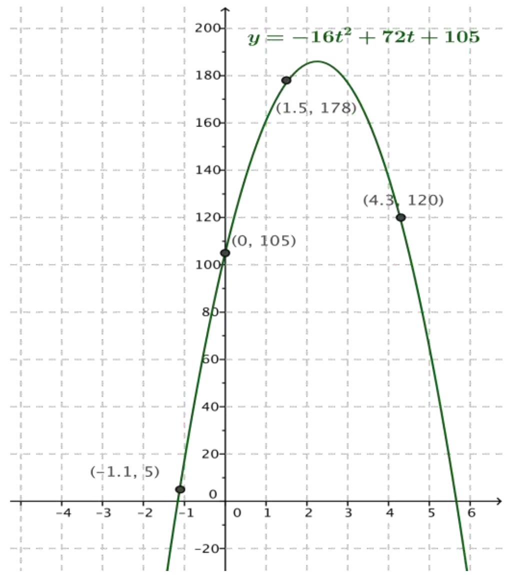
Sketch Randall's model on the axes at right on the next page.

d. **How does the graph of Randall's model $g(t) = -16t^2 + 72t + 105$ relate to the four points? Is his model a good fit to the data?**

The curve passes through or close to all four of those points, so this model also fits the data well.



Alexandra's Model



Randall's Model

e. **Suppose the four points represent positions of a projectile fired into the air. Which of the two models is more appropriate in that situation, and why?**

The quadratic curve of Randall's model makes more sense in this context. Some students may know that the acceleration of the projectile due to gravity warrants a quadratic term in the equation for the function, and all students should understand that the motion of the projectile is not cyclic—once it hits the ground, it stays there.

f. **In general, how do we know which model to choose?**

It entirely depends on the context and what we know about what the data represent.

Discussion (8 minutes)

- As the previous exercise showed, just knowing the coordinates of the data points does not tell us which type of function to use to model them. We need to know something about the context in which the data is gathered before we can decide what type of function to use as a model.
- We focus on linear, quadratic, sinusoidal, and exponential models in this lesson.
- Some things we need to think about: What is the end behavior of this type of function? How does the function change? If the input value (x or t) increases by 1 unit, what happens to the output value (y)? Is the function increasing or decreasing, or does it do both? Are there any relative maximum or minimum values? What is the range of the function?

- What are the characteristics of a nonconstant linear function? *(Allow students to suggest characteristics, but be sure that the traits listed below have been mentioned before moving on to quadratic models.)*
  - *If x increases by 1, then y changes by a fixed amount.*
  - *The function is always increasing or always decreasing at the same rate. This rate is the slope of the line when the function is graphed.*
  - *There is no maximum and no minimum value of the function.*
  - *The range is all real numbers.*
  - *The end behavior is that the function increases to ∞ in one direction and decreases to $-\infty$ in the other direction.*
- What are the characteristics of a quadratic function?
  - *The second differences of the function are constant, meaning that if x increases by 1, then y increases linearly with x.*
  - *The function increases and decreases, changing direction one time.*
  - *There is either a maximum value or a minimum value.*
  - *The range is either $(-\infty, a)$ or (a, ∞) for some real number a.*
  - *The end behavior is that either the function increases to ∞ in both directions or the function decreases to $-\infty$ in both directions.*
- What are the characteristics of a sinusoidal function?
  - *The function is periodic; the function values repeat over fixed intervals.*
  - *There is one relative maximum value of the function and one relative minimum value of the function. The function attains these values periodically, alternating between the maximum and minimum value.*
  - *The range of the function is $[a, b]$, for some real numbers $a < b$.*
  - *The end behavior of a sinusoidal function is that it bounces between the relative maximum and relative minimum values as $x \to \infty$ and as $x \to -\infty$.*
- What are the characteristics of an exponential function?
  - *The function increases (or decreases) at a rate proportional to the current value of the function.*
  - *The function is either always increasing or always decreasing.*
  - *Either the function values approach a constant as $x \to -\infty$, and the function values approach $\pm\infty$ as $x \to \infty$, or the function values approach $\pm\infty$ as $x \to -\infty$, and the function values approach a constant as $x \to -\infty$. (The function flattens off in one direction and approaches either ∞ or $-\infty$ in the other direction.)*
- What are the clues in the context of a particular situation that suggest the use of a particular type of function as a model?
  - *If we expect from the context that each new term in the sequence of data is a constant added to the previous term, then we try a linear model.*
  - *If we expect from the context that each new term in the sequence of data is a constant multiple of the previous term, then we try an exponential model.*
  - *If we expect from the context that the second differences of the sequence are constant (meaning that the rate of change between terms either grows or shrinks linearly), then we try a quadratic model.*
  - *If we expect from the context that the sequence of terms is periodic, then we try a sinusoidal model.*

> *Scaffolding:*
> Have students record this information about when to use each type of function in a chart or graphic organizer.

Exercise 1 (6 minutes)

This exercise, like the Opening Exercise, provides students with an ambiguous set of data for which it is necessary to understand the context before we can select a model. After students have completed this exercise, go through students' responses as a class to be sure that all students are aware of the ambiguity before setting them to work on the rest of the exercises. Students should work on this exercise in pairs or small groups.

Scaffolding:

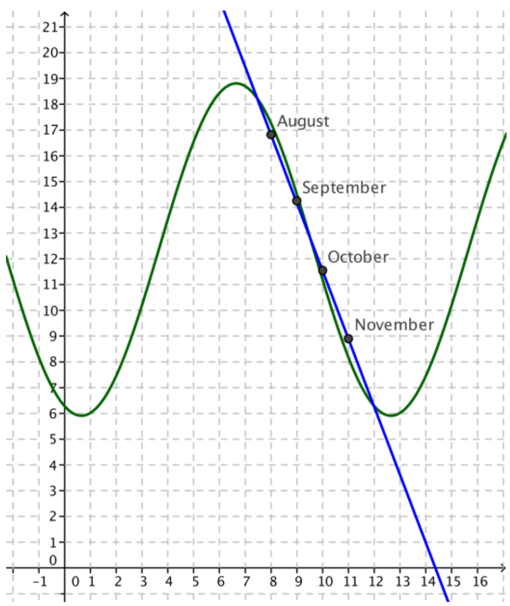
Tell advanced students that the maximum amount of daylight in Oslo is 18.83 hours on June 15, and challenge them to find an appropriate sinusoidal function to model the data.

Exercises

1. The table below contains the number of daylight hours in Oslo, Norway, on the specified dates.

Date	Hours and Minutes	Hours
August 1	$16:56$	16.93
September 1	$14:15$	14.25
October 1	$11:33$	11.55
November 1	$8:50$	8.83

a. Plot the data on the grid provided and decide how to best represent it.



b. Looking at the data, what type of function appears to be the best fit?

The data appears to lie on a straight line.

c. Looking at the context in which the data was gathered, what type of function should be used to model the data?

Since daylight hours increase and decrease with the season, being shortest in winter and longest in summer and repeating every 12 months, a linear model is not appropriate. We should model this data with a periodic function.

d. **Do you have enough information to find a model that is appropriate for this data? Either find a model or explain what other information you would need to do so.**

We cannot find a complete model for this scenario because we do not know the maximum and minimum number of daylight hours in Oslo. We do know that the maximum is less than 24 hours and the minimum is more than 0 hours. We could find a very rough model using a sinusoidal function, but we do not know the necessary amplitude. (The maximum number of daylight hours in Oslo is 18.83 hours and occurs in mid-June. With this information, the points can be modeled by the function $d(t) = -6.45\cos\left(\frac{\pi}{6}(t-0.65)\right) + 12.36$, as shown in green.)

Exercises 2–6 (12 minutes)

In these exercises, students are asked to determine which type of function should be used to model the given scenario. In some cases, students are given enough information to actually produce a model, and they are expected to do so, and in other cases the students can only specify the type of function that should be used. It is up to them to determine whether or not they have all of the needed information. Students should work on these exercises in pairs or small groups.

2. **The goal of the U.S. Centers for Disease Control and Prevention (CDC) is to protect public health and safety through the control and prevention of disease, injury, and disability. Suppose that 45 people have been diagnosed with a new strain of the flu virus and that scientists estimate that each person with the virus will infect 5 people every day with the flu.**

   a. **What type of function should the scientists at the CDC use to model the initial spread of this strain of flu to try to prevent an epidemic? Explain how you know.**

   Because each person infects 5 people every day, the number of infected people is multiplied by a factor of 5 each day. This would be best modeled by an exponential function, at least at the beginning of the outbreak.

   b. **Do you have enough information to find a model that is appropriate for this situation? Either find a model or explain what other information you would need to do so.**

   Yes. We know the initial number of infected people is 45, and we know that the number of infected people is multiplied by 5 each day. A model for the number of infected people on day t would be $F(t) = 45(5^t)$.

3. **An artist is designing posters for a new advertising campaign. The first poster takes 10 hours to design, but each subsequent poster takes roughly 15 minutes less time than the previous one as he gets more practice.**

   a. **What type of function models the amount of time needed to create n posters, for $n \leq 20$? Explain how you know.**

   Since the time difference between posters is decreasing linearly, we should model this scenario using a quadratic function.

Scaffolding:

Students may need to be reminded of the methods for finding the coefficients of a quadratic polynomial by solving a linear system as done in Module 1 Lesson 30.

b. **Do you have enough information to find a model that is appropriate for this situation? Either find a model or explain what other information you would need to do so.**

Yes. The number of hours needed to create n posters can be modeled by a quadratic function

$$T(n) = an^2 + bn + c,$$

where we know that $T(0) = 0$, $T(1) = 10$, and $T(2) = 19.75$. This gives us the three linear equations

$$c = 0$$
$$a + b + c = 10$$
$$4a + 2b + c = 19.75.$$

We can solve this system of three equations using the methods of Lesson 30 in Module 1, and we find

$$T(n) = -0.125n^2 + 10.125n.$$

4. **A homeowner notices that her heating bill is the lowest in the month of August and increases until it reaches its highest amount in the month of February. After February, the amount of the heating bill slowly drops back to the level it was in August, when it begins to increase again. The amount of the bill in February is roughly four times the amount of the bill in August.**

a. **What type of function models the amount of the heating bill in a particular month? Explain how you know.**

Because exterior temperatures repeat fairly periodically, with the coldest temperatures in the winter and the warmest temperatures in the summer, we would expect a periodic use of heating fuel that was highest in the winter and lowest in the summer. Thus, we should use a sinusoidal function to model this scenario.

b. **Do you have enough information to find a model that is appropriate for this situation? Either find a model or explain what other information you would need to do so.**

We cannot model this scenario because we do not know the highest or lowest amount of the heating bill. If we knew either of those amounts, we could find the other. Then, we could find the amplitude of the sinusoidal function and create a reasonable model.

5. **An online merchant sells used books for $\$5.00$ each, and the sales tax rate is 6% of the cost of the books. Shipping charges are a flat rate of $\$4.00$ plus an additional $\$1.00$ per book.**

a. **What type of function models the total cost, including the shipping costs, of a purchase of x books? Explain how you know.**

We would use a linear function to model this situation because the total cost increases at a constant rate when you increase the number of books purchased.

b. **Do you have enough information to find a model that is appropriate for this situation? Either find a model or explain what other information you would need to do so.**

We can model this situation exactly. If we buy x books, then the total cost of the purchase (in dollars) is given by

$$C(x) = 1.06(5x) + 4 + 1x$$
$$= 6.3x + 4.$$

6. **A stunt woman falls from a tall building in an action-packed movie scene. Her speed increases by 32 ft/s for every second that she is falling.**

a. **What type of function models her distance from the ground at time t seconds? Explain how you know.**

Because her speed is increasing by 32 ft/s every second, her rate at which she gets closer to the ground is increasing linearly; thus, we would model this situation with a quadratic function.

b. **Do you have enough information to find a model that is appropriate for this situation? Either find a model or explain what other information you would need to do so.**

We cannot create a model for this situation because we do not know the height of the building, so we do not know how far she will fall.

Discussion (4 minutes)

Either call on students or ask for volunteers to present their models for the scenarios in Exercises 2–6.

Closing (6 minutes)

Hold a discussion with the entire class in which students provide responses to the following questions.

- In this lesson, we have looked at four kinds of mathematical models: linear, quadratic, exponential, and sinusoidal. How is a linear model different from a quadratic model?
  - *Responses will vary. Sample responses include: A linear model has a constant rate of change. It has no maximum or minimum. The equation for a quadratic model has one or more squared terms. Its graph is a parabola, and it has a maximum or minimum.*
- How is a quadratic model different from an exponential model?
  - *A quadratic model has a maximum or minimum, whereas an exponential model is unbounded. A quantity increasing exponentially eventually exceeds a quantity increasing quadratically.*
- How is an exponential model different from a linear model?
  - *As x changes by 1 in a linear model, the y-value changes by a fixed amount, but as x changes by 1 in an exponential model, the y-value changes by a multiple of x. The range of a linear function (that is not constant) is all real numbers, and the range of an exponential function is either $(-\infty, a)$ or (a, ∞) for some real number a.*
- How is a quadratic model different from a sinusoidal model?
  - *A quadratic model has one relative maximum or minimum point and does not repeat, whereas a sinusoidal model has an infinite number of relative maximum and minimum points that repeat periodically.*
- How is a sinusoidal model different from an exponential model?
  - *A sinusoidal model is bounded and cyclic, whereas an exponential model either goes to $\pm\infty$ or to a constant value as $x \to \infty$. A sinusoidal model changes in a periodic fashion, but an exponential model changes at a rate proportional to the current value of the function.*

The four models are summarized in the table below, which can be reproduced and posted in the classroom.

Lesson Summary

- If we expect from the context that each new term in the sequence of data is a constant added to the previous term, then we try a linear model.
- If we expect from the context that the second differences of the sequence are constant (meaning that the rate of change between terms either grows or shrinks linearly), then we try a quadratic model.
- If we expect from the context that each new term in the sequence of data is a constant multiple of the previous term, then we try an exponential model.
- If we expect from the context that the sequence of terms is periodic, then we try a sinusoidal model.

Model	Equation of Function	Rate of Change
Linear	$f(t) = at + b$ for $a \neq 0$	Constant
Quadratic	$g(t) = at^2 + bt + c$ for $a \neq 0$	Changing linearly
Exponential	$h(t) = ab^{ct}$ for $0 < b < 1$ or $b > 1$	A multiple of the current value
Sinusoidal	$k(t) = A\sin\big(w(t-h)\big) + k$ for $A, w \neq 0$	Periodic

Exit Ticket (5 minutes)

Name ____ Date ____________________

Lesson 22: Choosing a Model

Exit Ticket

The amount of caffeine in a patient's bloodstream decreases by half every 3.5 hours. A latte contains 150 mg of caffeine, which is absorbed into the bloodstream almost immediately.

a. What type of function models the caffeine level in the patient's bloodstream at time t hours after drinking the latte? Explain how you know.

b. Do you have enough information to find a model that is appropriate for this situation? Either find a model or explain what other information you would need to do so.

Exit Ticket Sample Solutions

The amount of caffeine in a patient's bloodstream decreases by half every 3.5 hours. A latte contains 150 mg of caffeine, which is absorbed into the bloodstream almost immediately.

a. What type of function models the caffeine level in the patient's bloodstream at time t hours after drinking the latte? Explain how you know.

Because the amount of caffeine decreases by half of the current amount in the bloodstream over a fixed time period, we would model this scenario by a decreasing exponential function.

b. Do you have enough information to find a model that is appropriate for this situation? Either find a model or explain what other information you would need to do so.

Assuming that the latte was the only source of caffeine for this patient, we can model the amount of caffeine in the bloodstream (in mg) at time t (in hours) by

$$C(t) = 150\left(\frac{1}{2}\right)^{\frac{t}{3.5}}.$$

Problem Set Sample Solutions

1. A new car depreciates at a rate of about 20% per year, meaning that its resale value decreases by roughly 20% each year. After hearing this, Brett said that if you buy a new car this year, then after 5 years the car has a resale value of $\$0.00$. Is his reasoning correct? Explain how you know.

Brett is not correct. If the car loses 20% of its value each year, then it retains 80% of its resale value each year. In that case, the proper model to use is an exponential function,

$$V(t) = P(0.80)^t,$$

where P is the original price paid for the car when it was new, t is the number of years the car has been owned, and $V(t)$ is the resale value of the car in year t. Then, when $t = 5$, the value of the car is $V(5) = P(0.80)^5 \approx 0.33P$; so, after 5 years, the car is worth roughly 33% of its original price.

2. Alexei just moved to Seattle, and he keeps track of the average rainfall for a few months to see if the city deserves its reputation as the rainiest city in the United States.

Month	Average Rainfall
July	0.93 in.
September	1.61 in.
October	3.24 in.
December	6.06 in.

What type of function should Alexei use to model the average rainfall in month t?

Although the data appears to be exponential when plotted, an exponential model does not make sense for a seasonal phenomenon like rainfall. Alexei should use a sinusoidal function to model this data.

3. Sunny, who wears her hair long and straight, cuts her hair once per year on January 1, always to the same length. Her hair grows at a constant rate of 2 cm per month. Is it appropriate to model the length of her hair with a sinusoidal function? Explain how you know.

No. If we were to use a sinusoidal function to model the length of her hair, then that would imply that her hair grows longer and then slowly shrinks back to its original length. Even though the length of her hair can be represented by a periodic function, it abruptly gets cut off once per year and does not smoothly return to its shortest length. None of our models are appropriate for this situation.

4. On average, it takes 2 minutes for a customer to order and pay for a cup of coffee.

a. What type of function models the amount of time you wait in line as a function of how many people are in front of you? Explain how you know.

Because the wait time increases by a constant 2 minutes for each person in line, we can use a linear function to model this situation.

b. Find a model that is appropriate for this situation.

If there is no one ahead of you, then your wait time is zero. Thus, the wait time W, in minutes, can be modeled by

$$W(x) = 2x,$$

where x is the number of people in front of you in line.

5. An online ticket-selling service charges $\$50.00$ for each ticket to an upcoming concert. In addition, the buyer must pay 8% sales tax and a convenience fee of $\$6.00$ for the purchase.

a. What type of function models the total cost of the purchase of n tickets in a single transaction?

The complete price for each ticket is $1.08(\$50.00) = \$54.00$, so the total price of the purchase increases by $\$54.00$ per ticket. Thus, this should be modeled by a linear function.

b. Find a model that is appropriate for this situation.

The price (in dollars) for buying n tickets, including the convenience fee, is then $T(n) = 54n + 6$.

6. In a video game, the player must earn enough points to pass one level and progress to the next as shown in the table below.

To pass this level …	You need this many total points …
1	$5,000$
2	$15,000$
3	$35,000$
4	$65,000$

That is, the increase in the required number of points increases by $10,000$ points at each level.

a. What type of function models the total number of points you need to pass to level n? Explain how you know.

Because the increase in needed points is increasing linearly, we should use a quadratic function to model this situation.

b. **Find a model that is appropriate for this situation.**

The amount of points needed to pass level n can be modeled by a quadratic function

$$P(n) = an^2 + bn + c,$$

where we know that $P(1) = 5000$, $P(2) = 15000$, and $P(3) = 35000$. This gives us the three linear equations

$$a + b + c = 5000$$
$$4a + 2b + c = 15000$$
$$9a + 3b + c = 35000.$$

We can solve this system of three equations using the methods of Lesson 30 in Module 1, and we find

$$P(n) = 5000n^2 - 5000n + 5000.$$

7. **The southern white rhinoceros reproduces roughly once every 3 years, giving birth to one calf each time. Suppose that a nature preserve houses 100 white rhinoceroses, 50 of which are female. Assume that half of the calves born are female and that females can reproduce as soon as they are 1 year old.**

a. **What type of function should be used to model the population of female white rhinoceroses in the preserve?**

Because all female rhinoceroses give birth every 3 years, and half of those calves are assumed to be female, the population of female rhinoceroses increases by $\frac{1}{6}$ every year. Thus, we should use an exponential function to model the population of female southern white rhinoceroses.

b. **Assuming that there is no death in the rhinoceros population, find a function to model the population of female white rhinoceroses in the preserve.**

Since $1 + \frac{1}{6} \approx 1.17$ and the initial population is 50 female southern white rhinoceroses, we can model this by

$$R_1(t) = 50(1.17)^t.$$

c. **Realistically, not all of the rhinoceroses survive each year, so we assume a 5% death rate of all rhinoceroses. Now what type of function should be used to model the population of female white rhinoceroses in the preserve?**

We should still use an exponential function, but the growth rate needs to be altered to take the death rate into account.

d. **Find a function to model the population of female white rhinoceroses in the preserve, taking into account the births of new calves and the 5% death rate.**

Since 5% of the rhinoceroses die each year, that means that 95% of them survive. The new growth rate is then $0.95(1.17) \approx 1.11$. The new model would be

$$R_2(t) = 50(1.11)^t.$$

Lesson 23: Bean Counting

Student Outcomes

- Students gather experimental data and determine which type of function is best to model the data.
- Students use properties of exponents to interpret expressions for exponential functions.

Lesson Notes

In the main activity in this lesson, students work in pairs to gather their own data, plot it (MP.6), and apply the methods of Lesson 22 to decide which type of data to use in modeling the data (MP.4, MP.7). Students should use calculators (or other technological tools) to fit the data with an exponential function (MP.5). Since each group of students generates its own set of data, each group finds different functions to model the data, but all of those functions should be in the form $f(t) = a(b^t)$, where $a \approx 1$ and $b \approx 1.5$. Take time to discuss why the functions differ between groups and yet are closely related due to probability. If time permits, at the end of the activity average each group's values of the constants a and b; the averages should be very close to 1 and 1.5, respectively (**F-LE.B.5**).

In the Problem Set, students investigate the amount of time for quantities to double, triple, or increase by a factor of 10 using these functions and others like them. For example, students rewrite functions $f(t) = a(b^t)$ in an equivalent form $f(t) = a(2^{\log_2(b)t})$ and interpret the exponent (**F-IF.C.8b**).

Materials

Each team of two students gather data using the following materials:

- At least 50 small, flippable objects marked differently on the two sides. One inexpensive source for these objects would be dried beans, spray painted on one side and left unpainted on the other. Buttons or coins would also work. Throughout this lesson, these objects are referred to as beans.
- Two paper cups: one to hold the beans and the other to shake up and dump out the beans onto the paper plate.
- A paper plate (to keep the beans from ending up all over the floor).
- A calculator capable of plotting data and performing exponential regression.

Classwork

Opening (3 minutes)

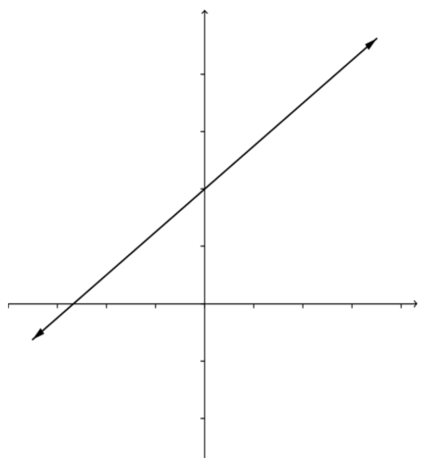
Divide the class into groups of 2–3 students. Smaller groups are better for this exercise, so have students work in pairs if possible. Before distributing the needed materials, model the process for gathering data for two trials. Then, provide each team with a cup containing at least 50 beans, an empty cup, a paper plate, and a calculator.

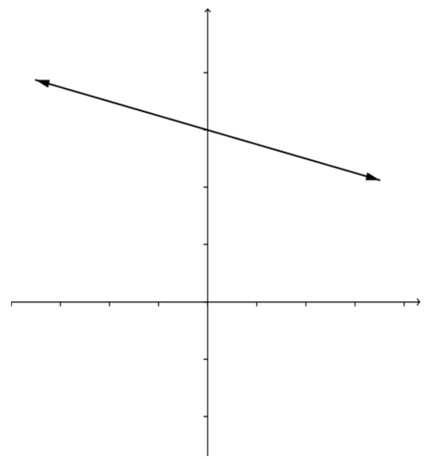
> *Scaffolding:*
>
> If students are collectively struggling with the concepts of exponential growth and modeling, consider extending this to a 2-day lesson. Devote the entire first day to Exercise 1 on exponential growth, and the second day to Exercise 2 on exponential decay.

- In this lesson, we gather some data and then decide which type of function to use to model them. In the previous lesson, we studied different types of functions we could use to model data in different situations. What were those types of functions?
  - *Linear, quadratic, sinusoidal, and exponential functions*
- Suppose you are gathering data from an experiment, measuring a quantity on evenly spaced time intervals. How can you recognize from the context that data should be modeled by a linear function?

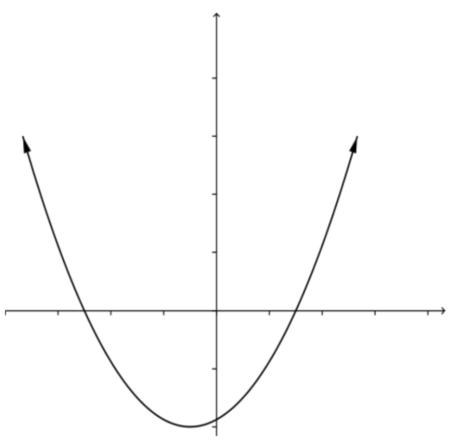
Accompany the discussion of different model types with visuals showing graphs of each type.

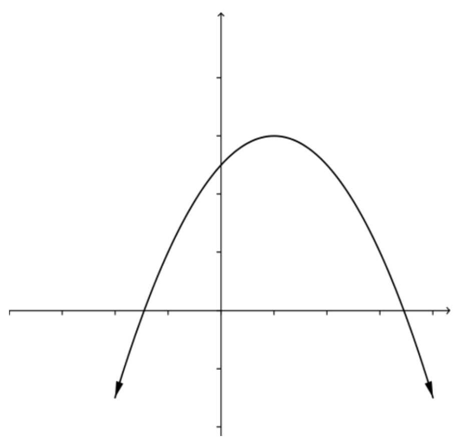
  - *If we expect the data values to increase (or decrease) at an even rate, then the data points should roughly lie on a line.*



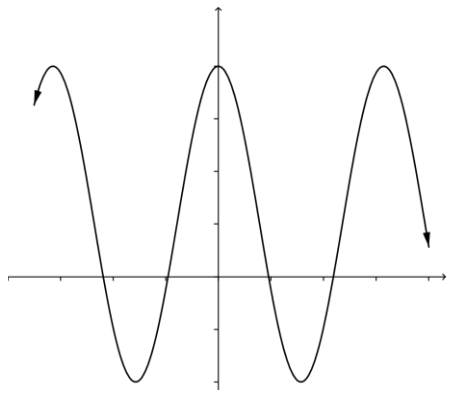


- How do you recognize that data should be modeled by a quadratic function?
  - *If we expect the distance between data values to increase or decrease at a constant rate, then the data points should roughly lie on a parabola.*

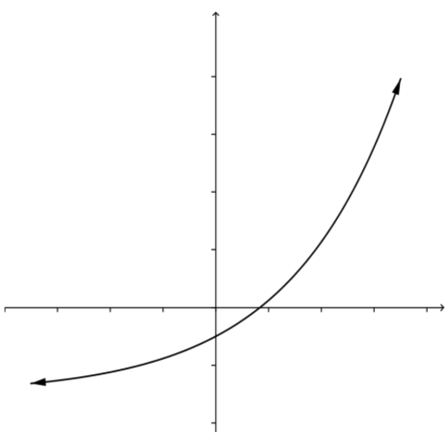


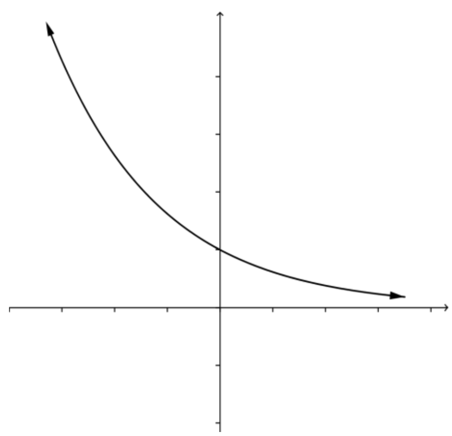


- How do you recognize that data should be modeled by a sinusoidal function?
  - *If we expect the data values to repeat periodically due to a repeating phenomenon, then the data points should roughly lie on a sinusoidal curve.*



- How do you recognize that data should be modeled by an exponential function?
  - *If we expect the data values to increase (or decrease) proportionally to the current quantity, then the data points should roughly lie on an exponential curve.*





- To gather data, we do the following: Start with one bean in one cup, and keep the other beans in the spare cup. Dump the single bean onto the paper plate, and record in the table if it lands marked-side up. If it lands marked-side up, then add another bean to the cup. This is the first trial.
- You now have either one or two beans in your cup. Shake the cup gently, then dump the beans onto the plate and record how many land marked-side up. Either 0, 1, or 2 beans should land marked-side up. Add that number of beans to your cup. This is the second trial.
- Repeat this process until you either have done 10 trials or have run out of beans in the spare cup.
- After you have gathered your data, plot the data on the coordinate grid provided. Plot the trial number on the horizontal axis and the number of beans in the cup at the start of the trial on the vertical axis.

Mathematical Modeling Exercise 1 (15 minutes)

Circulate around the room while students gather data for this exercise to ensure that they all understand the process for flipping the beans, adding new beans to the cup, and recording data.

Mathematical Modeling Exercises

1. **Working with a partner, you are going to gather some data, analyze the data, and find a function to use to model the data. Be prepared to justify your choice of function to the class.**

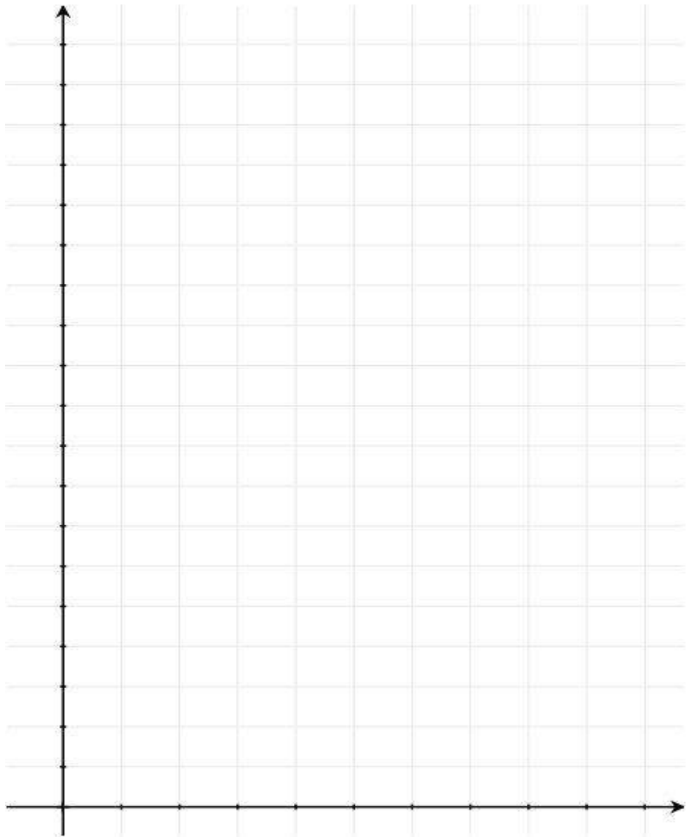
   a. **Gather your data: For each trial, roll the beans from the cup to the paper plate. Count the number of beans that land marked-side up, and add that many beans to the cup. Record the data in the table below. Continue until you have either completed 10 trials or the number of beans at the start of the trial exceeds the number that you have.**

Trial Number, t	Number of Beans at Start of Trial	Number of Beans That Landed Marked-Side Up
1	1	
2		
3		
4		
5		
6		
7		
8		
9		
10		

   b. **Based on the context in which you gathered this data, which type of function would best model your data points?**

   Since the number of beans we add at each toss is roughly half of the beans we had at the start of that turn, the data should be modeled by an exponential function.

c. **Plot the data: Plot the trial number on the horizontal axis and the number of beans in the cup at the start of the trial on the vertical axis. Be sure to label the axes appropriately and to choose a reasonable scale for the axes.**



d. **Analyze the data: Which type of function best fits your data? Explain your reasoning.**

Students should see that the data follows a clear pattern of exponential growth, and they should decide to model it with an exponential function.

e. **Model the data: Enter the data into the calculator and use the appropriate type of regression to find an equation that fits this data. Round the constants to two decimal places.**

Answers will vary but should be of the form $f(t) = a(b^t)$, where a is near 1 and b is near 1.5.

Discussion (5 minutes)

After students complete Exercise 1, have them write the equations that they found to model their data on the board, displayed by the document camera, or on poster board visible to all students. After all groups have reported their equations, debrief the class with questions like the following:

- What type of equation did you use to model your data?
  - *We used an exponential function.*

- Why did you choose this type of equation?
  - *The number of beans always increases, and the amount it increases depends on the current number of beans that we have.*
- Why did all of the teams get different equations?
  - *We all had different data because each roll of the beans is different. Each bean has a* 50% *chance of landing marked-side up, but that does not mean that half of the beans always land marked-side up.*

MP.2 & MP.7

- Look at the function $f(t) = a(b^t)$ that models your data and interpret the value of the constant a.
  - *The number* a *is the number of beans we started with (according to the model).*
- Look at the function $f(t) = a(b^t)$ that models your data and interpret the value of the base b.
  - *The base* b *is the growth factor. This means that the number of beans is multiplied by* b *with each toss.*
- If the beans landed perfectly with 50% of them marked-side up every time, what would we expect the value of b to be?
  - *If half of the beans always landed marked-side up, then the number of beans would increase by* 50% *with each trial, so that the new amount would be* 150% *of the old amount. That is, the number of beans would be multiplied by* 1.5 *at each trial. Then, the value of* b *would be* 1.5.
- From the situation we are modeling, what would we expect the value of the coefficient a to be?
  - *For an exponential function of the form* $f(t) = a(b^t)$, *we have* $f(0) = a$, *so* a *represents our initial number of beans. Thus, we should have a value of* a *near* 1.
- Why didn't your values of a and b turn out to be $a = 1$ and $b = 1.5$?
  - *Even though the probability says that we have a* 50% *chance of the beans landing marked-side up, this does not mean that half of the beans will always land marked-side up. Our experimental data is not going to lie perfectly on the curve* $f(t) = (1.5)^t$.
- Let's look at the constants in the equations $f(t) = a(b^t)$ that the groups found to fit their data.
- Have the students calculate the average values of the coefficients a and the base b from each group's equation.
  - *Answers will vary. The average value of* a *should be near* 1, *and the average value of* b *should be near* 1.5.

Create an exponential function $f(t) = a_{\text{avg}}\left(b_{\text{avg}}\right)^t$, where a_{avg} and b_{avg} represent the average values of the coefficient a and base b from each group's equation.

- What would happen to the values of a_{avg} and b_{avg} if we had data from $1{,}000$ groups?
  - *The value of* a_{avg} *should get very close to* 1 *and the value of* b_{avg} *should get very close to* 1.5.

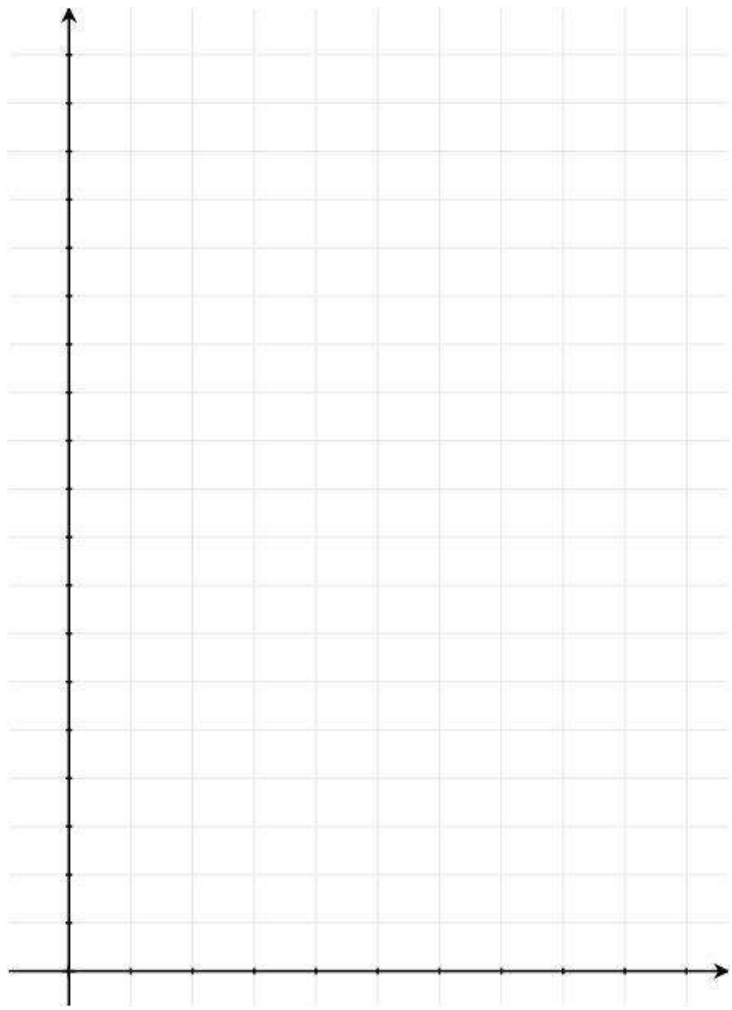
Mathematical Modeling Exercise 2 (10 minutes)

2. This time, we are going to start with 50 beans in your cup. Roll the beans onto the plate and remove any beans that land marked-side up. Repeat until you have no beans remaining.

   a. Gather your data: For each trial, roll the beans from the cup to the paper plate. Count the number of beans that land marked-side up, and remove that many beans from the plate. Record the data in the table below. Repeat until you have no beans remaining.

Trial Number, t	Number of Beans at Start of Trial	Number of Beans That Landed Marked-Side Up
1	50	
2		
3		
4		
5		
6		
7		
8		
9		
10		

b. **Plot the data: Plot the trial number on the horizontal axis and the number of beans in the cup at the start of the trial on the vertical axis. Be sure to label the axes appropriately and choose a reasonable scale for the axes.**



c. **Analyze the data: Which type of function would best fit your data? Explain your reasoning.**

Students should see that the data follows a clear pattern of exponential decay, and they should decide to model it with an exponential function.

d. **Make a prediction: What do you expect the values of a and b to be for your function? Explain your reasoning.**

Using an exponential function $f(t) = a(b^t)$, the value of a is the initial number of beans, so we should expect a to be near 50. The number of beans decreases by half each time, so we would expect $b = 0.5$.

e. **Model the data: Enter the data into the calculator. Do not enter your final data point of 0 beans. Use the appropriate type of regression to find an equation that fits this data. Round the constants to two decimal places.**

Answers will vary but should be of the form $f(t) = a(b^t)$, where a is near 50 and b is near 0.5.

Discussion (5 minutes)

After students complete Exercise 2, have them write the equations that they found to model their data in a shared location visible to all students. Prompt students to look for patterns in the equations produced by different groups and to revisit their answers from part (d) if necessary.

- What type of equation did you use to model your data?
  - *We used an exponential function.*
- Why did you choose this type of equation?
  - *The number of beans should always decrease, and the amount it decreases depends on the current number of beans that we have.*
- Let's look at the different constants in the equations $f(t) = a(b^t)$ that the groups found to fit the data. Have students calculate the average values of the coefficients a and the base b from each group's equation.
  - *Answers will vary. The average value of a should be near 50, and the average value of b should be near 0.5.*

MP.2 & MP.7

- From the situation we are modeling, what would we expect the values of the coefficient a and the base b to be?
  - *For an exponential function of the form $f(t) = a(b^t)$, we have $f(0) = a$. Thus, a represents our initial number of beans, and we should have a value of a near 50. The number of beans should be cut in half at each trial, so we should have a value of b near 0.5.*
- How many trials did it take before you had no beans left?
  - *Answers will vary but should be somewhere around seven trials.*
- What is the range of your exponential function? Does it include zero?
  - *The range of the exponential function is $(0, \infty)$, so it does not include zero.*
- How can we explain the discrepancy between the fact that the function that you are using to model the number of beans can never be zero, but the number of beans left at the end of the activity is clearly zero?
  - *The exponential function only approximates the number of beans. The number of beans is always an integer, while the values taken on by the exponential function are real numbers. That is, our function takes on values that are closer and closer to zero, without ever actually being zero, but if we were to round it to integers, then it rounds to zero.*

Closing (3 minutes)

Have students respond to the following questions individually in writing or orally with a partner.

- What sort of function worked best to model the data we gathered from Mathematical Modeling Exercise 1 where we added beans?
  - *We used an increasing exponential function.*
- Why was this the best type of function?
  - *The number of beans added was roughly a multiple of the current number of beans; if the beans had behaved perfectly and half of them always landed with the marked-side up, we would have added half the number of beans each time.*

- Why could we expect to model this data with a function $f(t) = (1.5)^t$?
    - *When we add half of the beans, that means that the current number of beans is being multiplied by* $1 + 0.5 = 1.5$*. With an exponential function, we get from one data point to the next by multiplying by a constant. For this function, the constant is* 1.5*.*
- Why did an exponential function work best to model the data from Mathematical Modeling Exercise 2, in which the number of beans was reduced at each trial?
    - *The number of beans removed at each trial was roughly half the current number of beans. The function that models this is* $f(t) = 50\left(\frac{1}{2}\right)^t$*, which is a decreasing exponential function.*

Exit Ticket (4 minutes)

Name ______________________________ Date ______________

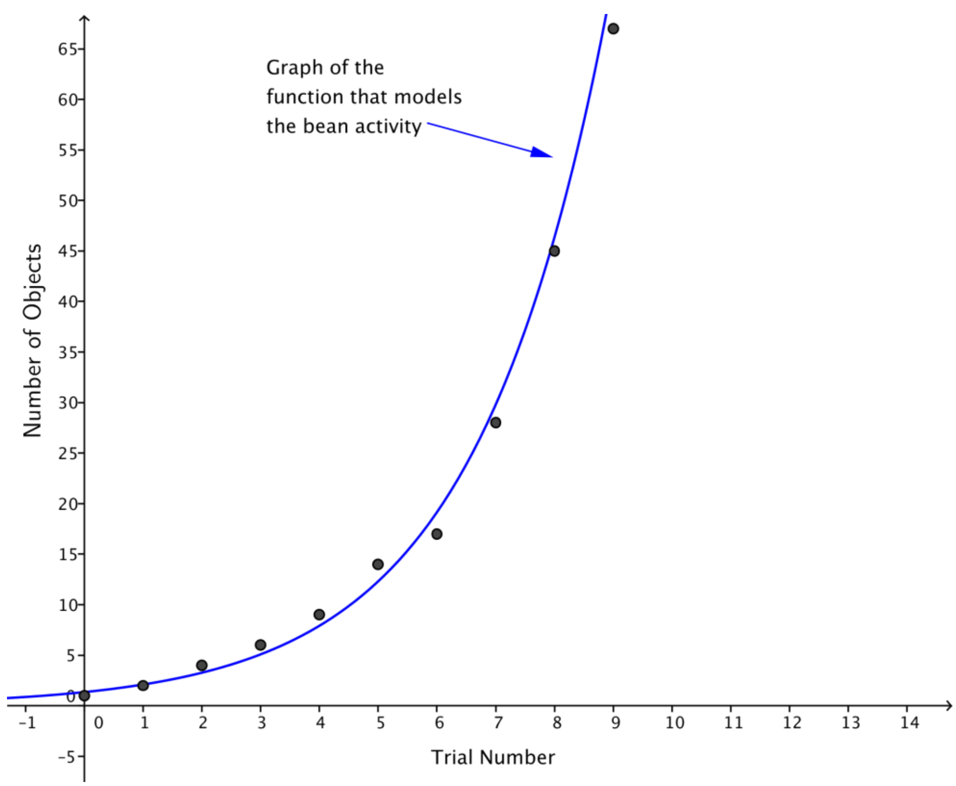
Lesson 23: Bean Counting

Exit Ticket

Suppose that you were to repeat the bean activity, but in place of beans, you were to use six-sided dice. Starting with one die, each time a die is rolled with a 6 showing, you add a new die to your cup.

a. Would the number of dice in your cup grow more quickly or more slowly than the number of beans did? Explain how you know.

b. A sketch of one sample of data from the bean activity is shown below. On the same axes, draw a rough sketch of how you would expect the curve through the data points from the dice activity to look.



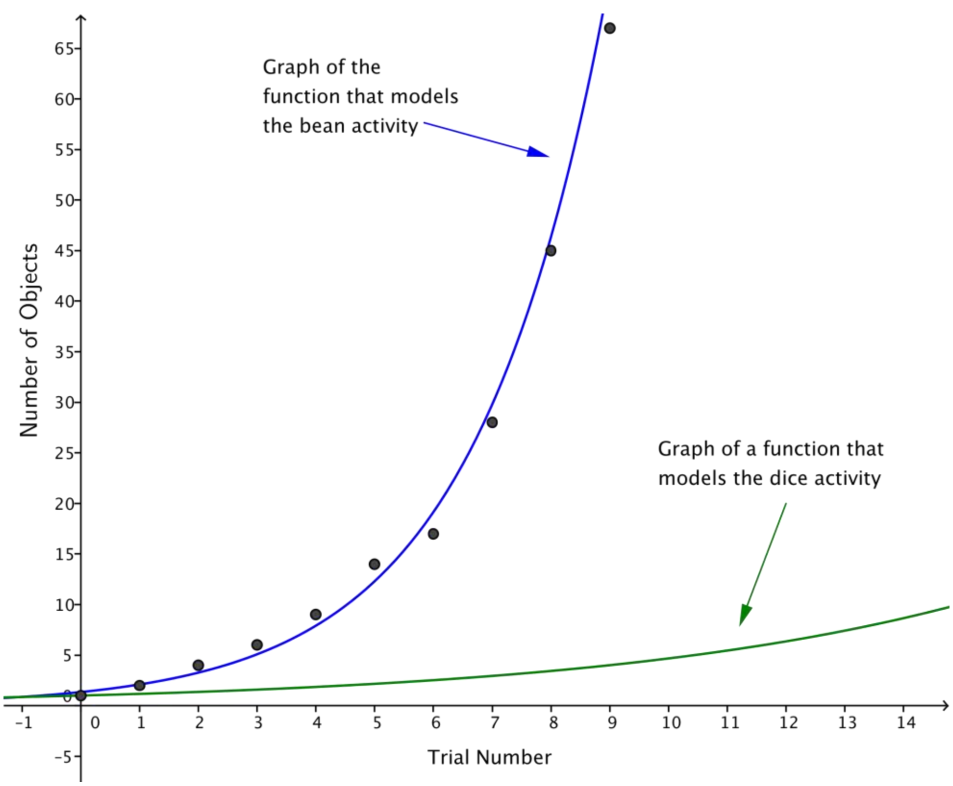
Exit Ticket Sample Solutions

Suppose that you were to repeat the bean activity, but in place of beans, you were to use six-sided dice. Starting with one die, each time a die is rolled with a 6 showing, you add a new die to your cup.

a. **Would the number of dice in your cup grow more quickly or more slowly than the number of beans did? Explain how you know.**

The number of dice in the cup should grow much more slowly because the probability of rolling a 6 is $\frac{1}{6}$, while the probability of flipping a bean marked-side up is roughly $\frac{1}{2}$. Thus, the beans should land marked-side up, and thus, increase the number of beans in our cup about half of the time, while the dice would show a 6 only $\frac{1}{6}$ of the time. As an example, it could be expected to take one or two flips of the first bean to get a bean to show the marked side, causing us to add one, but it could be expected to take six rolls of the first die to get a 6, causing us to add another die.

b. **A sketch of one sample of data from the bean activity is shown below. On the same axes, draw a rough sketch of how you would expect the curve through the data points from the dice activity to look.**



Problem Set Sample Solutions

1. For this problem, we consider three scenarios for which data have been collected and functions have been found to model the data, where a, b, c, d, p, q, r, s, t, and u are positive real number constants.
   i. The function $f(t) = a \cdot b^t$ models the original bean activity (Mathematical Modeling Exercise 1). Each bean is painted or marked on one side, and we start with one bean in the cup. A trial consists of throwing the beans in the cup and adding one more bean for each bean that lands marked-side up.
   ii. The function $g(t) = c \cdot d^t$ models a modified bean activity. Each bean is painted or marked on one side, and we start with one bean in the cup. A trial consists of throwing the beans in the cup and adding two more beans for each bean that lands marked-side up.
   iii. The function $h(t) = p \cdot q^t$ models the dice activity from the Exit Ticket. Start with one six-sided die in the cup. A trial consists of rolling the dice in the cup and adding one more die to the cup for each die that lands with a 6 showing.
   iv. The function $j(t) = r \cdot s^t$ models a modified dice activity. Start with one six-sided die in the cup. A trial consists of rolling the dice in the cup and adding one more die to the cup for each die that lands with a 5 or a 6 showing.
   v. The function $k(t) = u \cdot v^t$ models a modified dice activity. Start with one six-sided die in the cup. A trial consists of rolling the dice in the cup and adding one more die to the cup for each die that lands with an even number showing.

   a. What values do you expect for a, c, p, r, and u?

   The values of these four constants should each be around 1 because the first data point in all four cases is $(1, 0)$.

   b. What value do you expect for the base b in the function $f(t) = a \cdot b^t$ in scenario (i)?

   We know from the class activity that $b \approx 1.5$ because the number of beans grows by roughly half of the current amount at each trial.

   c. What value do you expect for the base d in the function $g(t) = c \cdot d^t$ in scenario (ii)?

   Suppose we have 4 beans in the cup. We should expect half of them to land marked-side up. Then, we would add $2 \cdot 2 = 4$ beans to the cup, doubling the amount that we had. This is true for any number of beans; if we had n beans, then $\frac{n}{2}$ should land marked-side up, so we would add n beans, doubling the amount. Thus, values of the function g should double, so $d \approx 2$.

   d. What value do you expect for the base q in the function $h(t) = p \cdot q^t$ in scenario (iii)?

   For this function h, we expect that the quantity increases by $\frac{1}{6}$ of the current quantity at each trial. Then, $q \approx 1 + \frac{1}{6}$, so $q \approx \frac{7}{6}$.

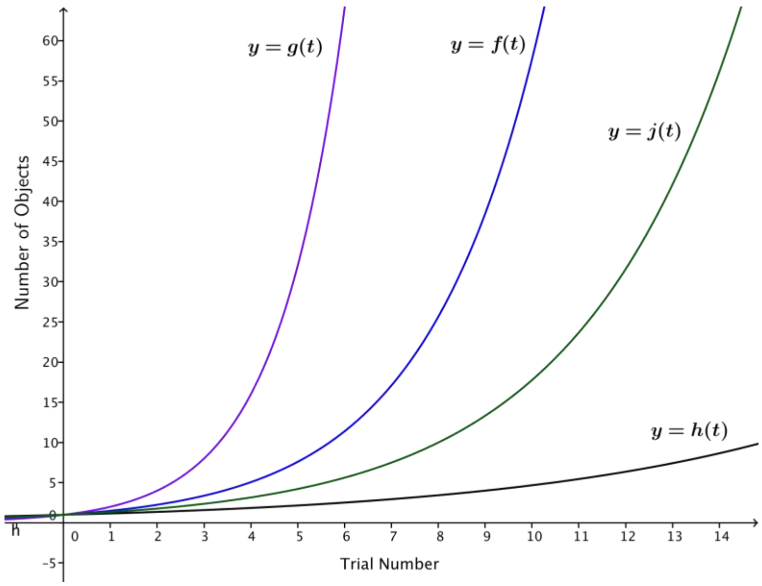
   e. What value do you expect for the base s in the function $j(t) = r \cdot s^t$ in scenario (iv)?

   The probability of rolling a 5 or 6 is $\frac{1}{3}$, so we would expect that the number of dice increases by $\frac{1}{3}$ of the current quantity at each trial. Thus, we expect $s \approx 1 + \frac{1}{3}$, which means that $s \approx \frac{4}{3}$.

f. What value do you expect for the base v in the function $k(t) = u \cdot v^t$ in scenario (v)?

The probability of rolling an even number on a six-sided die is the same as the probability of getting the marked-side up on a bean, so we would expect that f and k are the same function. Thus, $v \approx 1.5$.

g. The following graphs represent the four functions f, g, h, and j. Identify which graph represents which function.



2. Teams 1, 2, and 3 gathered data as shown in the tables below, and each team modeled the data using an exponential function of the form $f(t) = a \cdot b^t$.

a. Which team should have the highest value of b? Which team should have the lowest value of b? Explain how you know.

Team 1		Team 2		Team 3	
Trial Number, t	Number of Beans	Trial Number, t	Number of Beans	Trial Number, t	Number of Beans
0	1	0	1	0	1
1	1	1	1	1	2
2	2	2	1	2	3
3	2	3	2	3	5
4	4	4	2	4	8
5	6	5	3	5	14
6	8	6	5	6	26
7	14	7	7	7	46
8	22	8	12	8	76
9	41	9	18	9	
10	59	10	27	10	

The larger the value of the base b, the larger the function will be at $t = 10$. Since team 3 had the most beans at the end (they used their 50 beans first), their equation should have the highest value of the base b. Team 2 has the smallest number of beans after 10 trials, so team 2 should have the smallest value of the base b.

b. **Use a graphing calculator to find the equation that best fits each set of data. Do the equations of the functions provide evidence that your answer in part (a) is correct?**

Team 1's equation: $f_1(t) = 0.7378(1.5334)^t$

Team 2's equation: $f_2(t) = 0.6495(1.4245)^t$

Team 3's equation: $f_3(t) = 1.0328(1.7068)^t$

As predicted, team 2 has the smallest value of b, and team 3 has the largest value of b.

3. **Omar has devised an activity in which he starts with 15 dice in his cup. A trial consists of rolling the dice in the cup and adding one more die to the cup for each die that lands with a 1, 2, or 3 showing.**

a. **Find a function $f(t) = a(b^t)$ that Omar would expect to model his data.**

$$f(t) = 15\left(\frac{3}{2}\right)^t$$

b. **Solve the equation $f(t) = 30$. What does the solution mean?**

$$15\left(\frac{3}{2}\right)^t = 30$$
$$\left(\frac{3}{2}\right)^t = 2$$
$$t = \frac{\log(2)}{\log\left(\frac{3}{2}\right)}$$
$$t \approx 1.71$$

So, Omar should have more than 30 dice by the second trial, after rolling and adding dice twice.

c. **Omar wants to know in advance how many trials it should take for his initial quantity of 15 dice to double. He uses properties of exponents and logarithms to rewrite the function from part (a) as the exponential function $f(t) = 15\left(2^{t\cdot\log_2\left(\frac{3}{2}\right)}\right)$. Has Omar correctly applied the properties of exponents and logarithms to obtain an equivalent expression for his original equation in part (a)? Explain how you know.**

Yes. The expressions are equal by the property of exponents.

$$\left(2^{t\cdot\log_2\left(\frac{3}{2}\right)}\right) = \left(2^{\log_2\left(\frac{3}{2}\right)}\right)^t = \left(\frac{3}{2}\right)^t$$

Thus,

$$15\left(2^{t\cdot\log_2\left(\frac{3}{2}\right)}\right) = 15\left(\frac{3}{2}\right)^t.$$

d. **Explain how the modified formula from part (c) allows Omar to easily find the expected amount of time, t, for the initial quantity of dice to double.**

The quantity is doubled at a time t for which $t\cdot\log_2\left(\frac{3}{2}\right) = 1$. Thus, we solve the equation $t\cdot\log_2\left(\frac{3}{2}\right) = 1$ to find $t = \frac{1}{\log_2\left(\frac{3}{2}\right)} \approx 1.71$. This agrees with our answer to part (b).

4. Brenna has devised an activity in which she starts with 10 dice in her cup. A trial consists of rolling the dice in the cup and adding one more die to the cup for each die that lands with a 6 showing.

a. Find a function $f(t) = a(b^t)$ that you would expect to model her data.

$$f(t) = 10\left(\frac{7}{6}\right)^t$$

b. Solve the equation $f(t) = 30$. What does your solution mean?

$$10\left(\frac{7}{6}\right)^t = 30$$
$$\left(\frac{7}{6}\right)^t = 3$$
$$t = \frac{\log(3)}{\log\left(\frac{7}{6}\right)}$$
$$t \approx 7.13$$

Brenna's quantity of dice should reach 30 by the eighth trial.

c. Brenna wants to know in advance how many trials it should take for her initial quantity of 10 dice to triple. Use properties of exponents and logarithms to rewrite your function from part (a) as an exponential function of the form $f(t) = a(3^{ct})$.

Since $\frac{7}{6} = 3^{\log_3\left(\frac{7}{6}\right)}$, we have $\left(\frac{7}{6}\right)^t = \left(3^{\log_3\left(\frac{7}{6}\right)}\right)^t = 3^{t\cdot\log_3\left(\frac{7}{6}\right)}$. Then, $f(t) = 10\left(3^{t\cdot\log_3\left(\frac{7}{6}\right)}\right)$.

d. Explain how your formula from part (c) allows you to easily find the expected amount of time, t, for the initial quantity of dice to triple.

The quantity triples when $3^{t\cdot\log_3\left(\frac{7}{6}\right)} = 3$, so that $t \cdot \log_3\left(\frac{7}{6}\right) = 1$. Then, we solve that equation to find the value of t: $t = \frac{1}{\log_3\left(\frac{7}{6}\right)} \approx 7.13$.

e. Rewrite the formula for the function f using a base-10 exponential function.

Since $\frac{7}{6} = 10^{\log\left(\frac{7}{6}\right)}$, we have $\left(\frac{7}{6}\right)^t = \left(10^{\log\left(\frac{7}{6}\right)}\right)^t = 10^{t\cdot\log\left(\frac{7}{6}\right)}$. Then, $f(t) = 10\left(10^{t\cdot\log\left(\frac{7}{6}\right)}\right)$.

f. Use your formula from part (e) to find out how many trials it should take for the quantity of dice to grow to 100 dice.

The quantity will be 100 when $10^{t\cdot\log\left(\frac{7}{6}\right)} = 10$, so that $t \cdot \log\left(\frac{7}{6}\right) = 1$. Then, $t = \frac{1}{\log\left(\frac{7}{6}\right)} \approx 14.94$ so that the quantity should exceed 100 by the 15th trial.

5. Suppose that one bacteria population can be modeled by the function $P_1(t) = 500(2^t)$ and a second bacteria population can be modeled by the function $P_2(t) = 500(2.83^t)$, where t measures time in hours. Keep four digits of accuracy for decimal approximations of logarithmic values.

   a. What does the 500 mean in each function?

   In each function, the 500 means that each population has 500 bacteria at the onset of the experiment.

   b. Which population should double first? Explain how you know.

   Since $2.83 > 2$, the second population is growing at a faster rate than the first, so it should double more quickly.

   c. How many hours and minutes should it take for the first population to double?

   The first population doubles every hour, since the base of the exponential function is 2. Thus, the first population doubles in one hour.

   d. Rewrite the formula for $P_2(t)$ in the form $P_2(t) = a(2^{ct})$, for some real numbers a and c.

$$\begin{aligned} P_2(t) &= 500(2.83)^t \\ &= 500\left(2^{\log_2(2.83)}\right)^t \\ &= 500\left(2^{t\cdot\log_2(2.83)}\right) \end{aligned}$$

   e. Use your formula in part (d) to find the time, t, in hours and minutes until the second population doubles.

   The second population doubles when $t \cdot \log_2(2.83) = 1$, which happens when $t = \frac{1}{\log_2(2.83)}$. Thus, $t \approx 0.6663$ hours, so the population doubled after approximately 40 minutes.

6. Copper has antibacterial properties, and it has been shown that direct contact with copper alloy C11000 at $20°\text{C}$ kills 99.9% of all methicillin-resistant *Staphylococcus aureus* (MRSA) bacteria in about 75 minutes. Keep four digits of accuracy for decimal approximations of logarithmic values.

   a. A function that models a population of $1,000$ MRSA bacteria t minutes after coming in contact with copper alloy C11000 is $P(t) = 1000(0.912)^t$. What does the base 0.912 mean in this scenario?

   The base 0.912 means that 91.2% of the MRSA bacteria remain at the end of each minute.

   b. Rewrite the formula for P as an exponential function with base $\frac{1}{2}$.

   Since $0.912 = \left(\frac{1}{2}\right)^{\log_{\frac{1}{2}}(0.912)} = \left(\frac{1}{2}\right)^{-\log_2(0.912)}$, we have

$$\begin{aligned} P(t) &= 1000(0.912^t) \\ &= 1000\left(\frac{1}{2}\right)^{-t\cdot\log_2(0.912)}. \end{aligned}$$

   c. Explain how your formula from part (b) allows you to easily find the time it takes for the population of MRSA to be reduced by half.

   The population of MRSA is reduced by half when the exponent is 1. This happens when $-t \cdot \log_2(0.912) = 1$, so $t = -\frac{1}{\log_2(0.912)} \approx 7.52$. Thus, half of the MRSA bacteria die every $7\frac{1}{2}$ minutes.

Lesson 24: Solving Exponential Equations

Student Outcomes

- Students apply properties of logarithms to solve exponential equations.
- Students relate solutions to $f(x) = g(x)$ to the intersection point(s) on the graphs of $y = f(x)$ and $y = g(x)$ in the case where f and g are constant or exponential functions.

Lesson Notes

Much of our previous work with logarithms in Topic B provided students with the particular skills needed to manipulate logarithmic expressions and solve exponential equations. Although students have solved exponential equations in earlier lessons in Topic B, this is the first time that they solve such equations in the context of exponential functions. In this lesson, students solve exponential equations of the form $ab^{ct} = d$ using properties of logarithms developed in Lessons 12 and 13 (**F-LE.A.4**). For an exponential function f, students solve equations of the form $f(x) = c$ and write a logarithmic expression for the inverse (**F-BF.B.4a**). Additionally, students solve equations of the form $f(x) = g(x)$ where f and g are either constant or exponential functions (**A-REI.D.11**). Examples of exponential functions in this lesson draw from Lesson 7, in which the growth of a bacteria population was modeled by the function $P(t) = 2^t$, and Lesson 23, in which students modeled the growth of an increasing number of beans with a function $f(t) = a(b^t)$, where $a \approx 1$ and $b \approx 1.5$.

Students use technology to calculate logarithmic values and to graph linear and exponential functions.

Classwork

Opening Exercise (4 minutes)

The Opening Exercise is a simple example of solving an exponential equation of the form $ab^{ct} = d$. Allow students to work independently or in pairs to solve this problem. Circulate around the room to check that all students know how to apply a logarithm to solve this problem. Students may choose to use either a base-2 or base-10 logarithm.

Opening Exercise

In Lesson 7, we modeled a population of bacteria that doubled every day by the function $P(t) = 2^t$, where t was the time in days. We wanted to know the value of t when there were 10 bacteria. Since we did not know about logarithms at the time, we approximated the value of t numerically, and we found that $P(t) = 10$ when $t \approx 3.32$.

Use your knowledge of logarithms to find an exact value for t when $P(t) = 10$, and then use your calculator to approximate that value to 4 decimal places.

Since $P(t) = 2^t$, we need to solve $2^t = 10$.

$$2^t = 10$$
$$t\log(2) = \log(10)$$
$$t = \frac{1}{\log(2)}$$
$$t \approx 3.3219$$

Thus, the population will reach 10 bacteria in approximately 3.3219 days.

Discussion (2 minutes)

Ask students to describe their solution method for the Opening Exercise. Make sure that solutions are discussed using both base-10 and base-2 logarithms. If all students used the common logarithm to solve this problem, then present the following solution using the base-2 logarithm:

$$\begin{aligned} 2^t &= 10 \\ \log_2(2^t) &= \log_2(10) \\ t &= \log_2(10) \\ t &= \frac{\log(10)}{\log(2)} \\ t &= \frac{1}{\log(2)} \\ t &\approx 3.3219 \end{aligned}$$

The remaining exercises ask students to solve equations of the form $f(x) = c$ or $f(x) = g(x)$, where f and g are exponential functions (**F-LE.A.4**, **F-BF.B.4a**, **A-REI.D.11**). For the remainder of the lesson, allow students to work either independently or in pairs or small groups on the exercises. Circulate to ensure students are on task and solving the equations correctly. After completing Exercises 1–4, debrief students to check for understanding, and ensure they are using appropriate strategies to complete problems accurately before moving on to Exercises 5–10.

Exercises 1–4 (25 minutes)

Exercises

MP.2

1. **Fiona modeled her data from the bean-flipping experiment in Lesson 23 by the function $f(t) = 1.263(1.357)^t$, and Gregor modeled his data with the function $g(t) = 0.972(1.629)^t$.**
   a. **Without doing any calculating, determine which student, Fiona or Gregor, accumulated 100 beans first. Explain how you know.**

      Since the base of the exponential function for Gregor's model, 1.629, is larger than the base of the exponential function for Fiona's model, 1.357, Gregor's model will grow more quickly than Fiona's, and he will accumulate 100 beans before Fiona does.

   b. **Using Fiona's model …**
      i. **How many trials would be needed for her to accumulate 100 beans?**

         We need to solve the equation $f(t) = 100$ for t.

$$\begin{aligned} 1.263(1.357)^t &= 100 \\ 1.357^t &= \frac{100}{1.263} \\ t\log(1.357) &= \log\left(\frac{100}{1.263}\right) \\ t\log(1.357) &= \log(100) - \log(1.263) \\ t &= \frac{2-\log(1.263)}{\log(1.357)} \\ t &\approx 14.32 \end{aligned}$$

         So, it takes 15 trials for Fiona to accumulate 100 beans.

Scaffolding:

Have struggling students begin this exercise with functions $f(t) = 7(2^t)$ and $g(t) = 4(3^t)$.

ii. **How many trials would be needed for her to accumulate $1,000$ beans?**

We need to solve the equation $f(t) = 1000$ for t.

$$1.263(1.357)^t = 1000$$
$$1.357^t = \frac{1000}{1.263}$$
$$t\log(1.357) = \log\left(\frac{1000}{1.263}\right)$$
$$t\log(1.357) = \log(1000) - \log(1.263)$$
$$t = \frac{3-\log(1.263)}{\log(1.357)}$$
$$t \approx 21.86$$

So, it takes 22 trials for Fiona to accumulate 1000 beans.

c. **Using Gregor's model …**

i. **How many trials would be needed for him to accumulate 100 beans?**

We need to solve the equation $g(t) = 100$ for t.

$$0.972(1.629)^t = 100$$
$$1.629^t = \frac{100}{0.972}$$
$$t\log(1.629) = \log\left(\frac{100}{0.972}\right)$$
$$t\log(1.629) = \log(100) - \log(0.972)$$
$$t = \frac{2-\log(0.972)}{\log(1.629)}$$
$$t \approx 9.50$$

So, it takes 10 trials for Gregor to accumulate 100 beans.

ii. **How many trials would be needed for him to accumulate $1,000$ beans?**

We need to solve the equation $g(t) = 1000$ for t.

$$0.972(1.629)^t = 1000$$
$$1.629^t = \frac{1000}{0.972}$$
$$t\log(1.629) = \log\left(\frac{1000}{0.972}\right)$$
$$t\log(1.629) = \log(1000) - \log(0.972)$$
$$t = \frac{3-\log(0.972)}{\log(1.629)}$$
$$t \approx 14.21$$

So, it takes 15 trials for Gregor to accumulate 1000 beans.

d. **Was your prediction in part (a) correct? If not, what was the error in your reasoning?**

Responses will vary. Either students made the correct prediction, or they did not recognize that the base b determines the growth rate of the exponential function so the larger base 1.629 causes Gregor's function to grow much more quickly than Fiona's.

2. Fiona wants to know when her model $f(t) = 1.263(1.357)^t$ predicts accumulations of 500, $5,000$, and $50,000$ beans, but she wants to find a way to figure it out without doing the same calculation three times.

   a. Let the positive number c represent the number of beans that Fiona wants to have. Then solve the equation $1.263(1.357)^t = c$ for t.

$$1.263(1.357)^t = c$$
$$1.357^t = \frac{c}{1.263}$$
$$t\log(1.357) = \log\left(\frac{c}{1.263}\right)$$
$$t\log(1.357) = \log(c) - \log(1.263)$$
$$t = \frac{\log(c) - \log(1.263)}{\log(1.357)}$$

   b. Your answer to part (a) can be written as a function M of the number of beans c, where $c > 0$. Explain what this function represents.

   The function $M(c) = \frac{\log(c) - \log(1.263)}{\log(1.357)}$ calculates the number of trials it will take for Fiona to accumulate c beans.

   c. When does Fiona's model predict that she will accumulate …

   i. 500 beans?

$$M(500) = \frac{\log(500) - \log(1.263)}{\log(1.357)} \approx 19.59$$

   According to her model, it will take Fiona 20 trials to accumulate 500 beans.

   ii. $5,000$ beans?

$$M(5000) = \frac{\log(5000) - \log(1.263)}{\log(1.357)} \approx 27.14$$

   According to her model, it will take Fiona 28 trials to accumulate 5000 beans.

   iii. $50,000$ beans?

$$M(50000) = \frac{\log(50000) - \log(1.263)}{\log(1.357)} \approx 34.68$$

   According to her model, it will take Fiona 35 trials to accumulate 50000 beans.

3. Gregor states that the function g that he found to model his bean-flipping data can be written in the form $g(t) = 0.972(10^{\log(1.629)t})$. Since $\log(1.629) \approx 0.2119$, he is using $g(t) = 0.972(10^{0.2119t})$ as his new model.

   a. Is Gregor correct? Is $g(t) = 0.972\left(10^{\log(1.629)t}\right)$ an equivalent form of his original function? Use properties of exponents and logarithms to explain how you know.

   Yes, Gregor is correct. Since $10^{\log(1.629)} = 1.629$, and $10^{\log(1.629)t} = \left(10^{\log(1.629)}\right)^t \approx 10^{0.2119t}$, Gregor is right that $g(t) = 0.972(10^{0.2119t})$ is a reasonable model for his data.

b. Gregor also wants to find a function to help him to calculate the number of trials his function g predicts it should take to accumulate 500, $5,000$, and $50,000$ beans. Let the positive number c represent the number of beans that Gregor wants to have. Solve the equation $0.972(10^{0.2119t}) = c$ for t.

$$0.972(10^{0.2119t}) = c$$
$$10^{0.2119t} = \frac{c}{0.972}$$
$$0.2119t = \log\left(\frac{c}{0.972}\right)$$
$$t = \frac{\log(c) - \log(0.972)}{0.2119}$$

c. Your answer to part (b) can be written as a function N of the number of beans c, where $c > 0$. Explain what this function represents.

The function $N(c) = \frac{\log(c) - \log(0.972)}{0.2119}$ calculates the number of trials it will take for Gregor to accumulate c beans.

d. When does Gregor's model predict that he will accumulate …

i. 500 beans?

$$N(500) = \frac{\log(500) - \log(0.972)}{0.2119} \approx 12.80$$

According to his model, it will take Gregor 13 trials to accumulate 500 beans.

ii. $5,000$ beans?

$$N(500) = \frac{\log(5000) - \log(0.972)}{0.2119} \approx 17.51$$

According to his model, it will take Gregor 18 trials to accumulate $5,000$ beans.

iii. $50,000$ beans?

$$N(50000) = \frac{\log(50000) - \log(0.972)}{0.2119} \approx 22.23$$

According to his model, it will take Gregor 23 trials to accumulate $50,000$ beans.

4. Helena and Karl each change the rules for the bean experiment. Helena started with four beans in her cup and added one bean for each that landed marked-side up for each trial. Karl started with one bean in his cup but added two beans for each that landed marked-side up for each trial.

a. Helena modeled her data by the function $h(t) = 4.127(1.468^t)$. Explain why her values of $a = 4.127$ and $b = 1.468$ are reasonable.

Since Helena starts with four beans, we should expect that $a \approx 4$, so a value $a = 4.127$ is reasonable. Because she is using the same rule for adding beans to the cup as we did in Lesson 23, we should expect that $b \approx 1.5$. Thus, her value of $b = 1.468$ is reasonable.

b. Karl modeled his data by the function $k(t) = 0.897(1.992^t)$. Explain why his values of $a = 0.897$ and $b = 1.992$ are reasonable.

Since Karl starts with one bean, we should expect that $a \approx 1$, so a value $a = 0.897$ is reasonable. Because Karl adds two beans to the cup for each that lands marked-side up, we should expect that the number of beans roughly doubles with each trial. That is, we should expect $b \approx 2$. Thus, his value of $b = 1.992$ is reasonable.

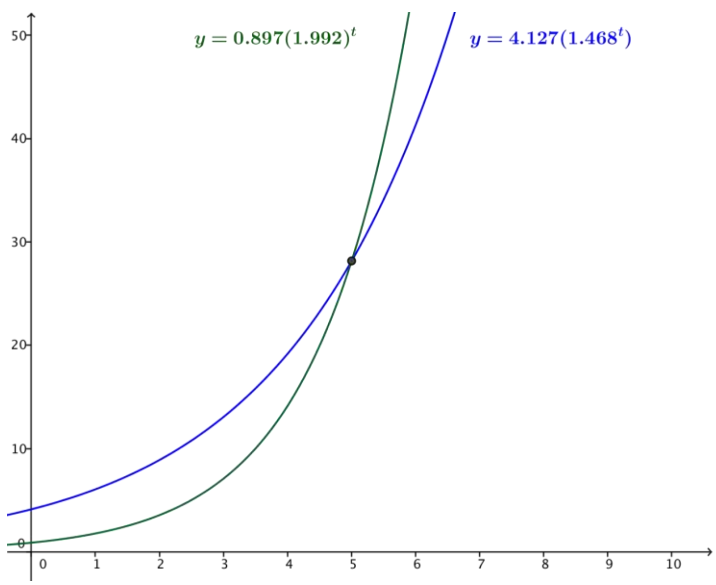
c. **At what value of t do Karl and Helena have the same number of beans?**

We need to solve the equation $h(t) = k(t)$ for t.

$$4.127(1.468^t) = 0.897(1.992^t)$$
$$\log(4.127(1.468^t)) = \log(0.897(1.992^t))$$
$$\log(4.127) + \log(1.468^t) = \log(0.897) + \log(1.992^t)$$
$$\log(4.127) + t\log(1.468) = \log(0.897) + t\log(1.992)$$
$$t\log(1.992) - t\log(1.468) = \log(4.127) - \log(0.897)$$
$$t(\log(1.992) - \log(1.468)) = \log(4.127) - \log(0.897)$$
$$t\left(\log\left(\frac{1.992}{1.468}\right)\right) = \log\left(\frac{4.127}{0.897}\right)$$
$$t(0.13256) \approx 0.66284$$
$$t \approx 5.0003$$

Thus, after trial number 5, Karl and Helena have the same number of beans.

d. **Use a graphing utility to graph $y = h(t)$ and $y = k(t)$ for $0 < t < 10$.**



e. **Explain the meaning of the intersection point of the two curves $y = h(t)$ and $y = k(t)$ in the context of this problem.**

The two curves intersect at the t-value where Helena and Karl have the same number of beans. The y-value indicates the number of beans they both have after five trials.

f. **Which student reaches 20 beans first? Does the reasoning used in deciding whether Gregor or Fiona would get 100 beans first hold true here? Why or why not?**

Helena reaches 20 beans first. Although the function modeling Helena's beans has a smaller base, Karl's does not catch up to Helena until after five trials. After five trials, Karl's will always be greater, and he will reach 100 beans first. The logic we applied to comparing Gregor's model and Fiona's model does not apply here because Helena and Karl do not start with the same initial number of beans.

Debrief students after they complete Exercises 1–4 to ensure understanding of the exercises and strategies used to solve the exercises before continuing. In Exercises 5–10, students solve exponential functions using what they know about logarithms. After completing Exercises 5–10, debrief students about when it is necessary to use logarithms to solve exponential equations and when it is not. Exercises 7, 8, and 9 are examples of exercises that do not require logarithms to solve but may be appropriate to solve with logarithms depending on the approach used by students.

Exercise 5–10 (7 minutes)

For the following functions f and g, solve the equation $f(x) = g(x)$. Express your solutions in terms of logarithms.

5. $f(x) = 10(3.7)^{x+1}$, $g(x) = 5(7.4)^x$

$$10(3.7)^{x+1} = 5(7.4)^x$$
$$2(3.7)^{x+1} = 7.4^x$$
$$\log(2) + \log(3.7^{x+1}) = \log(7.4^x)$$
$$\log(2) + (x+1)\log(3.7) = x\log(7.4)$$
$$\log(2) + x\log(3.7) + \log(3.7) = x\log(7.4)$$
$$\log(2) + \log(3.7) = x\big(\log(7.4) - \log(3.7)\big)$$
$$\log(7.4) = x\log\left(\frac{7.4}{3.7}\right)$$
$$\log(7.4) = x\log(2)$$
$$x = \frac{\log(7.4)}{\log(2)}$$

6. $f(x) = 135(5)^{3x+1}$, $g(x) = 75(3)^{4-3x}$

$$135(5)^{3x+1} = 75(3)^{4-3x}$$
$$9(5)^{3x+1} = 5(3)^{4-3x}$$
$$\log(9) + (3x+1)\log(5) = \log(5) + (4-3x)\log(3)$$
$$2\log(3) + 3x\log(5) + \log(5) = \log(5) + 4\log(3) - 3x\log(3)$$
$$3x\big(\log(5) + \log(3)\big) = 4\log(3) - 2\log(3)$$
$$3x\log(15) = 2\log(3)$$
$$x = \frac{2\log(3)}{3\log(15)}$$

7. $f(x) = 100^{x^3+x^2-4x}$, $g(x) = 10^{2x^2-6x}$

$$100^{x^3+x^2-4x} = 10^{2x^2-6x}$$
$$(10^2)^{x^3+x^2-4x} = 10^{2x^2-6x}$$
$$2(x^3+x^2-4x) = 2x^2 - 6x$$
$$x^3 + x^2 - 4x = x^2 - 3x$$
$$x^3 - x = 0$$
$$x(x^2-1) = 0$$
$$x(x+1)(x-1) = 0$$
$$x = 0,\ x = -1,\ \text{or}\ x = 1$$

Scaffolding:

- Challenge advanced students to solve Exercise 6 in more than one way, for example, by using first the logarithm base 5 and then the logarithm base 3, and comparing the results.
- Advanced students should be able to solve Exercises 7–9 without logarithms by expressing each function with a common base, but using logarithms may be a more reliable approach for students struggling with the exponential properties.

8. $f(x) = 48(4^{x^2+3x})$, $g(x) = 3(8^{x^2+4x+4})$

$$48(4^{x^2+3x}) = 3(8^{x^2+4x+4})$$
$$16(4^{x^2+3x}) = 8^{x^2+4x+4}$$
$$2^4((2^2)^{x^2+3x}) = (2^3)^{x^2+4x+4}$$
$$2^{2x^2+6x+4} = 2^{3x^2+12x+12}$$
$$2x^2 + 6x + 4 = 3x^2 + 12x + 12$$
$$x^2 + 6x + 8 = 0$$
$$(x+4)(x+2) = 0$$
$$x = -4 \text{ or } x = -2$$

9. $f(x) = e^{\sin^2(x)}$, $g(x) = e^{\cos^2(x)}$

$$e^{\sin^2(x)} = e^{\cos^2(x)}$$
$$\sin^2(x) = \cos^2(x)$$
$$\sin(x) = \cos(x) \text{ or } \sin(x) = -\cos(x)$$
$$x = \frac{\pi}{4} + k\pi \text{ or } x = \frac{3\pi}{4} + k\pi \text{ for all integers } k$$

10. $f(x) = (0.49)^{\cos(x)+\sin(x)}$, $g(x) = (0.7)^{2\sin(x)}$

$$(0.49)^{\cos(x)+\sin(x)} = (0.7)^{2\sin(x)}$$
$$\log\left((0.49)^{\cos(x)+\sin(x)}\right) = \log\left(0.7)^{2\sin(x)}\right)$$
$$(\cos(x) + \sin(x))\log(0.49) = 2\sin(x)\log(0.7)$$
$$(\cos(x) + \sin(x))\log(0.7^2) = 2\sin(x)\log(0.7)$$
$$2(\cos(x) + \sin(x))\log(0.7) = 2\sin(x)\log(0.7)$$
$$2\cos(x) + 2\sin(x) = 2\sin(x)$$
$$\cos(x) = 0$$
$$x = \frac{\pi}{2} + k\pi \text{ for all integers } k$$

Closing (3 minutes)

Ask students to respond to the following prompts either in writing or orally to a partner.

- Describe two different approaches to solving the equation $2^{x+1} = 3^{2x}$. Do not actually solve the equation.
  - *You could begin by taking the logarithm base 10 of both sides, or the logarithm base 2 of both sides. You could even take the logarithm base 3 of both sides.*
- Could the graphs of two exponential functions $f(x) = 2^{x+1}$ and $g(x) = 3^{2x}$ ever intersect at more than one point? Explain how you know.
  - *No. The graphs of these functions are always increasing. They intersect at one point, but once they cross once they cannot cross again. For large values of x, the quantity 3^{2x} is always greater than 2^{x+1}, so the graph of g ends up above the graph of f after they cross.*

- Discuss how the starting value and base affect the graph of an exponential function and how this can help you compare exponential functions.
  - *The starting value determines the y-intercept of an exponential function, so it determines how large or small the function is when $x = 0$. The base is ultimately more important and determines how quickly the function increases (or decreases). When comparing exponential functions, the function with the larger base always overtakes the function with the smaller base no matter how large the value when $x = 0$.*
- If $f(x) = 2^{x+1}$ and $g(x) = 3^{2x}$, is it possible for the equation $f(x) = g(x)$ to have more than one solution?
  - *No. Solutions to the equation $f(x) = g(x)$ correspond to x-values of intersection points of the graphs of $y = f(x)$ and $y = g(x)$. Since these graphs can intersect no more than once, the equation can have no more than one solution.*

Exit Ticket (4 minutes)

Name ____ Date ____________________

Lesson 24: Solving Exponential Equations

Exit Ticket

Consider the functions $f(x) = 2^{x+6}$ and $g(x) = 5^{2x}$.

a. Use properties of logarithms to solve the equation $f(x) = g(x)$. Give your answer as a logarithmic expression, and approximate it to two decimal places.

b. Verify your answer by graphing the functions $y = f(x)$ and $y = g(x)$ in the same window on a calculator, and sketch your graphs below. Explain how the graph validates your solution to part (a).

Exit Ticket Sample Solutions

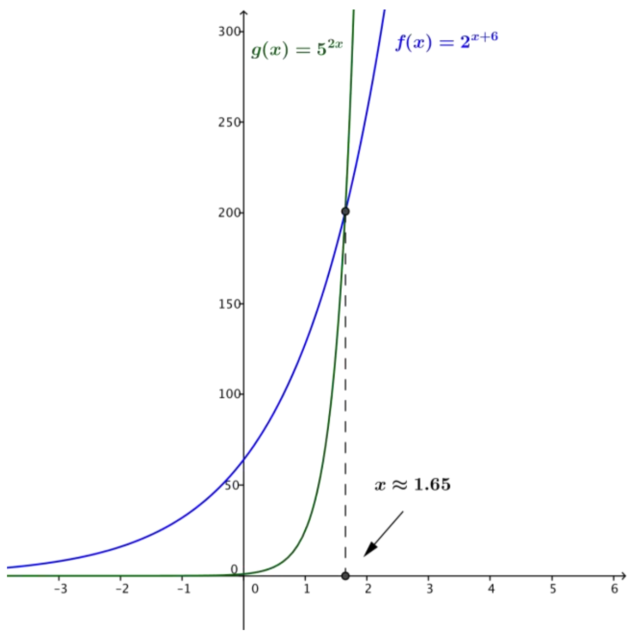
Consider the functions $f(x) = 2^{x+6}$ and $g(x) = 5^{2x}$.

a. **Use properties of logarithms to solve the equation $f(x) = g(x)$. Give your answer as a logarithmic expression, and approximate it to two decimal places.**

$$2^{x+6} = 5^{2x}$$
$$(x+6)\log(2) = 2x\log(5)$$
$$2x\log(5) - x\log(2) = 6\log(2)$$
$$x = \frac{6\log(2)}{2\log(5) - \log(2)}$$
$$x = \frac{\log(64)}{\log(25) - \log(2)}$$
$$x = \frac{\log(64)}{\log\left(\frac{25}{2}\right)}$$
$$x \approx 1.65$$

Any of the final three forms are acceptable, and other correct forms using logarithms with other bases (such as base 2) are possible.

b. **Verify your answer by graphing the functions $y = f(x)$ and $y = g(x)$ in the same window on a calculator, and sketch your graphs below. Explain how the graph validates your solution to part (a).**



Because the graphs of $y = f(x)$ and $y = g(x)$ intersect when $x \approx 1.65$, we know that the equation $f(x) = g(x)$ has a solution at approximately $x = 1.65$.

Problem Set Sample Solutions

1. Solve the following equations.

a. $2 \cdot 5^{x+3} = 6250$

$$\begin{aligned} 5^{x+3} &= 3125 \\ 5^{x+3} &= 5^5 \\ x+3 &= 5 \\ x &= 2 \end{aligned}$$

b. $3 \cdot 6^{2x} = 648$

$$\begin{aligned} 6^{2x} &= 216 \\ 6^{2x} &= 6^3 \\ 2x &= 3 \\ x &= \frac{3}{2} \end{aligned}$$

c. $5 \cdot 2^{3x+5} = 10240$

$$\begin{aligned} 2^{3x+5} &= 2048 \\ 2^{3x+5} &= 2^{11} \\ 3x+5 &= 11 \\ 3x &= 6 \\ x &= 2 \end{aligned}$$

d. $4^{3x-1} = 32$

$$\begin{aligned} 4^{3x-1} &= 2^5 \\ 2^{2\cdot(3x-1)} &= 2^5 \\ 6x-2 &= 5 \\ 6x &= 7 \\ x &= \frac{7}{6} \end{aligned}$$

e. $3 \cdot 2^{5x} = 216$

$$\begin{aligned} 2^{5x} &= 72 \\ 5x \cdot \ln(2) &= \ln(72) \\ x &= \frac{\ln(72)}{5 \cdot \ln(2)} \\ x &\approx 1.234 \end{aligned}$$

Note: Students can also use the common logarithm to find the solution.

f. $5 \cdot 11^{3x} = 120$

$$11^{3x} = 24$$
$$3x \cdot \ln(11) = \ln(24)$$
$$x = \frac{\ln(24)}{3 \cdot \ln(11)}$$
$$x \approx 0.442$$

Note: Students can also use the common logarithm to find the solution.

g. $7 \cdot 9^x = 5405$

$$9^x = \frac{5\,405}{7}$$
$$x \cdot \ln(9) = \ln\left(\frac{5\,405}{7}\right)$$
$$x = \frac{\ln\left(\frac{5\,405}{7}\right)}{\ln(9)}$$
$$x \approx 3.026$$

Note: Students can also use the common logarithm to find the solution.

h. $\sqrt{3} \cdot 3^{3x} = 9$

Solution using properties of exponents:

$$3^{\frac{1}{2}} \cdot 3^{3x} = 3^2$$
$$3^{\frac{1}{2}+3x} = 3^2$$
$$\frac{1}{2} + 3x = 2$$
$$x = \frac{1}{2}$$

i. $\log(400) \cdot 8^{5x} = \log(160000)$

$$8^{5x} = \frac{\log(160000)}{\log(400)}$$
$$8^{5x} = 2$$
$$8^{5x} = 8^{\frac{1}{3}}$$
$$5x = \frac{1}{3}$$
$$x = \frac{1}{15}$$

2. Lucy came up with the model $f(t) = 0.701(1.382)^t$ for the first bean activity. When does her model predict that she would have $1,000$ beans?

$$1000 = 0.701(1.382)^t$$
$$\log(1000) = \log(0.701) + t\log(1.382)$$
$$t = \frac{\log(1000) - \log(0.701)}{\log(1.382)}$$
$$t \approx 22.45$$

Lucy's model predicts that it will take 23 trials to have over 1000 beans.

3. Jack came up with the model $g(t) = 1.033(1.707)^t$ for the first bean activity. When does his model predict that he would have $50,000$ beans?

$$50000 = 1.033(1.707)^t$$
$$\log(50000) = \log(1.033) + t\log(1.707)$$
$$t = \frac{\log(50000) - \log(1.033)}{\log(1.707)}$$
$$t \approx 20.17$$

Jack's model predicts that it will take 21 trials to have over $50,000$ beans.

4. If instead of beans in the first bean activity you were using fair pennies, when would you expect to have $\$1,000,000$?

One million dollars is 10^8 pennies. Using fair pennies, we can model the situation by $f(t) = 1.5^t$.

$$10^8 = 1.5^t$$
$$8 = t\log(1.5)$$
$$t = \frac{8}{\log(1.5)}$$
$$t \approx 45.43$$

We should expect it to take 46 trials to reach more than $\$1$ million using fair pennies.

5. Let $f(x) = 2 \cdot 3^x$ and $g(x) = 3 \cdot 2^x$.

 a. Which function is growing faster as x increases? Why?

 The function f is growing faster due to its larger base, even though $g(0) > f(0)$.

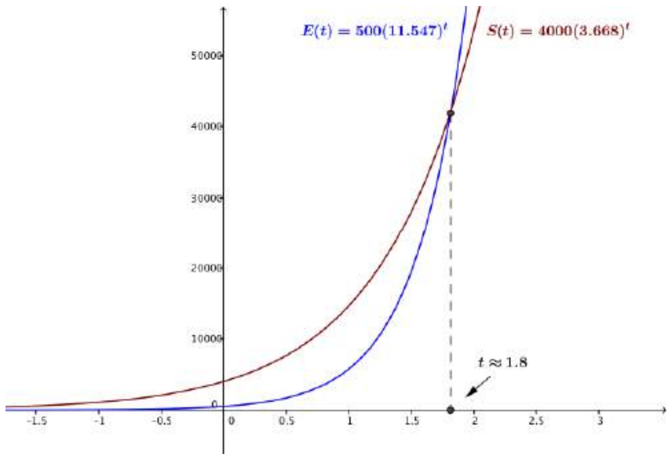
 b. When will $f(x) = g(x)$?

$$f(x) = g(x)$$
$$2 \cdot 3^x = 3 \cdot 2^x$$
$$\ln(2 \cdot 3^x) = \ln(3 \cdot 2^x)$$
$$\ln(2) + x\ln(3) = \ln(3) + x\ln(2)$$
$$x\ln(3) - x\ln(2) = \ln(3) - \ln(2)$$
$$x\ln\left(\frac{3}{2}\right) = \ln\left(\frac{3}{2}\right)$$
$$x = 1$$

 Note: Students can also use the common logarithm to find the solution.

6. The growth of a population of *E. coli* bacteria can be modeled by the function $E(t) = 500(11.547)^t$, and the growth of a population of *Salmonella* bacteria can be modeled by the function $S(t) = 4000(3.668)^t$, where t measures time in hours.

   a. Graph these two functions on the same set of axes. At which value of t does it appear that the graphs intersect?



From the graph, it appears that the two curves intersect at $t \approx 1.8$.

   b. Use properties of logarithms to find the time t when these two populations are the same size. Give your answer to two decimal places.

$$E(t) = S(t)$$
$$500(11.547)^t = 4000(3.668)^t$$
$$11.547^t = 8(3.668)^t$$
$$t\log(11.547) = \log(8) + t\log(3.668)$$
$$t\big(\log(11.547) - \log(3.668)\big) = \log(8)$$
$$t = \frac{\log(8)}{\log(11.547) - \log(3.668)}$$
$$t \approx 1.81329$$

It takes approximately 1.81 *hours for the populations to be the same size.*

7. Chain emails contain a message suggesting you will have bad luck if you do not forward the email to others. Suppose a student started a chain email by sending the message to 10 friends and asking those friends to each send the same email to 3 more friends exactly one day after receiving the message. Assuming that everyone that gets the email participates in the chain, we can model the number of people who receive the email on the n^{th} day by the formula $E(n) = 10(3^n)$, where $n = 0$ indicates the day the original email was sent.

   a. If we assume the population of the United States is 318 million people and everyone who receives the email sends it to 3 people who have not received it previously, how many days until there are as many emails being sent out as there are people in the United States?

$$318(10^6) = 10 \cdot 3^n$$
$$318(10^5) = 3^n$$
$$\log(318) + \log(10^5) = n \cdot \log(3)$$
$$\log(318) + 5 = n \cdot \log(3)$$
$$n = \frac{5 + \log(318)}{\log(3)}$$
$$n \approx 15.72$$

So by the 16^{th} *day, more than* 318 *million emails are being sent out.*

b. The population of earth is approximately 7.1 billion people. On what day will 7.1 billion emails be sent out?

$$7.1(10^9) = 10(3^n)$$
$$7.1(10^8) = 3^n$$
$$\log(7.1(10^8)) = n \cdot \log(3)$$
$$n = \frac{8 + \log(7.1)}{\log(3)}$$
$$n \approx 18.5514$$

By the 19^{th} day, more than 7.1 billion emails will be sent.

8. Solve the following exponential equations.

a. $10^{(3x-5)} = 7^x$

$$10^{3x-5} = 7^x$$
$$3x - 5 = x\log(7)$$
$$x\big(3 - \log(7)\big) = 5$$
$$x = \frac{5}{3 - \log(7)}$$

b. $3^{\frac{x}{5}} = 2^{4x-2}$

$$3^{\frac{x}{5}} = 2^{4x-2}$$
$$\frac{x}{5}\log(3) = (4x-2)\log(2)$$
$$4x\log(2) - x\frac{\log(3)}{5} = 2\log(2)$$
$$x\left(4\log(2) - \frac{\log(3)}{5}\right) = 2\log(2)$$
$$x = \frac{2\log(2)}{4\log(2) - \frac{\log(3)}{5}}$$

c. $10^{x^2+5} = 100^{2x^2+x+2}$

$$10^{x^2+5} = 100^{2x^2+x+2}$$
$$x^2 + 5 = (2x^2 + x + 2)\log(100)$$
$$x^2 + 5 = 4x^2 + 2x + 4$$
$$3x^2 + 2x - 1 = 0$$
$$(3x - 1)(x + 1) = 0$$
$$x = \frac{1}{3} \text{ or } x = -1$$

d. $4^{x^2-3x+4} = 2^{5x-4}$

$$4^{x^2-3x+4} = 2^{5x-4}$$
$$(x^2-3x+4)\log_2(4) = (5x-4)\ \log_2(2)$$
$$2(x^2-3x+4) = 5x-4$$
$$2x^2-6x+8 = 5x-4$$
$$2x^2-11x+12 = 0$$
$$(2x-3)(x-4) = 0$$
$$x = \frac{3}{2}\ or\ x = 4$$

9. Solve the following exponential equations.

a. $(2^x)^x = 8^x$

$$2^{x^2} = 8^x$$
$$x^2\log_2(2) = x\log_2(8)$$
$$x^2 = 3x$$
$$x^2-3x = 0$$
$$x(x-3) = 0$$
$$x = 0\ or\ x = 3$$

b. $(3^x)^x = 12$

$$3^{x^2} = 12$$
$$x^2\log(3) = \log(12)$$
$$x^2 = \frac{\log(12)}{\log(3)}$$
$$x = \sqrt{\frac{\log(12)}{\log(3)}}\ or\ x = -\sqrt{\frac{\log(12)}{\log(3)}}$$

10. Solve the following exponential equations.

a. $10^{x+1} - 10^{x-1} = 1287$

$$10^{x+1} - 10^{x-1} = 1287$$
$$100(10^{x-1}) - 10^{x-1} = 1287$$
$$10^{x-1}(100-1) = 1287$$
$$99(10^{x-1}) = 1287$$
$$10^{x-1} = 13$$
$$x-1 = \log(13)$$
$$x = \log(13)+1$$

b. $2(4^x) + 4^{x+1} = 342$

$$2(4^x) + 4^{x+1} = 342$$
$$2(4^x) + 4(4^x) = 342$$
$$6(4^x) = 342$$
$$4^x = 57$$
$$x = \log_4(57) = \frac{\log(57)}{\log(4)} = \frac{1}{2}\log_2(57)$$

11. **Solve the following exponential equations.**

a. $(10^x)^2 - 3(10^x) + 2 = 0$ **Hint: Let $u = 10^x$, and solve for u before solving for x.**

Let $u = 10^x$. Then

$$u^2 - 3u + 2 = 0$$
$$(u-2)(u-1) = 0$$
$$u = 2 \text{ or } u = 1$$

If $u = 2$, we have $2 = 10^x$, and then $x = \log(2)$.

If $u = 1$, we have $1 = 10^x$, and then $x = 0$.

Thus, the two solutions to this equation are 0 and $\log(2)$.

b. $(2^x)^2 - 3(2^x) - 4 = 0$

Let $u = 2^x$.

$$u^2 - 3u - 4 = 0$$
$$(u-4)(u+1) = 0$$
$$u = 4 \text{ or } u = -1$$

If $u = 4$, we have $2^x = 4$, and then $x = 2$.

If $u = -1$, we have $2^x = -1$, which has no solution.

Thus, the only solution to this equation is 2.

c. $3(e^x)^2 - 8(e^x) - 3 = 0$

Let $u = e^x$.

$$3u^2 - 8u - 3 = 0$$
$$(u-3)(3u+1) = 0$$
$$u = 3 \text{ or } u = -\frac{1}{3}$$

If $u = 3$, we have $e^x = 3$, and then $x = \ln(3)$.

If $u = -\frac{1}{3}$, we have $e^x = -\frac{1}{3}$, which has no solution because $e^x > 0$ for every value of x.

Thus, the only solution to this equation is $\ln(3)$.

d. $4^x + 7(2^x) + 12 = 0$

Let $u = 2^x$.

$$(2^x)^2 + 7(2^x) + 12 = 0$$
$$u^2 + 7u + 12 = 0$$
$$(u+3)(u+4) = 0$$
$$u = -3 \text{ or } u = -4$$

But $2^x > 0$ for every value of x, thus there are no solutions to this equation.

e. $(10^x)^2 - 2(10^x) - 1 = 0$

Let $u = 10^x$.

$$u^2 - 2u - 1 = 0$$
$$u = 1 + \sqrt{2} \text{ or } u = 1 - \sqrt{2}$$

If $u = 1 + \sqrt{2}$, *we have* $10^x = 1 + \sqrt{2}$, *and then* $x = \log(1 + \sqrt{2})$.

If $u = 1 - \sqrt{2}$, *we have* $10^x = 1 - \sqrt{2}$, *which has no solution because* $1 - \sqrt{2} < 0$.

Thus, the only solution to this equation is $\log(1 + \sqrt{2})$.

12. Solve the following systems of equations.

a. $2^{x+2y} = 8$
$4^{2x+y} = 1$

$$2^{x+2y} = 2^3$$
$$4^{2x+y} = 4^0$$

$$x + 2y = 3$$
$$2x + y = 0$$

$$x + 2y = 3$$
$$4x + 2y = 0$$

$$y = 2$$
$$x = -1$$

b. $2^{2x+y-1} = 32$
$4^{x-2y} = 2$

$$2^{2x+y-1} = 2^5$$
$$(2^2)^{x-2y} = 2^1$$

$$2x + y - 1 = 5$$
$$2(x - 2y) = 1$$

$$2x + y = 6$$
$$2x - 4y = 1$$

$$y = 1$$
$$x = \frac{5}{2}$$

c. $2^{3x} = 8^{2y+1}$
$9^{2y} = 3^{3x-9}$

$$2^{3x} = (2^3)^{2y+1}$$
$$(3^2)^{2y} = 3^{3x-9}$$

$$3x = 3(2y + 1)$$
$$2(2y) = (3x - 9)$$

$$3x - 6y = 3$$
$$3x - 4y = 9$$

$$y = 3$$
$$x = 7$$

13. Because $f(x) = \log_b(x)$ is an increasing function, we know that if $p < q$, then $\log_b(p) < \log_b(q)$. Thus, if we take logarithms of both sides of an inequality, then the inequality is preserved. Use this property to solve the following inequalities.

a. $4^x > \frac{5}{3}$

$$4^x > \frac{5}{3}$$
$$\log(4^x) > \log\left(\frac{5}{3}\right)$$
$$x\log(4) > \log(5) - \log(3)$$
$$x > \frac{\log(5) - \log(3)}{\log(4)}$$

b. $\left(\frac{2}{7}\right)^x > 9$

$$\left(\frac{2}{7}\right)^x > 9$$
$$x\log\left(\frac{2}{7}\right) > \log(9)$$

But, remember that $\log\left(\frac{2}{7}\right) < 0$, so we need to divide by a negative number. We then have $x < \frac{\log(9)}{\log(2) - \log(7)}$.

c. $4^x > 8^{x-1}$

$$(2^2)^x > (2^3)^{x-1}$$
$$2^{2x} > 2^{3x-3}$$
$$2x > 3x - 3$$
$$3 > x$$

d. $3^{x+2} > 5^{3-2x}$

$$3^{x+2} > 5^{3-2x}$$
$$(x+2)\log(3) > (3-2x)\log(5)$$
$$2x\log(5) + x\log(3) > 3\log(5) - 2\log(3)$$
$$x > \frac{3\log(5) - 2\log(3)}{2\log(5) + \log(3)}$$
$$x > \frac{\log\left(\frac{125}{9}\right)}{\log(75)}$$

e. $\left(\frac{3}{4}\right)^x > \left(\frac{4}{3}\right)^{x+1}$

$$\left(\frac{3}{4}\right)^x > \left(\frac{4}{3}\right)^{x+1}$$
$$x\log\left(\frac{3}{4}\right) > (x+1)\log\left(\frac{4}{3}\right)$$
$$x\left(\log\left(\frac{3}{4}\right) - \log\left(\frac{4}{3}\right)\right) > \log\left(\frac{4}{3}\right)$$

But, $\log\left(\frac{3}{4}\right) = -\log\left(\frac{4}{3}\right)$*, so we have*

$$x\left(-\log\left(\frac{4}{3}\right) - \log\left(\frac{4}{3}\right)\right) > \log\left(\frac{4}{3}\right)$$
$$x\left(-2\log\left(\frac{4}{3}\right)\right) > \log\left(\frac{4}{3}\right).$$

But, $-2\log\left(\frac{4}{3}\right) < 0$*, so we need to divide by a negative number, so we have*

$$x < \frac{\log\left(\frac{4}{3}\right)}{-2\log\left(\frac{4}{3}\right)}$$
$$x < -\frac{1}{2}.$$

Lesson 25: Geometric Sequences and Exponential Growth and Decay

Student Outcomes

- Students use geometric sequences to model situations of exponential growth and decay.
- Students write geometric sequences explicitly and recursively and translate between the two forms.

Lesson Notes

In Algebra I, students learned to interpret arithmetic sequences as linear functions and geometric sequences as exponential functions but both in simple contexts only. In this lesson, which focuses on exponential growth and decay, students construct exponential functions to solve multi-step problems. In the homework, they do the same with linear functions. The lesson addresses focus standard **F-BF.A.2**, which asks students to write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms. These skills are also needed to develop the financial formulas in Topic E.

In general, a *sequence* is defined by a function f from a domain of positive integers to a range of numbers that can be either integers or real numbers depending on the context, or other nonmathematical objects that satisfy the equation $f(n) = a_n$. When that function is expressed as an algebraic function of the index variable n, then that expression of the function is called an *explicit form of the sequence (or explicit formula).* For example, the function f with domain the positive integers and which satisfies $f(n) = 3^n$ for all $n \geq 1$ is an explicit form for the sequence $3, 9, 27, 81, \ldots$. If the function is expressed in terms of the previous terms of the sequence and an initial value, then that expression of the function is called the *recursive form of the sequence (or recursive formula).* A recursive formula for the sequence $3, 9, 27, 81, \ldots$ is $a_n = 3a_{n-1}$, with $a_0 = 3$.

It is important to note that sequences can be indexed by starting with any integer. The convention in Algebra I was that the indices of a sequence usually started at 1. In Algebra II, we often—but not always—start our indices at 0. In this way, we start counting at the zero term, and count $0, 1, 2, \ldots$ instead of $1, 2, 3, \ldots$.

Classwork

Opening Exercise (8 minutes)

The Opening Exercise is essentially a reprise of the use in Algebra I of an exponential decay model with a geometric sequence.

Scaffolding:

- Students who struggle in calculating the heights, even with a calculator, should have a much easier time getting the terms and seeing the pattern if the rebound is changed to 50% instead of 60%.
- Ask advanced students to develop a model without the scaffolded questions presented here.

Opening Exercise

Suppose a ball is dropped from an initial height h_0 and that each time it rebounds, its new height is 60% of its previous height.

a. **What are the first four rebound heights h_1, h_2, h_3, and h_4 after being dropped from a height of $h_0 = 10$ ft.?**

The rebound heights are $h_1 = 6$ ft., $h_2 = 3.6$ ft., $h_3 = 2.16$ ft., and $h_4 = 1.296$ ft.

b. **Suppose the initial height is A ft. What are the first four rebound heights? Fill in the following table:**

Rebound	Height (ft.)
1	$0.6A$
2	$0.36A$
3	$0.216A$
4	$0.1296A$

c. **How is each term in the sequence related to the one that came before it?**

Each term is 0.6 times the previous term.

d. **Suppose the initial height is A ft. and that each rebound, rather than being 60% of the previous height, is r times the previous height, where $0 < r < 1$. What are the first four rebound heights? What is the n^{th} rebound height?**

The rebound heights are $h_1 = Ar$ ft, $h_2 = Ar^2$ ft, $h_3 = Ar^3$ ft, and $h_4 = Ar^4$ ft. The n^{th} rebound height is $h_n = ar^n$ ft.

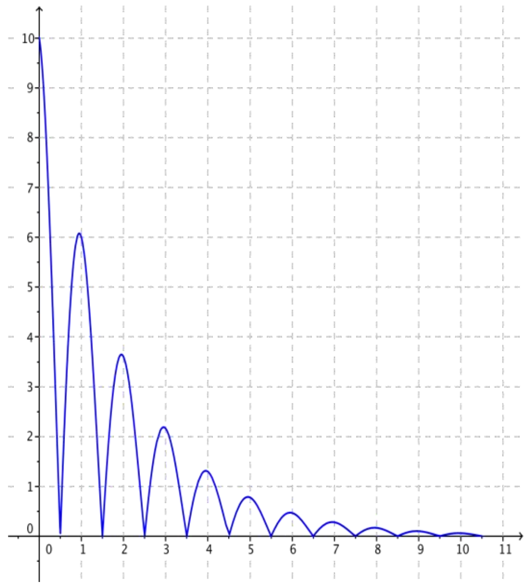
e. **What kind of sequence is the sequence of rebound heights?**

The sequence of rebounds is geometric (geometrically decreasing).

f. **Suppose that we define a function f with domain the positive integers so that $f(1)$ is the first rebound height, $f(2)$ is the second rebound height, and continuing so that $f(k)$ is the k^{th} rebound height for positive integers k. What type of function would you expect f to be?**

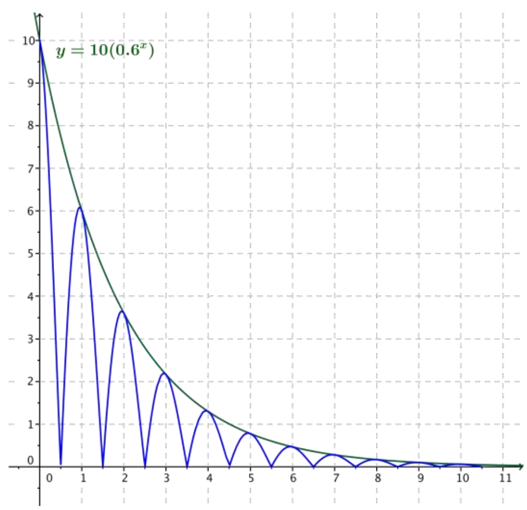
Since each bounce has a rebound height of r times the previous height, the function f should be exponentially decreasing.

g. **On the coordinate plane below, sketch the height of the bouncing ball when $A = 10$ and $r = 0.60$, assuming that the highest points occur at $x = 1, 2, 3, 4, \ldots$.**



h. **Does the exponential function $f(x) = 10(0.60)^x$ for real numbers x model the height of the bouncing ball? Explain how you know.**

No. Exponential functions do not have the same behavior as a bouncing ball. The graph of f is the smooth curve that connects the points at the "top" of the rebounds, as shown in the graph at right.



i. **What does the function $f(n) = 10(0.60)^n$ for integers $n \geq 0$ model?**

The exponential function $f(n) = 10(0.60)^n$ models the height of the rebounds for integer values of n.

Exercise 1 (4 minutes)

While students are working on Exercise 1, circulate around the classroom to ensure student comprehension. After students complete the exercise, debrief to make sure that everyone understands that the salary model is linear and not exponential.

Scaffolding:

If students struggle with calculating the earnings or visualizing the graph, have them calculate the salary for the first five days and plot the points corresponding to those earnings.

Exercises

1. **Jane works for a video game development company that pays her a starting salary of $\$100$ per day, and each day she works she earns $\$100$ more than the day before.**

   a. **How much does she earn on day 5?**

      On day 5, she earns $\$500$.

   b. **If you were to graph the growth of her salary for the first 10 days she worked, what would the graph look like?**

      The graph would be a set of points lying on a straight line.

   c. **What kind of sequence is the sequence of Jane's earnings each day?**

      The sequence of her earnings is arithmetic (that is, the sequence is arithmetically increasing).

Discussion (2 minutes)

Pause here to ask students the following questions:

- What have we learned so far? What is the point of the previous two exercises?
  - *There are two different types of sequences, arithmetic and geometric, that model different ways that quantities can increase or decrease.*
- What do you recall about geometric and arithmetic sequences from Algebra I?
  - *To get from one term of an arithmetic sequence to the next, you add a number d, called the common difference. To get from one term of a geometric sequence to the next you multiply by a number r, called the common quotient (or common ratio).*

For historical reasons, the number r that we call the *common quotient* is often referred to as the *common ratio*, which is not fully in agreement with our definition of *ratio*. Using the term is acceptable because its use is so standardized in mathematics.

Exercise 2 (9 minutes)

Students use a geometric sequence to model the following situation and develop closed and recursive formulas for the sequence. Then they find an exponential model first using base 2 and then using base e and solving for the doubling time. Students should work in pairs on these exercises, using a calculator as needed. They should be introduced to P_0 as the notation for the original number of bacteria (at time $t = 0$) and also the first term of the sequence, which we refer to as the *zero term*. Counting terms starting with 0 means that if we represent our sequence by a function f, then $P_n = f(n)$ for integers $n \geq 0$.

This is an appropriate time to mention to students that we often use a continuous function to model a discrete phenomenon. In this example, the function that we use to represent the bacteria population takes on non-integer values. We need to interpret these function values according to the situation—it is not appropriate to say that the population consists of a non-integer number of bacteria at a certain time, even if the function value is non-integer. In these cases, students should round their answers to an integer that makes sense in the context of the problem.

Scaffolding:

Students may need the hint that, in the Opening Exercise, they wrote the terms of a geometric sequence so they can begin with the first three terms of such a sequence and use it to find r.

2. A laboratory culture begins with $1,000$ bacteria at the beginning of the experiment, which we denote by time 0 hours. By time 2 hours, there are $2,890$ bacteria.

a. If the number of bacteria is increasing by a common factor each hour, how many bacteria are there at time 1 hour? At time 3 hours?

If P_0 is the original population, the first three terms of the geometric sequence are P_0, P_0r, and P_0r^2. In this case, $P_0 = 1000$ and $P_2 = P_0r^2 = 2890$, so $r^2 = 2.89$ and $r = 1.7$. Therefore, $P_1 = P_0r = 1700$ and $P_3 = P_0r^3 = 2890 \cdot 1.7 = 4913$.

b. Find the explicit formula for term P_n of the sequence in this case.

$$P_n = P_0r^n = 1000(1.7)^n$$

c. How would you find term P_{n+1} if you know term P_n? Write a recursive formula for P_{n+1} in terms of P_n.

You would multiply the n^{th} term by r, which in this case is 1.7. We have $P_{n+1} = 1.7\,P_n$.

d. If P_0 is the initial population, the growth of the population P_n at time n hours can be modeled by the sequence $P_n = P(n)$, where P is an exponential function with the following form:

$$P(n) = P_0 2^{kn}, \text{ where } k > 0.$$

Find the value of k and write the function P in this form. Approximate k to four decimal places.

We know that $P(n) = 1000(1.7)^n$ and $1.7 = 2^{\log_2(1.7)}$, with $k = \log_2(1.7) = \dfrac{\log(1.7)}{\log(2)} \approx 0.7655$.

Thus, we can express P in the form:

$$P(n) = 1000(2^{0.7655n}).$$

e. Use the function in part (d) to determine the value of t when the population of bacteria has doubled.

We need to solve $2000 = 1000(2^{0.7655t})$, which happens when the exponent is 1.

$$0.7655t = 1$$
$$t = \frac{1}{0.7655}$$
$$t \approx 1.306$$

This population doubles in roughly 1.306 hours, which is about 1 hour and 18 minutes.

f. If P_0 is the initial population, the growth of the population P at time t can be expressed in the following form:

$$P(n) = P_0 e^{kn}, \text{ where } k > 0.$$

Find the value of k, and write the function P in this form. Approximate k to four decimal places.

Substituting in the formula for $t = 2$, we get $2890 = 1000e^{2k}$. Solving for k, we get $k = \ln(1.7) \approx 0.5306$. Thus, we can express P in the form: $P(t) = 1000(e^{0.5306t})$.

g. **Use the formula in part (d) to determine the value of t when the population of bacteria has doubled.**

Substituting in the formula with $k = 0.5306$, we get $2000 = 1000e^{0.5306t}$. Solving for t, we get $t = \frac{\ln(2)}{0.5306} \approx 1.306$, which is the same value we found in part (e).

Discussion (4 minutes)

MP.3 Students should share their solutions to Exercise 2 with the rest of the class, giving particular attention to parts (b) and (c).

Part (b) of Exercise 2 presents what is called the *explicit formula* (or *closed form*) for a geometric sequence, whereas part (c) introduces the idea of a *recursive formula*. Students need to understand that given any two terms in a geometric (or arithmetic) sequence, they can derive the explicit formula. Recursion provides a way of defining a sequence given one or more initial terms by using one term of the sequence to find the next term.

Discuss with students the distinction between the two functions:

$$P(n) = 1000(2^{0.7655n}) \text{ for integers } n \geq 0, \text{ and}$$

$$P(t) = 1000(2^{0.7655t}) \text{ for real numbers } t \geq 0.$$

In the first case, the function P as a function of an integer n represents the population at discrete times $n = 0, 1, 2, \ldots$, while P as a function of a real number t represents the population at any time $t \geq 0$, regardless of whether that time is an integer. If we graphed these two functions, the first graph would be the points $(0, P(0))$, $(1, P(1))$, $(2, P(2))$, etc., and the second graph would be the smooth curve drawn through the points of the first graph. We can use either statement of the function to define a sequence $P_n = P(n)$ for integers n. This was discussed in Opening Exercise part (h), as the distinction between the graph of the points at the top of the rebounds of the bouncing ball and the graph of the smooth curve through those points.

Our work earlier in the module that extended the laws of exponents to the set of all real numbers applies here to extend a discretely defined function such as $P(n) = 1000(2^{0.7655n})$ for integers $n \geq 0$ to the continuously defined function $P(t) = 1000(2^{0.7655t})$ for real numbers $t \geq 0$. Then, we can solve exponential equations involving sequences using our logarithmic tools.

Students may question why we could find two different exponential representations of the function P in parts (d) and (f) of Exercise 2. We can use the properties of exponents to express an exponential function in terms of any base. In Lesson 6 earlier in the module, we saw that the functions $H(t) = ae^t$ for real numbers a have rate of change equal to 1. For this reason, which is important in Calculus and beyond, we usually prefer to use base e for exponential functions.

Exercises 3–4 (5 minutes)

Students should work on these exercises in pairs. They can take turns calculating terms in the sequences. Circulate the room and observe students to call on to share their work with the class before proceeding to the next and final set of exercises.

3. The first term a_0 of a geometric sequence is -5, and the common ratio r is -2.
   a. What are the terms a_0, a_1, and a_2?

   $a_0 = -5$
   $a_1 = 10$
   $a_2 = -20$

   b. Find a recursive formula for this sequence.

   The recursive formula is $a_{n+1} = -2a_n$, *with* $a_0 = -5$.

   c. Find an explicit formula for this sequence.

   The explicit formula is $a_n = -5(-2)^n$, *for* $n \geq 0$.

   d. What is term a_9?

   Using the explicit formula, we find: $a_9 = (-5) \cdot (-2)^9 = 2560$.

   e. What is term a_{10}?

   One solution is to use the explicit formula: $a_{10} = (-5) \cdot (-2)^{10} = -5120$.

   Another solution is to use the recursive formula: $a_{10} = a_9 \cdot (-2) = -5120$.

4. Term a_4 of a geometric sequence is 5.8564, and term a_5 is -6.44204.
   a. What is the common ratio r?

   We have $r = \frac{-6.44204}{5.8564} = -1.1$. *The common ratio is* -1.1.

   b. What is term a_0?

   From the definition of a geometric sequence, $a_4 = a_0 r^4 = 5.8564$, *so* $a_0 = \frac{5.8564}{(-1.1)^4} = \frac{5.8564}{1.4641} = 4$.

   c. Find a recursive formula for this sequence.

   The recursive formula is $a_{n+1} = -1.1(a_n)$ *with* $a_0 = 4$.

   d. Find an explicit formula for this sequence.

   The explicit formula is $a_n = 4(-1.1)^n$, *for* $n \geq 0$.

Scaffolding:

Students may need the hint that in the Opening Exercise, they wrote the terms of a geometric sequence so they can begin with the first three terms of such a sequence and use it to find r.

Exercises 5–6 (4 minutes)

This final set of exercises in the lesson attends to **F-BF.A.2**, and asks students to translate between explicit and recursive formulas for geometric sequences. Students should continue to work in pairs on these exercises.

5. The recursive formula for a geometric sequence is $a_{n+1} = 3.92(a_n)$ with $a_0 = 4.05$. Find an explicit formula for this sequence.

   The common ratio is 3.92, *and the initial value is* 4.05, *so the explicit formula is*

   $$a_n = 4.05(3.92)^n \text{ for } n \geq 0.$$

6. **The explicit formula for a geometric sequence is $a_n = 147(2.1)^{3n}$. Find a recursive formula for this sequence.**

First, we rewrite the sequence as $a_n = 147(2.1^3)^n = 147(9.261)^n$. We then see that the common ratio is 9.261, and the initial value is 147, so the recursive formula is

$$a_{n+1} = (9.261)a_n \text{ with } a_0 = 147.$$

Closing (4 minutes)

Debrief students by asking the following questions and taking answers as a class:

- If we know that a situation can be described using a geometric sequence, how can we create the geometric sequence for that model? How is the geometric sequence related to an exponential function with base e?
  - *The terms of the geometric sequence are determined by letting $P_n = P(n)$ for an exponential function $P(n) = P_0e^{kn}$, where P_0 is the initial amount, n indicates the term of the sequence, and e^k is the growth rate of the function. Depending on the data given in the situation, we can use either the explicit formula or the recursive formula to find the common ratio $r = e^k$ of the geometric sequence and its initial term P_0.*
- Do we need to use an exponential function base e?
  - *No. We can choose any base that we want for an exponential function, but mathematicians often choose base e for exponential and logarithm functions.*

Although arithmetic sequences are not emphasized in this lesson, they do make an appearance in the Problem Set. For completeness, the lesson summary includes both kinds of sequences. Explicit and recursive formulas for each type of sequence are summarized in the box below, which can be reproduced and posted in the classroom:

Lesson Summary

ARITHMETIC SEQUENCE: A sequence is called *arithmetic* if there is a real number d such that each term in the sequence is the sum of the previous term and d.

- ***Explicit formula:*** **Term a_n of an arithmetic sequence with first term a_0 and common difference d is given by $a_n = a_0 + nd$, for $n \geq 0$.**
- ***Recursive formula:*** **Term a_{n+1} of an arithmetic sequence with first term a_0 and common difference d is given by $a_{n+1} = a_n + d$, for $n \geq 0$.**

GEOMETRIC SEQUENCE: A sequence is called *geometric* if there is a real number r such that each term in the sequence is a product of the previous term and r.

- ***Explicit formula:*** **Term a_n of a geometric sequence with first term a_0 and common ratio r is given by $a_n = a_0r^n$, for $n \geq 0$.**
- ***Recursive formula:*** **Term a_{n+1} of a geometric sequence with first term a_0 and common ratio r is given by $a_{n+1} = a_nr$.**

Exit Ticket (5 minutes)

Name ___________________________________ Date ____________________

Lesson 25: Geometric Sequences and Exponential Growth and Decay

Exit Ticket

1. Every year, Mikhail receives a 3% raise in his annual salary. His starting annual salary was $\$40,000$.
   a. Does a geometric or arithmetic sequence best model Mikhail's salary in year n? Explain how you know.

   b. Find a recursive formula for a sequence, S_n, which represents Mikhail's salary in year n.

2. Carmela's annual salary in year n can be modeled by the recursive sequence $C_{n+1} = 1.05\, C_n$, where $C_0 = \$75,000$.
   a. What does the number 1.05 represent in the context of this problem?

   b. What does the number $\$75,000$ represent in the context of this problem?

   c. Find an explicit formula for a sequence that represents Carmela's salary.

Exit Ticket Sample Solutions

1. Every year, Mikhail receives a 3% raise in his annual salary. His starting annual salary was $\$40,000$.
   a. Does a geometric or arithmetic sequence best model Mikhail's salary in year n? Explain how you know.

      Because Mikhail's salary increases by a multiple of itself each year, a geometric sequence is an appropriate model.

   b. Find a recursive formula for a sequence, S_n, which represents Mikhail's salary in year n.

      Mikhail's annual salary can be represented by the sequence $S_{n+1} = 1.03\, S_n$ *with* $S_0 = \$40,000$.

2. Carmela's annual salary in year n can be modeled by the recursive sequence $C_{n+1} = 1.05\, C_n$, where $C_0 = \$75,000$.
   a. What does the number 1.05 represent in the context of this problem?

      The number 1.05 *represents the growth rate of her salary with time; it indicates that she is receiving a* 5% *raise each year.*

   b. What does the number $\$75,000$ represent in the context of this problem?

      Carmela's starting annual salary was $\$75,000$, *before she earned any raises.*

   c. Find an explicit formula for a sequence that represents Carmela's salary.

      Carmela's salary can be represented by the sequence $C_n = \$75,000\,(1.05)^n$.

Problem Set Sample Solutions

1. Convert the following recursive formulas for sequences to explicit formulas.
   a. $a_{n+1} = 4.2 + a_n$ with $a_0 = 12$

      $a_n = 12 + 4.2n$ *for* $n \geq 0$

   b. $a_{n+1} = 4.2a_n$ with $a_0 = 12$

      $a_n = 12(4.2)^n$ *for* $n \geq 0$

   c. $a_{n+1} = \sqrt{5}\, a_n$ with $a_0 = 2$

      $a_n = 2\left(\sqrt{5}\right)^n$ *for* $n \geq 0$

   d. $a_{n+1} = \sqrt{5} + a_n$ with $a_0 = 2$

      $a_n = 2 + n\sqrt{5}$ *for* $n \geq 0$

   e. $a_{n+1} = \pi\, a_n$ with $a_0 = \pi$

      $a_n = \pi(\pi)^n = \pi^{n+1}$ *for* $n \geq 0$

2. Convert the following explicit formulas for sequences to recursive formulas.

a. $a_n = \frac{1}{5}(3^n)$ for $n \geq 0$

$a_{n+1} = 3a_n$ with $a_0 = \frac{1}{5}$

b. $a_n = 16 - 2n$ for $n \geq 0$

$a_{n+1} = a_n - 2$ with $a_0 = 16$

c. $a_n = 16\left(\frac{1}{2}\right)^n$ for $n \geq 0$

$a_{n+1} = \frac{1}{2}a_n$ with $a_0 = 16$

d. $a_n = 71 - \frac{6}{7}n$ for $n \geq 0$

$a_{n+1} = a_n - \frac{6}{7}$ with $a_0 = 71$

e. $a_n = 190(1.03)^n$ for $n \geq 0$

$a_{n+1} = 1.03\, a_n$ with $a_0 = 190$

3. If a geometric sequence has $a_1 = 256$ and $a_8 = 512$, find the exact value of the common ratio r.

The recursive formula is $a_{n+1} = a_n \cdot r$, so we have

$$\begin{aligned} a_8 &= a_7(r) \\ &= a_6(r^2) \\ &= a_5(r^3) \\ &\vdots \\ &= a_1(r^7) \\ 512 &= 256(r^7) \\ 2 &= r^7 \\ r &= \sqrt[7]{2} \end{aligned}$$

4. If a geometric sequence has $a_2 = 495$ and $a_6 = 311$, approximate the value of the common ratio r to four decimal places.

The recursive formula is $a_{n+1} = a_n \cdot r$, so we have

$$\begin{aligned} a_6 &= a_5(r) \\ &= a_4(r^2) \\ &= a_3(r^3) \\ &= a_2(r^4) \\ 311 &= 495(r^4) \\ r^4 &= \frac{311}{495} \\ r &= \sqrt[4]{\frac{311}{495}} \approx 0.8903. \end{aligned}$$

5. Find the difference between the terms a_{10} of an arithmetic sequence and a geometric sequence, both of which begin at term a_0 and have $a_2 = 4$ and $a_4 = 12$.

Arithmetic: The explicit formula has the form $a_n = a_0 + nd$, so $a_2 = a_0 + 2d$ and $a_4 = a_0 + 4d$. Then $a_4 - a_2 = 12 - 4 = 8$ and $a_4 - a_2 = (a_0 + 4d) - (a_0 + 2d)$, so that $8 = 2d$ and $d = 4$. Since $d = 4$, we know that $a_0 = a_2 - 2d = 4 - 8 = -4$. So, the explicit formula for this arithmetic sequence is $a_n = -4 + 4n$. We then know that $a_{10} = -4 + 40 = 36$.

Geometric: The explicit formula has the form $a_n = a_0(r^n)$, so $a_2 = a_0(r^2)$ and $a_4 = a_0(r^4)$, so $\frac{a_4}{a_2} r^2$ and $\frac{a_4}{a_2} = \frac{12}{4} = 3$. Thus, $r^2 = 3$, so $r = \pm\sqrt{3}$. Since $r^2 = 3$, we have $a_2 = 4 = a_0(r^2)$, so that $a_0 = \frac{4}{3}$. Then the explicit formula for this geometric sequence is $a_n = \frac{4}{3}\left(\pm\sqrt{3}\right)^n$. We then know that $a_{10} = \frac{4}{3}\left(\pm\sqrt{3}\right)^{10} = \frac{4}{3}(3^5) = 4(3^4) = 324$.

Thus, the difference between the terms a_{10} of these two sequences is $324 - 36 = 288$.

6. Given the geometric sequence defined by the following values of a_0 and r, find the value of n so that a_n has the specified value.

a. $a_0 = 64, r = \frac{1}{2}, a_n = 2$

The explicit formula for this geometric sequence is $a_n = 64\left(\frac{1}{2}\right)^n$ and $a_n = 2$.

$$\begin{aligned} 2 &= 64\left(\frac{1}{2}\right)^n \\ \frac{1}{32} &= \left(\frac{1}{2}\right)^n \\ \left(\frac{1}{2}\right)^5 &= \left(\frac{1}{2}\right)^n \\ n &= 5 \end{aligned}$$

Thus, $a_5 = 2$.

b. $a_0 = 13, r = 3, a_n = 85293$

The explicit formula for this geometric sequence is $a_n = 13(3)^n$, and we have $a_n = 85293$.

$$\begin{aligned} 13(3)^n &= 85293 \\ 3^n &= 6561 \\ 3^n &= 3^8 \\ n &= 8 \end{aligned}$$

Thus, $a_8 = 85293$.

c. $a_0 = 6.7, r = 1.9, a_n = 7804.8$

The explicit formula for this geometric sequence is $a_n = 6.7(1.9)^n$, and we have $a_n = 7804.8$.

$$\begin{aligned} 6.7(1.9)^n &= 7804.8 \\ (1.9)^n &= 1164.9 \\ n\log(1.9) &= \log(1164.9) \\ n &= \frac{\log(1164.9)}{\log(1.9)} = 11 \end{aligned}$$

Thus, $a_{11} = 7804.8$.

d. $a_0 = 10958, r = 0.7, a_n = 25.5$

The explicit formula for this geometric sequence is $a_n = 10958(0.7)^n$, and we have $a_n = 25.5$.

$$10958(0.7)^n = 25.5$$
$$\log(10958) + n\log(0.7) = \log(25.5)$$
$$n = \frac{\log(25.5) - \log(10958)}{\log(0.7)}$$
$$n = 17$$

Thus, $a_{17} = 25.5$.

7. Jenny planted a sunflower seedling that started out 5 cm tall, and she finds that the average daily growth is 3.5 cm.

   a. Find a recursive formula for the height of the sunflower plant on day n.

   $$h_{n+1} = 3.5 + h_n \text{ with } h_0 = 5$$

   b. Find an explicit formula for the height of the sunflower plant on day $n \geq 0$.

   $$h_n = 5 + 3.5n$$

MP.2

8. Kevin modeled the height of his son (in inches) at age n years for $n = 2, 3, \ldots, 8$ by the sequence $h_n = 34 + 3.2(n - 2)$. Interpret the meaning of the constants 34 and 3.2 in his model.

   At age 2, Kevin's son was 34 inches tall, and between the ages of 2 and 8 he grew at a rate of 3.2 inches per year.

MP.2

9. Astrid sells art prints through an online retailer. She charges a flat rate per order for an order processing fee, sales tax, and the same price for each print. The formula for the cost of buying n prints is given by $P_n = 4.5 + 12.6n$.

   a. Interpret the number 4.5 in the context of this problem.

   The number 4.5 represents a $\$4.50$ order processing fee.

   b. Interpret the number 12.6 in the context of this problem.

   The number 12.6 represents the cost of each print, including the sales tax.

   c. Find a recursive formula for the cost of buying n prints.

   $P_n = 12.6 + P_{n-1}$ with $P_1 = 17.10$

   (Notice that it makes no sense to start the sequence with $n = 0$, since that would mean you need to pay the processing fee when you do not place an order.)

10. A bouncy ball rebounds to 90% of the height of the preceding bounce. Craig drops a bouncy ball from a height of 20 feet.

   a. Write out the sequence of the heights h_1, h_2, h_3, and h_4 of the first four bounces, counting the initial height as $h_0 = 20$.

   $$h_1 = 18$$
   $$h_2 = 16.2$$
   $$h_3 = 14.58$$
   $$h_4 = 13.122$$

b. Write a recursive formula for the rebound height of a bouncy ball dropped from an initial height of 20 feet.

$$h_{n+1} = 0.9\, h_n \text{ with } h_0 = 20$$

c. Write an explicit formula for the rebound height of a bouncy ball dropped from an initial height of 20 feet.

$$h_n = 20(0.9)^n \text{ for } n \geq 0$$

d. How many bounces does it take until the rebound height is under 6 feet?

$$\begin{aligned} 20(0.9)^n &< 6 \\ n\log(0.9) &< \log(6) - \log(20) \\ n &> \frac{\log(6) - \log(20)}{\log(0.9)} \\ n &> 11.42 \end{aligned}$$

So, it takes 12 bounces for the bouncy ball to rebound under 6 feet.

e. Extension: Find a formula for the minimum number of bounces needed for the rebound height to be under y, feet, for a real number $0 < y < 20$.

$$\begin{aligned} 20(0.9)^n &< y \\ n\log(0.9) &< \log(y) - \log(20) \\ n &> \frac{\log(y) - \log(20)}{\log(0.9)} \end{aligned}$$

Rounding this up to the next integer with the ceiling function, it takes $\left\lceil \frac{\log(y) - \log(20)}{\log(0.9)} \right\rceil$ bounces for the bouncy ball to rebound under y feet.

11. Show that when a quantity $a_0 = A$ is increased by $x\%$, its new value is $a_1 = A\left(1 + \frac{x}{100}\right)$. If this quantity is again increased by $x\%$, what is its new value a_2? If the operation is performed n times in succession, what is the final value of the quantity a_n?

We know that $x\%$ of a number A is represented by $\frac{x}{100}A$. Thus, when $a_0 = A$ is increased by $x\%$, the new quantity is

$$\begin{aligned} a_1 &= A + \frac{x}{100}A \\ &= A\left(1 + \frac{x}{100}\right). \end{aligned}$$

If we increase it again by $x\%$, we have

$$\begin{aligned} a_2 &= a_1 + \frac{x}{100}a_1 \\ &= \left(1 + \frac{x}{100}\right)a_1 \\ &= \left(1 + \frac{x}{100}\right)\left(1 + \frac{x}{100}\right)a_0 \\ &= \left(1 + \frac{x}{100}\right)^2 a_0. \end{aligned}$$

If we repeat this operation n times, we find that

$$a_n = \left(1 + \frac{x}{100}\right)^n a_0.$$

12. When Eli and Daisy arrive at their cabin in the woods in the middle of winter, the interior temperature is $40°\text{F}$.

a. Eli wants to turn up the thermostat by $2°\text{F}$ every 15 minutes. Find an explicit formula for the sequence that represents the thermostat settings using Eli's plan.

Let n represent the number of 15-minute increments. The function $E(n) = 40 + 2n$ models the thermostat settings using Eli's plan.

b. Daisy wants to turn up the thermostat by 4% every 15 minutes. Find an explicit formula for the sequence that represents the thermostat settings using Daisy's plan.

Let n represent the number of 15-minute increments. The function $D(n) = 40(1.04)^n$ models the thermostat settings using Daisy's plan

c. Which plan gets the thermostat to $60°\text{F}$ most quickly?

Making a table of values, we see that Eli's plan sets the thermostat to $60°\text{F}$ first.

n	Elapsed Time	$E(n)$	$D(n)$
0	0 minutes	40	40.00
1	15 minutes	42	41.60
2	30 minutes	44	43.26
3	45 minutes	46	45.00
4	1 hour	48	46.79
5	1 hour 15 minutes	50	48.67
6	1 hour 30 minutes	52	50.61
7	1 hour 45 minutes	54	52.64
8	2 hours	56	54.74
9	2 hours 15 minutes	58	56.93
10	2 hours 30 minutes	60	59.21

d. Which plan gets the thermostat to $72°\text{F}$ most quickly?

Continuing the table of values from part (c), we see that Daisy's plan sets the thermostat to $72°\text{F}$ first.

n	Elapsed Time	$E(n)$	$D(n)$
11	2 hours 45 minutes	62	61.58
12	3 hours	64	64.04
13	3 hours 15 minutes	66	66.60
14	3 hours 30 minutes	68	69.27
15	3 hours 45 minutes	70	72.04

13. In nuclear fission, one neutron splits an atom causing the release of two other neutrons, each of which splits an atom and produces the release of two more neutrons, and so on.

a. Write the first few terms of the sequence showing the numbers of atoms being split at each stage after a single atom splits. Use $a_0 = 1$.

$a_0 = 1$, $a_1 = 2$, $a_2 = 4$, $a_3 = 8$

b. **Find the explicit formula that represents your sequence in part (a).**

$a_n = 2^n$

c. **If the interval from one stage to the next is one-millionth of a second, write an expression for the number of atoms being split at the end of one second.**

At the end of one second $n = 1000000$, so $2^{1000000}$ atoms are being split.

d. **If the number from part (c) were written out, how many digits would it have?**

The number of digits in a number x is given by rounding up $\log(x)$ to the next largest integer; that is, by the ceiling of $\log(x)$, $\lceil \log(x) \rceil$. Thus, there are $\lceil \log(2^{1000000}) \rceil$ digits.

Since $\log(2^{1000000}) = 1000000 \log(2) \approx 301030$, there are 301030 digits in the number $2^{1000000}$.

Lesson 26: Percent Rate of Change

Student Outcomes

- Students develop a *general growth/decay rate formula* in the context of compound interest.
- Students compute *future values* of investments with continually compounding interest rates.

Lesson Notes

In this lesson, we develop a general growth/decay rate formula by investigating the compound interest formula. In Algebra I, the compound interest formula is described via sequences or functions whose domain is a subset of the integers. We start from this point (**F-IFA.3**) and extend the function to a domain of all real numbers. The function for compound interest is developed first using a recursive process to generate a geometric sequence, which is then rewritten in its explicit form (**F-BF.A.1a**, **F-BF.A.2**). Many of the situations and problems presented here were first encountered in Module 3 of Algebra I, but now students are able to use logarithms to find solutions, using technology appropriately to evaluate the logarithms (MP.5). Students also work on converting between different growth rates and time units (**A-SSE.B.3c**). Students continue to create equations in one variable from the exponential models to solve problems (**A-CED.A.3**).

Note: In this lesson, the letter r stands for *the percent rate of change,* which is different from how the letter r was used in Lesson 25 where it denoted the common ratio. These two concepts are slightly different (in this lesson, $1+r$ is *the common ratio*), and this difference might cause confusion for your students. We use the letter r to refer to both, due to historical reasons and because r is the notation most commonly used by adults in both situations. Help students understand how the context dictates whether r stands for the common ratio or the percent rate of change.

Classwork

Example 1 (8 minutes)

MP.4

Present the following situation, which was first seen in Algebra I, to the students. Some trigger questions are presented to help advance student understanding. A general exponential model of the form $F = P(1+r)^t$ is presented. This formula is appropriate in most applications that can be modeled using exponential functions and was introduced in Module 3 Lesson 4 of Algebra I. It has been a while since the students have seen this formula, so it is developed slowly through this example first using a recursive process before giving the explicit form (**F-BF.A.1a**, **F-BF.A.2**).

- A youth group has a yard sale to raise money for charity. The group earns $800 but decides to put the money in the bank for a while. Their local bank pays an interest rate of 3% per year, and the group decides to put all of the interest they earn back into the account to earn even more interest.

Scaffolding:

- Either present the following information explicitly or encourage students to write out the first few terms without evaluating to see the structure. Once they see that $P_2 = 800 \cdot 1.03^2$ and that $P_3 = 800 \cdot 1.03^3$, they should be able to see that $P_m = 800 \cdot 1.03^m$.
- Have advanced learners work on their own to develop the values for years 0–3 and year m.

- We refer to the time at which the money was deposited into the bank as year 0. At the end of each year, how can we calculate how much money is in the bank if we know the previous year's balance?
    - *Each year, multiply the previous year's balance by* 1.03. *For example, since* 3% *can be written* 0.03, *the amount at the end of the first year is* $800 + 800(0.03) = 800(1 + 0.03) = 800(1.03)$.
- How much money is in the bank at the following times?

Year	Balance in Terms of Last Year's Balance (in dollars)	Balance in Terms of the Year, m (in dollars)
0	800	800
1	$824 = 800(1.03)$	$824 = 800(1.03)$
2	$848.72 = 824(1.03)$	$848.72 = 800(1.03)(1.03)$
3	$874.18 \approx 848.72(1.03)$	$874.18 \approx 800(1.03)(1.03)(1.03)$
m	$b_{m-1} \cdot (1.03)$	$800(1.03)^m$

- If instead of evaluating, we write these balances out as mathematical expressions, what pattern do you notice?
    - *For instance, the second year would be* $800(1.03)(1.03) = 800(1.03)^2$. *From there we can see that the balance in the* m^{th} *year would be* $800(1.03)^m$ *dollars.*
- What kind of sequence do these numbers form? Explain how you know.
    - *They form a geometric sequence because each year's balance is the previous year's balance multiplied by* 1.03.
- Write a recursive formula for the balance in the $(m + 1)^{st}$ year, denoted by b_{m+1}, in terms of the balance of the m^{th} year, denoted by b_m.
    - $b_{m+1} = (1.03)b_m$

MP.8

- What is the explicit formula that gives the amount of money, F (i.e., future value), in the bank account after m years?
    - *The group started with* $\$800$ *and this increases* 3% *each year. After the first year, the group has* $\$800 \cdot 1.03$ *in the account, and after* m *years, they should have* $\$800 \cdot 1.03^m$. *Thus, the formula for the balance (in dollars) could be represented by* $F = 800(1.03)^m$.
- Let us examine the base of the exponent in the above problem a little more closely, and write it as $1.03 = 1 + 0.03$. Rewrite the formula for the amount they have in the bank after m years using $1 + 0.03$ instead of 1.03.
    - $F = 800(1 + 0.03)^m$

MP.2 & MP.7

- What does the 800 represent? What does the 1 represent? What does the 0.03 represent?
    - *The number* 800 *represents the* $\$800$ *starting amount. The* 1 *represents* 100% *of the previous balance that is maintained every year. The* 0.03 *represents the* 3% *of the previous balance that is added each year due to interest.*
- Let P be the present or starting value of 800, let r represent the interest rate of 3%, and t be the number of years, where we allow t to be any real number. Write a formula for the future value F in terms of P, r, and t.
    - $F = P(1 + r)^t$

Scaffolding:

For struggling classes, Example 2 may be omitted in lieu of developing fluency with the formula through practice exercises. The ending discussion questions in Example 2 should be discussed throughout the practice. Some practice exercises are presented below:

- Evaluate $300(1 + 0.12)^3$
- Find the future value of an investment of $\$1000$ growing at a rate of 3% per year, compounded monthly.
- Find the growth rate and how many days it would take to grow $\$2$ into $\$2$ million if the amount doubles every day.
- Find the growth rate per year necessary to grow $\$450$ into $\$900$ after ten years.

Discussion (5 minutes)

MP.8 Make three important points during this discussion: (1) that the formula $F = P(1+r)^t$ can be used in situations far more general than just finance, (2) that r is the percent rate of change expressed as a unit rate, and (3) that the domain of the function given by the formula now includes all real numbers. Note: r is expressed as a unit rate for a unit of time; in finance, that unit of time is typically a year. In the next examples, we investigate compounding interest problem with different compounding periods.

- This formula, $F = P(1+r)^t$, can be used in far more situations than just finance, including radioactive decay and population growth. Given a *percent rate of change* (i.e., the percentage increase or decrease of an amount over a unit of time); the number r is the unit rate of that percentage rate of change. For example, if a bank account grows by 2.5% a year, the unit rate is $\frac{2.5}{100}$, which means $r = 0.025$. What is the unit rate if the percent rate of change is a 12% increase? A 100% increase? A 0.2% increase? A 5% decrease?
  - $r = 0.12, r = 1, r = 0.002, r = -0.05$
- Given the value P and the percent rate of change represented by the unit rate r, we can think of the formula as function of time t, that is, $F(t) = P(1+r)^t$. In Algebra I, t represented a positive integer, but now we can think of the function as having a domain of all real numbers. Why can we think of the domain of this function as being all real numbers?
  - *Earlier in this module, we learned how to define the value of an exponent when the power is a rational number, and we showed how to use that definition to evaluate exponents when the power is an irrational number. Thus, we can assume that the domain of the function F can be any real number.*

Students can now use the fact that the function F has a domain of all real numbers and their knowledge of logarithms to solve equations involving the function $F = P(1+r)^t$.

- In Example 1, the group's goal is to save $\$1{,}000$ with the money they made from the yard sale. How many years does it take for the amount in the bank to be at least $\$1{,}000$?
  - *Substitute 1000 for F and solve for t using logarithms.*

$$
\begin{aligned}
1000 &= 800 \cdot 1.03^t \\
\frac{1000}{800} &= 1.03^t \\
1.25 &= 1.03^t \\
\ln(1.25) &= t \cdot \ln(1.03) \\
t &= \frac{\ln(1.25)}{\ln(1.03)} \approx 7.5
\end{aligned}
$$

Since they earn interest every year, it takes them 8 years to save more than $\$1{,}000$ with this money.

- What does the approximation 7.5 mean?
  - *The amount in the bank reaches $\$1{,}000$ after roughly 7 years, 6 months.*

The percent rate of change can also be negative, which usually corresponds to a negative unit rate r, with $-1 < r < 0$.

- Can you give an example of percent rate of change that we have studied before that has a negative rate of change?
  - *Radioactive decay and shrinking populations are examples that have negative rates of change. An interesting example is the bean-counting experiment where we started with lots of beans and removed beans after each trial.*

At this point in the lesson, you may want to work out one problem from the non-financial Problem Set as an example, or have students work one as an exercise.

Example 2 (8 minutes)

In the future value function $F(t) = P(1+r)^t$, the number r is the unit rate of the percent rate of change, and t is time. Frequently, the time units for the percent rate of change and the time unit for t do not agree and some calculation needs to be done so that they do. For instance, if the growth rate is an amount per hour and the time period is a number of days, these quantities need to be converted to use the same units.

In this example, students learn about compounding periods and percent rates of change that are based upon different units (**A-SSE.B.3c**). Students explore these concepts through some exercises immediately following the example.

- In finance, the interest rates are almost always tied to a specific time period and only accumulate once this has elapsed (called *compounding*). In this context, we refer to the time periods as compounding periods.
- Interest rates for accounts are frequently given in terms of what is called the *nominal annual percentage rate of change* or *nominal APR.* Specifically, the nominal APR is the percent rate of change per compounding period times the number of compounding periods per year. For example, if the percent rate of change is 0.5% per month, then the nominal APR is 6% since there are 12 months in a year. The nominal APR is an easy way of discussing a monthly or daily percent rate of change in terms of a yearly rate, but as is shown in the examples below, it does not necessarily reflect actual or effective percent rate of change per year.

Note about language: In this lesson and later lessons, we often use the phrase "an interest rate of 3% per year compounded monthly" to mean, a nominal APR of 3% compounded monthly. Both phrases refer to nominal APR.

- Frequently in financial problems and real-life situations, the nominal APR is given and the percent rate of change per compounding period is deducted from it. The following example shows how to deduce the future value function in this context.
- If the nominal APR is 6% and is compounded monthly, then monthly percent rate of change is $\frac{6\%}{12}$ or 0.5% per month. That means that, if a starting value of $\$800$ was deposited in a bank, after one month there would be $\$800\left(1+\frac{0.06}{12}\right)^1$ in the account, after two months there would be $\$800\left(1+\frac{0.06}{12}\right)^2$, and after 12 months in the bank there would be $\$800\left(1+\frac{0.06}{12}\right)^{12}$ in the account. In fact, since it is compounding 12 times a year, it would compound 12 times over 1 year, 24 times over 2 years, 36 times over 3 years, and $12t$ times over t years. Hence, a function that describes the amount in the account after t years is
$$F(t) = 800\left(1+\frac{0.06}{12}\right)^{12t}.$$
- Describe a function F that describes the amount that would be in an account after t years if P was deposited in an account with a nominal APR given by the unit rate r that is compounded n times a year.
  - $F(t) = P\left(1+\frac{r}{n}\right)^{nt}$
- In this form, $\frac{r}{n}$ is the unit rate per compounding period, and nt is the total number of compounding periods over time t.

- However, time t can be any real number; it does not have to be integer valued. For example, if a savings account earns 1% interest per year, compounded monthly, then we would say that the account compounds at a rate of $\frac{0.01}{12}$ per month. How much money would be in the account after $2\frac{1}{2}$ years with an initial deposit of $\$200$?
  - Because $F(2.5) = 200\left(1 + \frac{0.01}{12}\right)^{12(2.5)} \approx 205.06$, there is $\$205.06$ in the account after $2\frac{1}{2}$ years.

Exercise (8 minutes)

Have students work through the following problem to explore the consequences of having different compounding periods. After students finish, debrief them to ensure understanding.

Exercise

Answer the following questions.

The youth group from Example 1 is given the option of investing their money at 2.976% interest per year, compounded monthly instead of depositing it in the original account earning 3.0% compounded yearly.

a. With an initial deposit of $\$800$, how much would be in each account after two years?

The account from the beginning of the lesson would contain $\$848.72$, and the new account would contain $\$800\left(1 + \frac{0.029\,76}{12}\right)^{12 \cdot 2} \approx \$849.00$.

b. Compare the total amount from part (a) to how much they would have made using the interest rate of 3% compounded yearly for two years. Which account would you recommend the youth group invest its money in? Why?

The 3% compounded yearly yields $\$848.72$, while the 2.976% compounded monthly yields $\$849.00$ after two years. The yield from the two accounts are very close to each other, but the account compounded monthly earns $\$0.28$ more over the two-year period.

In part (b), the amount from both options is virtually the same: $\$848.72$ versus $\$849.00$. But point out that there is something strange about the numbers; even though the interest rate of 2.975% is less than the interest rate of 3%, the total amount is more. This is due to compounding every month versus every year.

To illustrate this, rewrite the expression $\left(1 + \frac{0.02976}{12}\right)^{12t}$ as $((1 + 0.00248)^{12})^t$, and take the 12^{th} power of 1.00248; the result is $((1.00248)^{12})^t$, which is approximately $(1.030169)^t$. This shows that when the nominal APR of 2.975% compounded monthly is written as a percent rate of change compounded yearly (the same compounding period as in Example 1), then the interest rate is approximately 3.0169%, which is more than 3%. In other words, interest rates can only be accurately compared when they have both been converted to the same compounding period.

Example 3 (10 minutes)

MP.7 & MP.8

In this example, students develop the $F(t) = Pe^{rt}$ model using a numerical approach. Have students perform the beginning calculations on their own as much as possible before transitioning into continuous compounding.

- Thus far, we have seen that the number of times a quantity compounds per year does have an effect on the future value. For instance, if someone tells you that one savings account gives you a nominal APR of 3% per year compounded yearly, and another gives you a nominal APR of 3% per year compounded monthly, which account generates more interest by the end of the year?
    - *The account that compounds monthly generates more interest.*
- How much more interest is generated? Does it give twelve times as much? How can we find out how much money we have at the end of the year if we deposit $\$100$?
    - *Calculate using the formula.*

$$F = 100(1 + 0.03)^1 = 103$$

$$F = 100\left(1 + \frac{0.03}{12}\right)^{12} \approx 103.04$$

The account compounding monthly earned 4 *cents more.*

- So, even though the second account compounded twelve times as much as the other, it only earned a fraction of a dollar more. Do we think that there is a limit to how much an account can earn through increasing the number of times compounding?
    - *Answers may vary. At this point, students have experience with logarithms that grow incredibly slowly but have no upper bound. This could be a situation with an upper bound or not.*
- Let's explore this idea of a limit using our calculators to do the work. Holding the principal, percent rate of change, and number of time units constant, is there a limit to how large the future value can become solely through increasing the number of compounding periods?
- We can simplify this question by setting $P = 1$, $r = 1$, and $t = 1$. What does the expression become for n compounding periods?
    - $F = 1\left(1 + \frac{1}{n}\right)^n$
- Then the question becomes: As $n \to \infty$, does F converge to a specific value, or does it also increase without bound?

- Let's rewrite this expression as something our calculators and computers can evaluate: $y = \left(1 + \frac{1}{x}\right)^x$. For now, go into the table feature of your graphing utility, and let x start at 1 and go up by 1. Can we populate the following table as a class?

x	$y = \left(1 + \frac{1}{x}\right)^x$
1	2
2	2.25
3	2.3704
4	2.4414
5	2.4883
6	2.5216
7	2.5465

- This demonstrates that although the value of the function is continuing to increase as x increases, it is increasing at a decreasing rate. Still, does this function ever start decreasing? Let's set our table to start at $10{,}000$ and increase by $10{,}000$.

x	$y = \left(1 + \frac{1}{x}\right)^x$
10,000	2.7181459
20,000	2.7182139
30,000	2.7182365
40,000	2.7182478
50,000	2.7182546
60,000	2.7182592
70,000	2.7182624

- It turns out that we are rapidly approaching the limit of what our calculators can reliably compute. Much past this point, the rounding that the calculator does to perform its calculations starts to insert horrible errors into the table. However, it is true that the value of the function increases forever but at a slower and slower rate. In fact, as $x \to \infty$, y does approach a specific value. You may have started to recognize that value from earlier in the module: Euler's number, e.
- Unfortunately, a proof that the expression $\left(1 + \frac{1}{x}\right)^x \to e$ as $x \to \infty$ requires mathematics more advanced than what we currently have available. Using calculus and other advanced mathematics, mathematicians have not only been able to show that as $x \to \infty$, $\left(1 + \frac{1}{x}\right)^x \to e$, but also that as $x \to \infty$, $\left(1 + \frac{r}{x}\right)^x \to e^r$!

Note: As an extension, you can hint at why the expression involving r converges to e^r: Rewrite the expression above as $\left(1 + \frac{r}{x}\right)^{\left(\frac{x}{r}\right)\cdot r}$ or $\left(\left(1 + \frac{r}{x}\right)^{\frac{x}{r}}\right)^r$, and substitute $u = \frac{x}{r}$. Then the expression in terms of u becomes $\left(\left(1 + \frac{1}{u}\right)^u\right)^r$. If $x \to \infty$, then u does also, but as $u \to \infty$, the expression $\left(1 + \frac{1}{u}\right)^u \to e$, so $\left(\left(1 + \frac{1}{u}\right)^u\right)^r \to e^r$.

- Revisiting our earlier application, what does x represent in our original formula, and what could it mean that $x \to \infty$?
  - The number x *represents the number of compounding periods in a year. If* x *was large (e.g.,* 365*), it would imply that interest was compounding once a day. If it were very large, say* $365{,}000$*, it would imply that interest was compounding* $1{,}000$ *times a day. As* $x \to \infty$*, the interest would be compounding continuously.*
- Thus, we have a new formula for when interest is compounding continuously: $F = Pe^{rt}$. This is just another representation of the exponential function that we have been using throughout the module.
- The formula is often called the *pert* formula.

Closing (2 minutes)

Have students summarize the key points of the lesson in writing. A sample is included which you may want to share with the class, or you can guide students to these conclusions on their own.

Lesson Summary

- **For application problems involving a percent rate of change represented by the unit rate r, we can write $F(t) = P(1 + r)^t$, where F is the future value (or ending amount), P is the present amount, and t is the number of time units. When the percent rate of change is negative, r is negative, and the quantity decreases with time.**
- **The nominal APR is the percent rate of change per compounding period times the number of compounding periods per year. If the nominal APR is given by the unit rate r and is compounded n times a year, then function $F(t) = P\left(1 + \frac{r}{n}\right)^{nt}$ describes the future value at time t of an account given that is given nominal APR and an initial value of P.**
- **For continuous compounding, we can write $F = Pe^{rt}$, where e is Euler's number and r is the unit rate associated to the percent rate of change.**

Exit Ticket (4 minutes)

Name ____ Date ____________________

Lesson 26: Percent Rate of Change

Exit Ticket

April would like to invest $\$200$ in the bank for one year. Three banks all have a nominal APR of 1.5%, but compound the interest differently.

a. Bank A computes interest just once at the end of the year. What would April's balance be after one year with this bank?

b. Bank B compounds interest at the end of each six-month period. What would April's balance be after one year with this bank?

c. Bank C compounds interest continuously. What would April's balance be after one year with this bank?

d. Each bank decides to double the nominal APR it offers for one year. That is, they offer a nominal APR of 3%. Each bank advertises, "DOUBLE THE AMOUNT YOU EARN!" For which of the three banks, if any, is this advertised claim correct?

Exit Ticket Sample Solutions

April would like to invest $\$200$ in the bank for one year. Three banks all have a nominal APR of 1.5%, but compound the interest differently.

a. **Bank A computes interest just once at the end of the year. What would April's balance be after one year with this bank?**

$$I = 200 \cdot 0.015 = 3$$

April would have $\$203$ at the end of the year.

b. **Bank B compounds interest at the end of each six-month period. What would April's balance be after one year with this bank?**

$$F = 200\left(1 + \frac{0.015}{2}\right)^2 \approx 203.01$$

April would have $\$203.01$ at the end of the year.

c. **Bank C compounds interest continuously. What would April's balance be after one year with this bank?**

$$F = 200e^{0.015} \approx 203.02$$

April would have $\$203.02$ at the end of the year.

d. **Each bank decides to double the nominal APR it offers for one year. That is, they offer a nominal APR of 3%. Each bank advertises, "DOUBLE THE AMOUNT YOU EARN!" For which of the three banks, if any, is this advertised claim correct?**

Bank A:

$$I = 200 \cdot 0.03 = 6$$

Bank B:

$$F = 200\left(1 + \frac{0.03}{2}\right)^2 \approx 206.045$$

Bank C:

$$F = 200e^{0.015} \approx 206.09$$

All three banks earn at least twice as much with a double interest rate. Bank A earns exactly twice as much, Bank B earns 2 cents more than twice as much, and Bank C earns 5 cents more than twice as much.

Problem Set Sample Solutions

Problems 1 and 2 provide students with more practice converting arithmetic and geometric sequences between explicit and recursive forms. Fluency with geometric sequences is required for the remainder of the lessons in this module.

1. Write each recursive sequence in explicit form. Identify each sequence as arithmetic, geometric, or neither.

a. $a_1 = 3\,,\ a_{n+1} = a_n + 5$

$a_n = 3 + 5(n-1)$*, arithmetic*

b. $a_1 = -1\,,\ a_{n+1} = -2a_n$

$a_n = -(-2)^{n-1}$*, geometric*

c. $a_1 = 30\,,\ a_{n+1} = a_n - 3$

$a_n = 30 - 3(n-1)$*, arithmetic*

d. $a_1 = \sqrt{2}\,,\ a_{n+1} = \frac{a_n}{\sqrt{2}}$

$a_n = \sqrt{2}\left(\frac{1}{\sqrt{2}}\right)^{n-1}$*, geometric*

e. $a_1 = 1\,,\ a_{n+1} = \cos(\pi a_n)$

$a_1 = 1,\ a_n = -1$ *for* $n > 1$*, neither.*

2. Write each sequence in recursive form. Assume the first term is when $n = 1$.

a. $a_n = \frac{3}{2}n + 3$

$a_1 = \frac{9}{2},\ a_{n+1} = a_n + \frac{3}{2}$

b. $a_n = 3\left(\frac{3}{2}\right)^n$

$a_1 = \frac{9}{2},\ a_{n+1} = \frac{3}{2} \cdot a_n$

c. $a_n = n^2$

$a_1 = 1,\ a_{n+1} = a_n + 2n + 1$

d. $a_n = \cos(2\pi n)$

$a_1 = 1,\ a_{n+1} = a_n$

3. Consider two bank accounts. Bank A gives simple interest on an initial investment in savings accounts at a rate of 3% per year. Bank B gives compound interest on savings accounts at a rate of 2.5% per year. Fill out the following table.

Number of Years, n	Bank A Balance, a_n (in dollars)	Bank B Balance, b_n (in dollars)
0	$1,000.00$	$1,000.00$
1	$1,030.00$	$1,025.00$
2	$1,060.00$	$1,050.63$
3	$1,090.00$	$1,076.89$
4	$1,120.00$	$1,103.81$
5	$1,150.00$	$1,131.41$

a. What type of sequence do the Bank A balances represent?

The balances in Bank A represent an arithmetic sequence with constant difference $\$30$.

b. Give both a recursive and an explicit formula for the Bank A balances.

Recursive: $a_0 = 1000$, $a_n = a_{n-1} + 30$

Explicit: $a_n = 1000 + 30n$ *or* $f_A(n) = 1000 + 30n$

c. What type of sequence do the Bank B balances represent?

The balances in Bank B represent a geometric sequence with common ratio 1.025.

d. Give both a recursive and an explicit formula for the Bank B balances.

Recursive: $b_1 = 1000$, $b_n = b_{n-1} \cdot 1.025$

Explicit: $b_n = 1000 \cdot 1.025^n$ *or* $f_B(n) = 1000 \cdot 1.025^n$

e. Which bank account balance is increasing faster in the first five years?

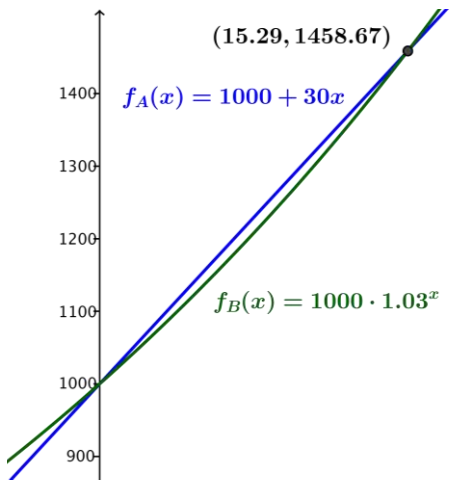
During the first five years, the balance in Bank A is increasing faster at a constant rate of $\$30$ *per year.*

f. If you were to recommend a bank account for a long-term investment, which would you recommend?

The balance in Bank B would eventually outpace the balance in Bank A since the balance in Bank B is increasing geometrically.

g. **At what point is the balance in Bank B larger than the balance in Bank A?**

Once the balance in Bank B overtakes the balance in Bank A, it will always be larger, so we just have to find when they are equal. This can only be done by graphing functions f_A and f_B and estimating the intersection point.



It appears that the balance in Bank B overtakes the balance in Bank A in the 16^{th} year and be larger from then on. Any investment made for 0 to 15 years would perform better in Bank A than Bank B.

4. **You decide to invest your money in a bank that uses continuous compounding at 5.5% interest per year. You have $\$500$.**

   a. **Ja'mie decides to invest $\$1,000$ in the same bank for one year. She predicts she will have double the amount in her account than you will have. Is this prediction correct? Explain.**

$$F = 1000 \cdot e^{0.055}$$
$$\approx 1056.54$$

$$F = 500 \cdot e^{0.055}$$
$$\approx 528.27$$

Her prediction is correct. Evaluating the formula with $1,000$, we can see that $1000e^{0.055} = 2 \cdot 500 \cdot e^{0.055}$.

   b. **Jonas decides to invest $\$500$ in the same bank as well, but for two years. He predicts that after two years he will have double the amount of cash that you will after one year. Is this prediction correct? Explain.**

Jonas will earn more than double the amount of interest since the value increasing is in the exponent but will not have more than double the amount of cash.

5. **Use the properties of exponents to identify the percent rate of change of the functions below, and classify them as modeling exponential growth or decay. (The first two problems are done for you.)**

   a. $f(t) = (1.02)^t$

The percent rate of change is 2% and models exponential growth.

b. $f(t) = (1.01)^{12t}$

Since $(1.01)^{12t} = ((1.01)^{12})^t \approx (1.1268)^t$, the percent rate of change is 12.68% and models exponential growth.

c. $f(t) = (0.97)^t$

Since $(0.97)^t = (1 - 0.03)^t$, the percent rate of change is -3% and models exponential decay.

d. $f(t) = 1000(1.2)^t$

The percent rate of change is 20% and models exponential growth.

e. $f(t) = \frac{(1.07)^t}{1000}$

The percent rate of change is 7% and models exponential growth.

f. $f(t) = 100 \cdot 3^t$

Since $3^t = (1+2)^t$, the percent rate of change is 200% and models exponential growth.

g. $f(t) = 1.05 \cdot \left(\frac{1}{2}\right)^t$

Since $\left(\frac{1}{2}\right)^t = (0.5)^t = (1 - 0.5)^t$, the percent rate of change is -50% and models exponential decay.

h. $f(t) = 80 \cdot \left(\frac{49}{64}\right)^{\frac{1}{2}t}$

Since $\left(\frac{49}{64}\right)^{\frac{1}{2}t} = \left(\left(\frac{49}{64}\right)^{\frac{1}{2}}\right)^t = \left(\frac{7}{8}\right)^t = \left(1 - \frac{1}{8}\right)^t = (1 - 0.125)^t$, the percent rate of change is -12.5% and models exponential decay.

i. $f(t) = 1.02 \cdot (1.13)^{\pi t}$

Since $(1.13)^{\pi t} = ((1.13)^{\pi})^t \approx (1.468)^t$, the percent rate of change is 46.8% and models exponential growth.

6. The effective rate of an investment is the percent rate of change per year associated with the nominal APR. The effective rate is very useful in comparing accounts with different interest rates and compounding periods. In general, the effective rate can be found with the following formula: $r_E = \left(1 + \frac{r}{k}\right)^k - 1$. The effective rate presented here is the interest rate needed for annual compounding to be equal to compounding n times per year.

a. For investing, which account is better: an account earning a nominal APR of 7% compounded monthly or an account earning a nominal APR of 6.875% compounded daily? Why?

The 7% account is better. The effective rate for the 7% account is $\left(1 + \frac{0.07}{12}\right)^{12} - 1 \approx 0.07229$ compared to the effective rate for the 6.875% account, which is 0.07116.

b. The effective rate formula for an account compounded continuously is $r_E = e^r - 1$. Would an account earning 6.875% interest compounded continuously be better than the accounts in part (a)?

The effective rate of the account continuously compounded at 6.875% is $e^{0.06875} - 1 \approx 0.07117$, which is less than the effective rate of the 7% account, so the 7% account is better.

7. Radioactive decay is the process in which radioactive elements decay into more stable elements. A half-life is the time it takes for half of an amount of an element to decay into a more stable element. For instance, the half-life of uranium-235 is 704 million years. This means that half of any sample of uranium-235 transforms into lead-207 in 704 million years. We can assume that radioactive decay is modeled by exponential decay with a constant decay rate.

a. Suppose we have a sample of A g of uranium-235. Write an exponential formula that gives the amount of uranium-235 remaining after m half-lives.

The decay rate is constant on average and is 0.5. If the present value is A, then we have $F = A(1 + (-0.50))^m$, which simplifies to $F = A\left(\frac{1}{2}\right)^m$.

b. Does the formula that you wrote in part (a) work for any radioactive element? Why?

Since m represents the number of half-lives, this should be an appropriate formula for any decaying element.

c. Suppose we have a sample of A g of uranium-235. What is the decay rate per million years? Write an exponential formula that gives the amount of uranium-235 remaining after t million years.

The decay rate is 0.5 every 704 million years. If the present value is A, then we have $F = A\big(1 + (-0.5)\big)^{\frac{t}{704}} = A(0.5)^{\frac{t}{704}}$. This tells us that the growth rate per million years is $(0.5)^{\frac{1}{704}} \approx 0.9990159005$, and the decay rate is 0.0009840995 per million years. Written with this decay rate, the formula becomes $F = A(0.9990159005)^t$.

d. How would you calculate the number of years it takes to get to a specific percentage of the original amount of material? For example, how many years will it take for there to be 80% of the original amount of uranium-235 remaining?

Set $F = 0.80A$ in our formula and solve for t. For this example, this gives

$$0.80A = A\left(\frac{1}{2}\right)^{\frac{t}{704}}$$
$$0.80 = \left(\frac{1}{2}\right)^{\frac{t}{704}}$$
$$\ln(0.80) = \frac{t}{704}\left(\ln\left(\frac{1}{2}\right)\right)$$
$$t = 704\frac{\ln(0.80)}{\ln(0.5)}$$
$$t \approx 226.637$$

Remember that t represents the number of millions of years. So, it takes approximately $227{,}000{,}000$ years.

e. How many millions of years would it take 2.35 kg of uranium-235 to decay to 1 kg of uranium?

For our formula, the future value is 1 kg, and the present value is 2.35.

$$1 = 2.35\left(\frac{1}{2}\right)^{\frac{t}{704}}$$
$$\frac{1}{2.35} = \left(\frac{1}{2}\right)^{\frac{t}{704}}$$
$$\ln\left(\frac{1}{2.35}\right) = \frac{t}{704}\cdot\ln\left(\frac{1}{2}\right)$$
$$t = 704\frac{\ln\left(\frac{1}{2.35}\right)}{\ln\left(\frac{1}{2}\right)}$$
$$t \approx 867.793$$

Since t is the number of millions of years, it would take approximately 868 million years for 2.35 kg of uranium-235 to decay to 1 kg.

MP.4

8. Doug drank a cup of tea with 130 mg of caffeine. Each hour, the caffeine in Doug's body diminishes by about 12%. (This rate varies between 6% and 14% depending on the person.)

a. Write a formula to model the amount of caffeine remaining in Doug's system after each hour.

$$c(t) = 130\cdot(1-0.12)^t$$
$$c(t) = 130\cdot(0.88)^t$$

b. About how long should it take for the level of caffeine in Doug's system to drop below 30 mg?

$$30 = 130\cdot(0.88)^t$$
$$\frac{3}{13} = 0.88^t$$
$$\ln\left(\frac{3}{13}\right) = t\cdot\ln(0.88)$$
$$t = \frac{\ln\left(\frac{3}{13}\right)}{\ln(0.88)}$$
$$t \approx 11.471$$

The caffeine level is below 30 mg after about 11 hours and 28 minutes.

c. The time it takes for the body to metabolize half of a substance is called its *half-life*. To the nearest 5 minutes, how long is the half-life for Doug to metabolize caffeine?

$$65 = 130\cdot(0.88)^t$$
$$\frac{1}{2} = 0.88^t$$
$$\ln\left(\frac{1}{2}\right) = t\cdot\ln(0.88)$$
$$t = \frac{\ln\left(\frac{1}{2}\right)}{\ln(0.88)}$$
$$t \approx 5.422$$

The half-life of caffeine in Doug's system is about 5 hours and 25 minutes.

d. Write a formula to model the amount of caffeine remaining in Doug's system after m half-lives.

$$c = 130 \cdot \left(\frac{1}{2}\right)^m$$

9. A study done from 1950 through 2000 estimated that the world population increased on average by 1.77% each year. In 1950, the world population was 2.519 billion.

a. Write a function p for the world population t years after 1950.

$$p(t) = 2.519 \cdot (1 + 0.0177)^t$$
$$p(t) = 2.519 \cdot (1.0177)^t$$

b. If this trend continued, when should the world population have reached 7 billion?

MP.4

$$7 = 2.519 \cdot (1.0177)^t$$
$$\frac{7}{2.519} = 1.0177^t$$
$$\ln\left(\frac{7}{2.519}\right) = t \cdot \ln(1.0177)$$
$$t = \frac{\ln\left(\frac{7}{2.519}\right)}{\ln(1.0177)}$$
$$t \approx 58.252$$

The model says that the population should reach 7 billion sometime roughly $58\frac{1}{4}$ years after 1950. This would be around April 2008.

c. The world population reached 7 billion October 31, 2011, according to the United Nations. Is the model reasonably accurate?

Student responses will vary. The model was accurate to within three years, so, yes, it is reasonably accurate.

d. According to the model, when should the world population be greater than 12 billion people?

$$12 = 2.519 \cdot (1.0177)^t$$
$$\frac{12}{2.519} = 1.0177^t$$
$$\ln\left(\frac{12}{2.519}\right) = t \cdot \ln(1.0177)$$
$$t = \frac{\ln\left(\frac{12}{2.519}\right)}{\ln(1.0177)}$$
$$t \approx 88.973$$

According to the model, it will take a little less than 89 years from 1950 to get a world population of 12 billion. This would be the year 2039.

10. A particular mutual fund offers 4.5% nominal APR compounded monthly. Trevor wishes to deposit $\$1,000$.

a. What is the percent rate of change per month for this account?

There are twelve months in a year, so $\frac{4.5\%}{12} = 0.375\% = 0.00375.$

b. Write a formula for the amount Trevor will have in the account after m months.

$$A = 1000 \cdot (1 + 0.00375)^m$$
$$A = 1000 \cdot (1.00375)^m$$

c. *Doubling time* is the amount of time it takes for an investment to double. What is the doubling time of Trevor's investment?

$$2000 = 1000 \cdot (1.00375)^m$$
$$2 = 1.00375^m$$
$$\ln(2) = m \cdot \ln(1.00375)$$
$$m = \frac{\ln(2)}{\ln(1.00375)}$$
$$m \approx 185.186$$

It will take 186 months for Trevor's investment to double. This is 15 years and 6 months.

11. When paying off loans, the monthly payment first goes to any interest owed before being applied to the remaining balance. Accountants and bankers use tables to help organize their work.

a. Consider the situation that Fred is paying off a loan of $\$125,000$ with an interest rate of 6% per year compounded monthly. Fred pays $\$749.44$ every month. Complete the following table:

Payment	Interest Paid	Principal Paid	Remaining Principal
$\$749.44$	$\$625.00$	$\$124.44$	$\$124,875.56$
$\$749.44$	$\$624.38$	$\$125.06$	$\$124,750.50$
$\$749.44$	$\$623.75$	$\$125.69$	$\$124,624.82$

b. Fred's loan is supposed to last for 30 years. How much will Fred end up paying if he pays $\$749.44$ every month for 30 years? How much of this is interest if his loan was originally for $\$125,000$?

$$\$749.44(30)(12) = \$269,798.40$$

Fred will pay $\$269,798.40$ for his loan. The interest paid will be $\$269,798.40 - \$125,000.00$, which is $\$144,798.40$.

Lesson 27: Modeling with Exponential Functions

Student Outcomes

- Students create exponential functions to model real-world situations.
- Students use logarithms to solve equations of the form $f(t) = a \cdot b^{ct}$ for t.
- Students decide which type of model is appropriate by analyzing numerical or graphical data, verbal descriptions, and by comparing different data representations.

Lesson Notes

In this summative lesson, students write exponential functions for different situations to describe the relationships between two quantities (**F-BF.A.1a**). This lesson uses real U.S. Census data to demonstrate how to create a function of the form $f(t) = a \cdot b^{ct}$ that can be used to model quantities that exhibit exponential growth or decay. Students must use properties of exponents to rewrite exponential expressions in order to interpret the properties of the function (**F-IF.C.8b**). They estimate populations at a given time and determine the time when a population reaches a certain value by writing exponential equations (**A-CED.A.1**) and solving them analytically (**F-LE.A.4**). In Algebra I, students solved these types of problems graphically or numerically, but we have developed the necessary skills in this module to solve these problems explicitly. The data is presented in different forms (**F-IF.C.9**), and students use average rate of change (**F-IF.B.6**) to decide whether a linear or an exponential function is a more appropriate model (**F-LE.A.1**). Students have several different methods for determining the formula for an exponential function from given data: using a calculator's regression feature, solving for the parameters in the function analytically, and estimating the growth rate from a table of data (as covered in this lesson). This lesson ties those methods together and asks students to determine which method is most suitable for a particular situation (MP.4).

Classwork

Opening (1 minute)

Pose this question, which recalls the work students did in Lesson 22:

- If you only have two data points, how should you decide which type of function to use to model the data?
  - *Two data points could be modeled using a linear, quadratic, sinusoidal, or exponential function. You would have to have additional information or know something about the real-world situation to make a decision about which model would be best.*

The Opening Exercise has students review how to find a linear and exponential model given two data points. Later in the lesson, students are then given more information about the data and asked to select and refine a model.

Scaffolding:

If students struggle with the opening question, use this problem to provide a more concrete approach:

Given the ordered pairs $(0,3)$ and $(3,6)$, we could write the following functions:

$$f(t) = 3 + t$$

$$g(t) = 3(2)^{\frac{t}{3}}$$

Match each function to the appropriate verbal description and explain how you made your choice.

A: A plant seedling is 3 feet tall, and each week the height increases by a fixed amount. After three weeks, the plant is 6 feet tall.

B: Bacteria are dividing in a petri dish. Initially there are 300 bacteria, and three weeks later, there are 600.

Opening Exercise (5 minutes)

Give students time to work this Opening Exercise either independently or with a partner. Observe whether they are able to successfully write a linear and an exponential function for this data. If a majority of students are struggling to complete these exercises, then you may need to make adjustments during the lesson to help them build fluency with writing a function from given numerical data.

> *Scaffolding:*
> Encourage students who struggle with algebraic manipulations to use the statistical features of a graphing calculator to create a linear regression and an exponential regression equation in part (ii) of each Opening Exercise.

MP.3

Opening Exercise

The following table contains U.S. population data for the two most recent census years, 2000 and 2010.

Census Year	U.S. Population (in millions)
2000	281.4
2010	308.7

a. **Steve thinks the data should be modeled by a linear function.**

i. **What is the average rate of change in population per year according to this data?**

The average rate of change is $\frac{308.7-281.4}{2010-2000} = 2.73$ *million people per year.*

ii. **Write a formula for a linear function, L, to estimate the population t years since the year 2000.**

$L(t) = 2.73t + 281.4$

b. **Phillip thinks the data should be modeled by an exponential function.**

i. **What is the growth rate of the population per year according to this data?**

Since $\frac{308.7}{281.4} = 1.097$*, the population will increase by the factor* 1.097 *every* 10 *years. To determine the yearly rate, we would need to express* 1.097 *as the product of* 10 *equal numbers (e.g.,* $1.097^{\frac{1}{10}} \cdot 1.097^{\frac{1}{10}} \cdot \ldots \cdot 1.097^{\frac{1}{10}}$ *ten times). The annual rate would be* $1.097^{\frac{1}{10}}$*, which is approximately* 1.0093*.*

ii. **Write a formula for an exponential function, E, to estimate the population t years since the year 2000.**

Start with $E(t) = a \cdot b^t$*. Substitute* $(0, 281.4)$ *into the formula to solve for* a*.*

$$281.4 = a \cdot b^0$$

Thus, $a = 281.4$*.*

Next, substitute the value of a *and the ordered pair* $(10, 308.7)$ *into the formula to solve for* b*.*

$$308.7 = 281.4b^{10}$$

$$b^{10} = 1.097$$

$$b = \sqrt[10]{1.097}$$

Thus, $b = 1.0093$ *when you round to the ten-thousandths place and*

$$E(t) = 281.4(1.0093)^t.$$

c. **Who has the correct model? How do you know?**

You cannot determine who has the correct model without additional information. However, populations over longer intervals of time tend to grow exponentially if environmental factors do not limit the growth, so Phillip's model is likely to be more appropriate.

Discussion (3 minutes)

Before students start working in pairs or small groups on the modeling exercises, debrief the Opening Exercise with the following discussion to ensure that all students are prepared to begin the Modeling Exercise.

MP.7

- What function best modeled the given data? Allow students to debate about whether they chose a linear or an exponential model, and encourage them to provide justification for their decision.
  - $E(t) = 281.4(1.0093)^t$
- What does the number 281.4 represent?
  - *The initial population in the year 2000 was* 281.4 *million people.*
- What does the variable t represent?
  - *The number of years since the year 2000*
- What does the number 1.0093 represent?
  - *The population is increasing by a factor of* 1.0093 *each year.*
- How does rewriting the base as $1 + 0.0093$ help us to understand the population growth rate?
  - *We can see the population is increasing by approximately* 0.93% *every year according to our model.*

Mathematical Modeling Exercises 1–14 (24 minutes)

These problems ask students to compare their model from the Opening Exercise to additional models created when given additional information about the U.S. population, and then ask students to use additional data to find a better model. Students should form small groups and work these exercises collaboratively. Provide time at the end of this portion of the lesson for different groups to share their rationale for the choices that they made. Students are exposed to both tabular and graphical data (**F-IF.C.9**) as they work through these exercises. They must use the properties of exponents to interpret and compare exponential functions (**F-IF.C.8b**).

Exercise 11 requires access to the Internet to look up the current population estimate for the U.S. If students do not have convenient Internet access, you can either display the U.S. population clock at http://www.census.gov/popclock, which would be an interesting way to introduce this exercise, or look up the current population estimate at the onset of class and provide this information to the students. The U.S. population clock is updated every 10 or 12 seconds, so it shows a dramatic population increase through a single class period.

Mathematical Modeling Exercises 1–14

This challenge continues to examine U.S. census data to select and refine a model for the population of the United States over time.

Scaffolding:

For students who are slow to recognize data as linear or exponential, create an additional column that shows the average rate of change and reinforce that unless those values are very close to a constant, a linear function is not the best model.

MP.3

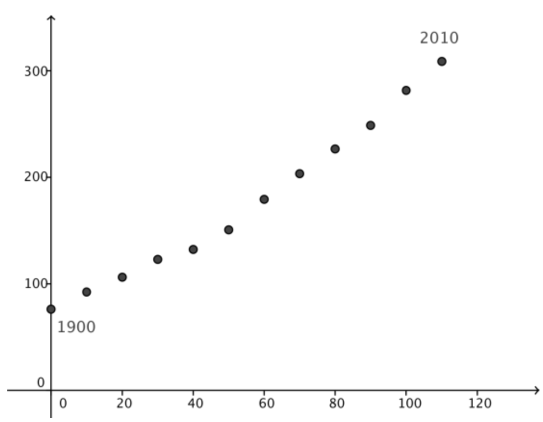
1. **The following table contains additional U.S. census population data. Would it be more appropriate to model this data with a linear or an exponential function? Explain your reasoning.**

Census Year	U.S. Population (in millions of people)
1900	76.2
1910	92.2
1920	106.0
1930	122.8
1940	132.2
1950	150.7
1960	179.3
1970	203.3
1980	226.5
1990	248.7
2000	281.4
2010	308.7

It is not clear by looking at a graph of this data whether it lies on an exponential curve or a line. However, from the context, we know that populations tend to grow as a constant factor of the previous population, so we should use an exponential function to model it. The graph below uses $t = 0$ to represent the year 1900.

OR

The differences between consecutive population values do not remain constant and in fact get larger as time goes on, but the quotients of consecutive population values are nearly constant around 1.1. This indicates that a linear model is not appropriate but an exponential model is.



After the work in Lesson 22, students should know that a situation such as this one involving population growth should be modeled by an exponential function. However, the reasoning used by each group of students may vary. Some may plot the data and note the characteristic shape of an exponential curve. Some may calculate the quotients and differences between consecutive population values. If time permits, have students share the reasoning they used to decide which type of function to use.

Scaffolding:

Students may need to be shown how to use the calculator to find the exponential regression function.

2. **Use a calculator's regression capability to find a function, f, that models the U.S. Census Bureau data from 1900 to 2010.**

Using a graphing calculator and letting the year 1900 correspond to $t = 0$ gives the following exponential regression equation.

$$P(t) = 81.1(1.0126)^t$$

3. **Find the growth factor for each 10-year period and record it in the table below. What do you observe about these growth factors?**

Census Year	U.S. Population (in millions of people)	Growth Factor (10-year period)
1900	76.2	--
1910	92.2	1.209974
1920	106.0	1.149675
1930	122.8	1.158491
1940	132.2	1.076547
1950	150.7	1.139939
1960	179.3	1.189781
1970	203.3	1.133854
1980	226.5	1.114117
1990	248.7	1.098013
2000	281.4	1.131484
2010	308.7	1.097015

The growth factors are fairly constant around 1.1.

4. **For which decade is the 10-year growth factor the lowest? What factors do you think caused that decrease?**

The 10-year growth factor is lowest in the 1930's, which is the decade of the Great Depression.

5. **Find an average 10-year growth factor for the population data in the table. What does that number represent? Use the average growth factor to find an exponential function, g, that can model this data.**

Averaging the 10-year growth factors gives 1.136; using our previous form of an exponential function; this means that the growth rate r satisfies $1 + r = 1.136$, so $r = 0.136$. This represents a 13.6% population increase every ten years. The function g has an initial value $g(0) = 76.2$, so g is then given by $g(t) = 76.2(1.136)^{\frac{t}{10}}$, where t represents year since 1900.

6. **You have now computed three potential models for the population of the United States over time: functions E, f, and g. Which one do you expect would be the most accurate model based on how they were created? Explain your reasoning.**

Student responses will vary. Potential responses:

- *I expect that function f that we found through exponential regression on the calculator is the most accurate because it uses all of the data points to compute the coefficients of the function.*
- *I expect that the function E is most accurate because it uses only the most recent population values.*

Students should notice that function g is expressed in terms of a 10-year growth rate (the exponent is $\frac{t}{10}$), while the other two functions are expressed in terms of single-year growth rates (the exponent is t). In Exercise 8, encourage students to realize that they need to use properties of exponents to rewrite the exponential expression in g in the form $g(t) = A(1+r)^t$ with an annual growth rate r so that the three functions can be compared in Exercise 10 (F-IF.C.8b). Through questioning, lead students to notice that time $t = 0$ does not have the same meaning for all three functions E, f, and g. In Exercise 9, they need to transform function E so that $t = 0$ corresponds to the year 1900 instead of 2000. This is the equivalent of translating the graph of $y = E(t)$ horizontally to the right by 100 units.

7. Summarize the three formulas for exponential models that you have found so far: Write the formula, the initial populations, and the growth rates indicated by each function. What is different between the structures of these three functions?

We have the three models:

- $E(t) = 281.4(1.0093)^t$: *Population is 281.4 million in the year 2000; annual growth rate is 0.93%.*
- $f(t) = 81.1(1.0126)^t$: *Population is 81.1 million in the year 1900; annual growth rate is 1.26%.*
- $g(t) = 76.2(13.6)^{\frac{t}{10}}$: *Population is 76.2 million in the year 1900; 10-year growth rate is 13.6%.*

In function E, $t = 0$ corresponds to the year 2000, while in functions f and g, $t = 0$ represents the year 1900. Function g is expressed in terms of a 10-year growth factor instead of an annual growth factor as in functions E and f. Function E has the year 2000 corresponding to $t = 0$, while in functions f and g the year $t = 0$ represents the year 1900.

8. Rewrite the functions E, f, and g as needed in terms of an annual growth rate.

We need to use properties of exponents to rewrite g.

$$\begin{aligned} g(t) &= 76.2(1.136)^{\frac{t}{10}} \\ &= 76.2\left((1.136)^{\frac{1}{10}}\right)^t \\ &\approx 76.2(1.0128)^t \end{aligned}$$

Scaffolding:
Struggling students may need to be explicitly told that they need to re-express g in the form $g(t) = A(1+r)^t$ with an annual growth rate r.

9. Transform the functions as needed so that the time $t = 0$ represents the same year in functions E, f, and g. Then compare the values of the initial populations and annual growth rates indicated by each function.

In function E, $t = 0$ represents the year 2000, and in functions f and g, $t = 0$ represents the year 1900.

Thus, we need to translate function E horizontally to the right by 100 years, giving a new function:

$$\begin{aligned} E(t) &= 281.4(1.0093)^{t-100} \\ &= 281.4(1.0093)^{-100}(1.0093)^t \\ &\approx 111.5(1.0093)^t. \end{aligned}$$

Then we have the three functions:

$$\begin{aligned} E(t) &= 111.5(1.0093)^t \\ f(t) &= 81.1(1.0126)^t \\ g(t) &= 76.2(1.0128)^t \end{aligned}$$

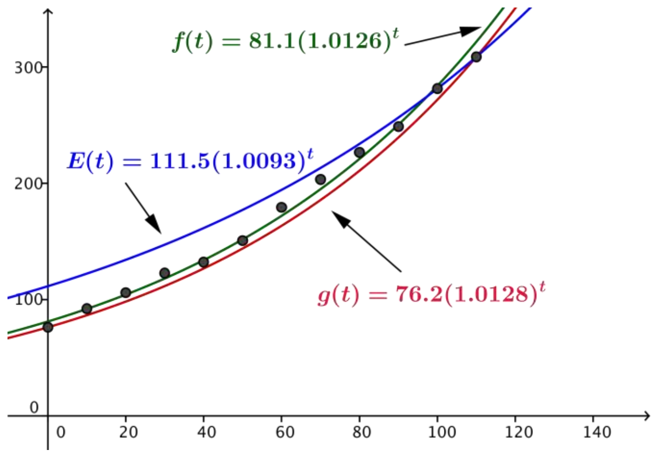
- *Function E has the largest initial population and the smallest growth rate at 0.93% increase per year.*
- *Function g has the smallest initial population and the largest growth rate at 1.28% increase per year.*

Scaffolding:
Struggling students may need to be explicitly told that they need to translate function E so that $t = 0$ represents the year 1900 for all three functions.

10. **Which of the three functions is the best model to use for the U.S. census data from 1900 to 2010? Explain your reasoning.**

Student responses will vary.

Possible response: Graphing all three functions together with the data, we see that function f appears to be the closest to all of the data points.



11. **The U.S. Census Bureau website http://www.census.gov/popclock displays the current estimate of both the United States and world populations.**

a. **What is today's current estimated population of the U.S.?**

This will vary by the date. The solution shown here uses the population 318.7 million and the date August 16, 2014.

b. **If time $t = 0$ represents the year 1900, what is the value of t for today's date? Give your answer to two decimal places.**

August 16 is the 228^{th} day of the year, so the time is $t = 114 + \frac{228}{365}$. We use $t = 114.62$.

c. **Which of the functions E, f, and g gives the best estimate of today's population? Does that match what you expected? Justify your reasoning.**

$$E(114.62) = 322.2$$
$$f(114.62) = 340.7$$
$$g(114.62) = 327.4$$

The function E gives the closest value to today's estimated population, but all three functions produce estimates that are too high. Possible response: I had expected that function f, which was obtained through regression, to produce the closest population estimate, so this is a surprise.

d. **With your group, discuss some possible reasons for the discrepancy between what you expected in Exercise 8 and the results of part (c) above.**

Student responses will vary.

MP.3

12. Use the model that most accurately predicted today's population in Exercise 9, part (c) to predict when the U.S. population will reach half a billion.

Half a billion is 500 million. Set the formula for E equal to 500 and solve for t.

$$\begin{aligned} 111.5(1.0093)^t &= 500 \\ 1.0093^t &= \frac{500}{111.5} \\ 1.0093^t &= 4.4843 \\ \log(1.0093)^t &= \log(4.4843) \\ t\log(1.0093) &= \log(4.4843) \\ t &= \frac{\log(4.4843)}{\log(1.0093)} \\ t &\approx 162 \end{aligned}$$

Assuming the same rate of growth, the population will reach half a billion people 162 years from the year 1900, in the year 2062.

13. Based on your work so far, do you think this is an accurate prediction? Justify your reasoning.

Student responses will vary. Possible response: From what we know of population growth, the data should most likely be fit with an exponential function, however the growth rate appears to be decreasing because the models that use all of the census data produce estimates for the current population that are too high. I think the population will reach half a billion sometime after the year 2062 because the U.S. Census Bureau expects the growth rate to slow down. Perhaps the United States is reaching its capacity and cannot sustain the same exponential rate of growth into the future.

14. Here is a graph of the U.S. population since the census began in 1790. Which type of function would best model this data? Explain your reasoning.

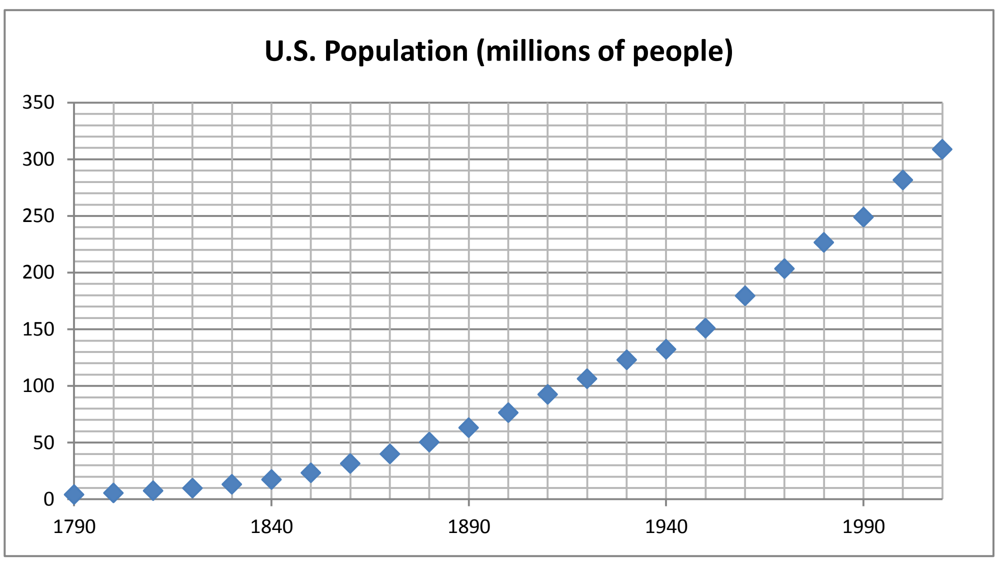


Figure 1: Source U.S. Census Bureau

The shape of the curve indicates that an exponential model would be the best choice. You could model the data for short periods of time using a series of piecewise linear functions, but the average rate of change in the early years is clearly less than that in later years. A linear model would also not make sense because at some point in the past you would have had a negative number of people living in the U.S.

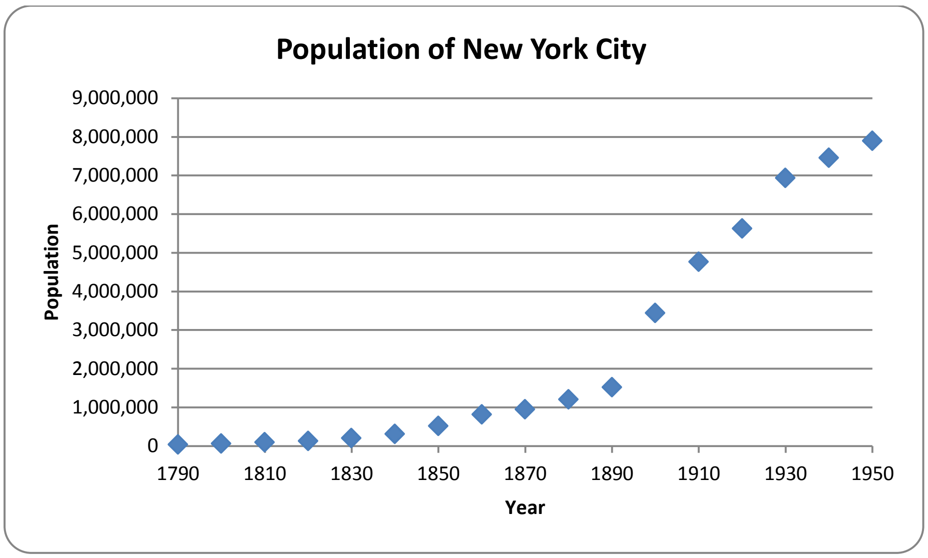
Exercises 15–16 (6 minutes)

Exercises 15–16 are provided for students who complete the Modeling Exercises. You might consider assigning these exercises as additional Problem Sets for the rest of the class.

In these two exercises, students are asked to compare different exponential population models. They need to rewrite them to interpret the parameters when they compare the functions and apply the formula to solve a variety of problems. They are asked to compare the functions that model this data with an actual graph of the data. These problems are examples of **F-IF.C.8b**, **F-LE.A.1**, **F-LE.A.4**, and **F-IF.C.9**.

Exercises 15–16

15. The graph below shows the population of New York City during a time of rapid population growth.



Finn averaged the 10-year growth rates and wrote the function $f(t) = 33131(1.44)^{\frac{t}{10}}$, where t is the time in years since 1790.

Gwen used the regression features on a graphing calculator and got the function $g(t) = 48661(1.036)^t$, where t is the time in years since 1790.

a. Rewrite each function to determine the annual growth rate for Finn's model and Gwen's model.

Finn's function: $f(t) = 33131\left(1.44^{\frac{1}{10}}\right)^t = 33131(1.037)^t$. *The annual growth rate is* 3.7%.

Gwen's function has a growth rate of 3.6%.

b. What is the predicted population in the year 1790 for each model?

It will be the value of the function when $t = 0$. *Finn:* $f(0) = 33131$. *Gwen:* $g(0) = 48661$.

c. Lenny calculated an exponential regression using his graphing calculator and got the same growth rate as Gwen, but his initial population was very close to 0. Explain what data Lenny may have used to find his function.

He may have used the actual year for his time values; where Gwen represented year 1790 by $t = 0$, Lenny may have represented year 1790 by $t = 1790$. If you translate Gwen's function 1790 units to the right write the resulting function in the form $f(t) = a \cdot b^t$, the value of a would be very small.

$$48661(1.036)^{t-1790} = \frac{48661(1.036)^t}{1.036^{1790}} \text{ and } \frac{48661}{1.036^{1790}} \approx 1.56 \times 10^{-23}$$

d. When does Gwen's function predict the population will reach $1{,}000{,}000$? How does this compare to the graph?

Solve the equation: $48661(1.036)^t = 1000000$.

$$1.036^t = \frac{1000000}{48\,661}$$
$$\log(1.036)^t = \log\left(\frac{1000000}{48\,661}\right)$$
$$t\log(1.036) = \log\left(\frac{1000000}{48\,661}\right)$$
$$t = \frac{\log\left(\frac{1000000}{48661}\right)}{\log(1.036)}$$
$$t \approx 85.5$$

Gwen's model predicts that the population will exceed one million after 86 years, which would be during the year 1867. It appears that the population was close to one million around 1870 so the model does a fairly good job of estimating the population.

e. Based on the graph, do you think an exponential growth function would be useful for predicting the population of New York in the years after 1950?

The graph appears to be increasing but curving downwards, and an exponential model with a base greater than 1 would always be increasing at an increasing rate, so its graph would curve upwards. The difference between the function and the data would be increasing, so this is probably not an appropriate model.

16. Suppose each function below represents the population of a different U.S. city since the year 1900.

a. Complete the table below. Use the properties of exponents to rewrite expressions as needed to help support your answers.

City Population Function (t is years since 1900)	Population in the Year 1900	Annual Growth/Decay Rate	Predicted in 2000	Between Which Years Did the Population Double?
$A(t) = 3000(1.1)^{\frac{t}{5}}$	3000	1.9% *growth*	20182	*Between* 1936 *and* 1937
$B(t) = \frac{(1.5)^{2t}}{2.25}$	1	125% *growth*	7.3×10^{34}	*Between* 1901 *and* 1902
$C(t) = 10000(1-0.01)^t$	10000	1% *decay*	475	*Never*
$D(t) = 900(1.02)^t$	900	2% *growth*	6520	*Between* 1935 *and* 1936

b. **Could the function $E(t) = 6520(1.219)^{\frac{t}{10}}$, where t is years since 2000 also represent the population of one of these cities? Use the properties of exponents to support your answer.**

Yes, it could represent the population in the city with function D. The expression $1.219^{\frac{t}{10}} \approx 1.02^t$ for any real number t. Also, $E(0) \approx D(100)$, which would make sense if the point of reference in time is 100 years apart.

c. **Which cities are growing in size, and which are decreasing according to these models?**

The cities represented by functions A, B, and D are growing because their base value is greater than 1. The city represented by function C is shrinking because $1 - 0.01$ is less than 1.

d. **Which of these functions might realistically represent city population growth over an extended period of time?**

Based on the United States and New York City data, it is unlikely that a city in the United States could sustain a 50% growth rate every two years for an extended period of time as indicated by function B and its predicted population in the year 2000. The other functions seem more realistic, with annual growth or decay rates similar to other city populations we examined.

Closing (2 minutes)

Have students respond to these questions either in writing or with a partner.

- How do you decide when an exponential function would be an appropriate model for a given situation?
  - *You must consider the real-world situation to determine whether growth or decay by a constant factor is appropriate or not. Analyzing patterns in the graphs or data tables can also help.*
- Which method do you prefer for determining a formula for an exponential function?
  - *Student responses will vary. A graphing calculator provides a statistical regression equation, but you have to type in the data to use that feature.*
- Why did we rewrite the expression for function g?
  - *We can more easily compare the properties of functions if they have the same structure.*

Lesson Summary

To model exponential data as a function of time:

- **Examine the data to see if there appears to be a constant growth or decay factor.**
- **Determine a growth factor and a point in time to correspond to $t = 0$.**
- **Create a function $f(t) = a \cdot b^{ct}$ to model the situation, where b is the growth factor every $\frac{1}{c}$ years and a is the value of f when $t = 0$.**

Logarithms can be used to solve for t when you know the value of $f(t)$ in an exponential function.

Exit Ticket (4 minutes)

Name ______________________________ Date ________________

Lesson 27: Modeling with Exponential Functions

Exit Ticket

1. The table below gives the average annual cost (e.g., tuition, room, and board) for four-year public colleges and universities. Explain why a linear model might not be appropriate for this situation.

Year	Average Annual Cost
1981	$\$2,550$
1991	$\$5,243$
2001	$\$8,653$
2011	$\$15,918$

2. Write an exponential function to model this situation.

3. Use the properties of exponents to rewrite the function from Problem 2 to determine an annual growth rate.

4. If this trend continues, when will the average annual cost of attendance exceed $\$35,000$?

Exit Ticket Sample Solutions

1. **The table below gives the average annual cost (e.g., tuition, room, and board) for four-year public colleges and universities. Explain why a linear model might not be appropriate for this situation.**

Year	Average Annual Cost
1981	$\$2,550$
1991	$\$5,243$
2001	$\$8,653$
2011	$\$15,918$

A linear function would not be appropriate because the average rate of change is not constant.

2. **Write an exponential function to model this situation.**

If you calculate the growth factor every 10 years, you get the following values.

$$1981-1991: \frac{5243}{2550}=2.056$$

$$1991-2001: \frac{8653}{5243}=1.650$$

$$2001-2011: \frac{15918}{8653}=1.840$$

The average of these growth factors is 1.85.

Then the average annual cost in dollars t years after 1981 is $C(t)=2550(1.85)^{\frac{t}{10}}$.

3. **Use the properties of exponents to rewrite the function from Problem 2 to determine an annual growth rate.**

We know that $2550(1.85)^{\frac{t}{10}}=2250\left(1.85^{\frac{1}{10}}\right)^{t}$ and $1.85^{\frac{1}{10}}\approx 1.063$. Thus the annual growth rate is 6.3%.

4. **If this trend continues, when will the average annual cost exceed $\$35,000$?**

We need to solve the equation $C(t)=35000$ for t.

$$2550(1.85)^{\frac{t}{10}}=35000$$

$$(1.85)^{\frac{t}{10}}=13.725$$

$$\log\left((1.85)^{\frac{t}{10}}\right)=\log(13.725)$$

$$\frac{t}{10}=\frac{\log(13.725)}{\log(1.85)}$$

$$t=10\left(\frac{\log(13.725)}{\log(1.85)}\right)$$

$$t\approx 42.6$$

The cost will exceed $\$35,000$ after 43 years, in the year 2024.

Problem Set Sample Solutions

1. Does each pair of formulas described below represent the same sequence? Justify your reasoning.

   a. $a_{n+1} = \frac{2}{3}a_n$, $a_0 = -1$ and $b_n = -\left(\frac{2}{3}\right)^n$ for $n \geq 0$.

   Yes, checking the first few terms in each sequence gives the same values. Both sequences start with -1 and are repeatedly multiplied by $\frac{2}{3}$.

   b. $a_n = 2a_{n-1} + 3$, $a_0 = 3$ and $b_n = 2(n-1)^3 + 4(n-1) + 3$ for $n \geq 1$.

   No, the first two terms are the same, but the third term is different.

   c. $a_n = \frac{1}{3}(3)^n$ for $n \geq 0$ and $b_n = 3^{n-2}$ for $n \geq 0$.

   Yes, the first terms are equal $a_0 = \frac{1}{3}$ and $b_0 = 3^{-1} = \frac{1}{3}$, and the next term is found by multiplying the previous term by 3 in both sequences.

2. Tina is saving her babysitting money. She has $\$500$ in the bank, and each month she deposits another $\$100$. Her account earns 2% interest compounded monthly.

   a. Complete the table showing how much money she has in the bank for the first four months.

Month	Amount (in dollars)
1	500
2	$500(1.00167) + 100 = 600.84$
3	$(500(1.00167) + 100)(1.00167) + 100 = 701.84$
4	$((500(1.00167) + 100)(1.00167) + 100)1.00167 + 100 = 803.01$

   b. Write a recursive sequence for the amount of money she has in her account after n months.

   $a_1 = 500$, $a_{n+1} = a_n\left(1 + \frac{0.02}{12}\right) + 100$

3. Assume each table represents values of an exponential function of the form $f(t) = a(b)^{ct}$ where b is a positive real number and a and c are real numbers. Use the information in each table to write a formula for f in terms of t for parts (a)–(d).

   a.

t	$f(t)$
0	10
4	50

   $f(t) = 10(5)^{\frac{t}{4}}$

   b.

t	$f(t)$
0	$1\,000$
5	750

   $f(t) = 1000(0.75)^{\frac{t}{5}}$

   c.

t	$f(t)$
6	25
8	45

   $f(t) = 4.287\left(\frac{9}{5}\right)^{\frac{t}{2}}$

   d.

t	$f(t)$
3	50
6	40

   $f(t) = 62.5\left(\frac{4}{5}\right)^{\frac{t}{3}}$

e. **Rewrite the expressions for each function in parts (a)–(d) to determine the annual growth or decay rate.**

For part (a), $5^{\frac{t}{4}} = \left(5^{\frac{1}{4}}\right)^t$ *so the annual growth factor is* $5^{\frac{1}{4}} \approx 1.495$*, and the annual growth rate is* 49.5%*.*

For part (b), $0.75^{\frac{t}{5}} = \left(0.75^{\frac{1}{5}}\right)^t$ *so the annual growth factor is* $0.75^{\frac{1}{5}} \approx 0.596$*, so the annual growth rate is* -40.4%*, meaning that the quantity is decaying at a rate of* 40.4%*.*

For part (c), $\left(\frac{9}{5}\right)^{\frac{t}{2}} = \left(\left(\frac{9}{5}\right)^{\frac{1}{2}}\right)^t$ *so the annual growth factor is* $\left(\frac{9}{5}\right)^{\frac{1}{2}} \approx 1.312$ *and the annual growth rate is* 31.2%*.*

For part (a), $\left(\frac{4}{5}\right)^{\frac{t}{3}} = \left(\left(\frac{4}{5}\right)^{\frac{1}{3}}\right)^t$ *so the annual growth factor is* $\left(\frac{4}{5}\right)^{\frac{1}{3}} \approx 0.928$ *and the annual growth rate is* -0.072*, which is a decay rate of* 7.2%*.*

f. **For parts (a) and (c), determine when the value of the function is double its initial amount.**

For part (a), solve the equation $2 = 5^{\frac{t}{4}}$ *for t.*

$$\begin{aligned} 2 &= 5^{\frac{t}{4}} \\ \log(2) &= \log\left(5^{\frac{t}{4}}\right) \\ \frac{t}{4} &= \frac{\log(2)}{\log(5)} \\ t &= 4\left(\frac{\log(2)}{\log(5)}\right) \\ t &\approx 1.723 \end{aligned}$$

For part (c), solve the equation $2 = \left(\frac{9}{5}\right)^{\frac{t}{2}}$ *for t. The solution is* 2.358*.*

g. **For parts (b) and (d), determine when the value of the function is half of its initial amount.**

For part (b), solve the equation $\frac{1}{2} = (0.75)^{\frac{t}{5}}$ *for t. The solution is* 12.047*.*

For part (d), solve the equation $\frac{1}{2} = \left(\frac{4}{5}\right)^{\frac{t}{3}}$ *for t. The solution is* 9.319*.*

4. **When examining the data in Example 1, Juan noticed the population doubled every five years and wrote the formula** $P(t) = 100(2)^{\frac{t}{5}}$**. Use the properties of exponents to show that both functions grow at the same rate per year.**

Using properties of exponents, $100(2)^{\frac{t}{5}} = 100\left(2^{\frac{1}{5}}\right)^t$*. The annual growth is* $2^{\frac{1}{5}}$*. In the other function, the annual growth is* $4^{\frac{1}{10}} = \left(4^{\frac{1}{2}}\right)^{\frac{1}{5}} = 2^{\frac{1}{5}}$*.*

5. **The growth of a tree seedling over a short period of time can be modeled by an exponential function. Suppose the tree starts out 3 feet tall and its height increases by** 15% **per year. When will the tree be 25 feet tall?**

We model the growth of the seedling by $h(t) = 3(1.15)^t$*, where t is measured in years, and we find that* $3(1.15)^t = 25$ *when* $t = \frac{\log\left(\frac{25}{3}\right)}{\log(1.15)}$*, so* $t \approx 15.17$ *years. The tree will be* 25 *feet tall when it is* 15 *years and* 2 *months old.*

6. Loggerhead turtles reproduce every 2–4 years, laying approximately 120 eggs in a clutch. Studying the local population, a biologist records the following data in the second and fourth years of her study:

Year	Population
2	50
4	1250

a. Find an exponential model that describes the loggerhead turtle population in year t.

From the table, we see that $P(2) = 50$ and $P(4) = 1250$. So, the growth rate over two years is $\frac{1250}{50} = 25$. Since $P(2) = 50$, and $P(t) = P_0(25)^{\frac{t}{2}}$, we know that $50 = P_0(25)$, so $P_0 = 2$. Then $50r^2 = P_0r^4$, so $50r^2 = 1250$. Thus, $r^2 = 25$ and then $r = 5$. Since $50 = P_0r^2$, we see that $P_0 = 2$. Therefore,

$$P(t) = 2(5^t).$$

b. According to your model, when will the population of loggerhead turtles be over $5,000$? Give your answer in years and months.

$$2(5^t) = 5000$$
$$5^t = 2500$$
$$t\log(5) = \log(2500)$$
$$t = \frac{\log(2500)}{\log(5)}$$
$$t \approx 4.86$$

The population of loggerhead turtles will be over $5,000$ after year 4.86, which is roughly 4 years and 11 months.

7. The radioactive isotope seaborgium-266 has a half-life of 30 seconds, which means that if you have a sample of A grams of seaborgium-266, then after 30 seconds half of the sample has decayed (meaning it has turned into another element), and only $\frac{A}{2}$ grams of seaborgium-266 remain. This decay happens continuously.

a. Define a sequence $a_0, a_1, a_2, \ldots$ so that a_n represents the amount of a 100-gram sample that remains after n minutes.

In one minute, the sample has been reduced by half two times, leaving only $\frac{1}{4}$ of the sample. We can represent this by the sequence $a_n = 100\left(\frac{1}{2}\right)^{2n} = 100\left(\frac{1}{4}\right)^n$. (Either form is acceptable.)

b. Define a function $a(t)$ that describes the amount of a 100-gram sample of seaborgium-266 that remains after t minutes.

$$a(t) = 100\left(\frac{1}{4}\right)^t = 100\left(\frac{1}{2}\right)^{2t}$$

c. Do your sequence from part (a) and your function from part (b) model the same thing? Explain how you know.

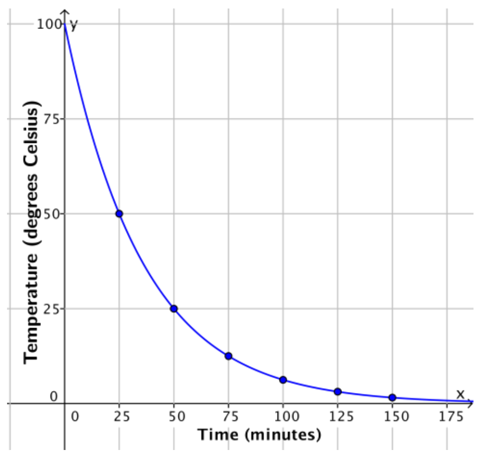
The function models the amount of seaborgium-266 as it constantly decreases every fraction of a second, and the sequence models the amount of seaborgium-266 that remains only in 30-second intervals. They model nearly the same thing, but not quite. The function is continuous and the sequence is discrete.

d. **How many minutes does it take for less than 1 g of seaborgium-266 to remain from the original 100 g sample? Give your answer to the nearest minute.**

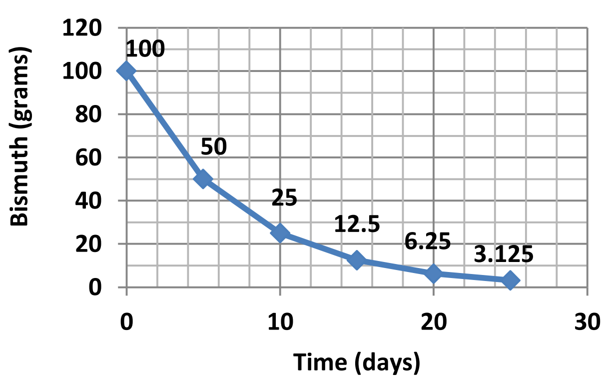
The sequence is $a_0 = 100$, $a_1 = 25$, $a_2 = 6.25$, $a_3 = 1.5625$, $a_4 = 0.390625$, so after 4 minutes there is less than 1 g of the original sample remaining.

8. **Strontium-90, magnesium-28, and bismuth all decay radioactively at different rates. Use the data provided in the graphs and tables below to answer the questions that follow.**

Strontium-90 (grams) vs. time (hours)



Radioactive Decay of Magnesium-28	
R grams	t hours
1	0
0.5	21
0.25	42
0.125	63
0.0625	84



a. **Which element decays most rapidly? How do you know?**

Magnesium-28 decays most rapidly. It loses half its amount every 21 hours.

b. Write an exponential function for each element that shows how much of a 100 g sample will remain after t days. Rewrite these expressions to show precisely how their exponential decay rates compare to confirm your answer to part (a).

- *Strontium-90: We model the remaining quantity by $f(t) = 100\left(\frac{1}{2}\right)^{\frac{t}{\frac{25}{24}}}$ where t is in days.*

  Rewriting the expression gives a growth factor of $\left(\frac{1}{2}\right)^{\frac{24}{25}} \approx 0.514$, so $f(t) = 100(0.514)^t$.

- *Magnesium-28: We model the remaining quantity by $f(t) = 100\left(\frac{1}{2}\right)^{\frac{t}{\frac{21}{24}}}$ where t is in days.*

  Rewriting the expression give a growth factor of $\left(\frac{1}{2}\right)^{\frac{24}{21}} \approx 0.453$, so $f(t) = 100(0.453)^t$

- *Bismuth: We model the remaining quantity by $f(t) = 100\left(\frac{1}{2}\right)^{\frac{t}{5}}$ where t is in days. Rewriting the expression gives a growth factor of $\left(\frac{1}{2}\right)^{\frac{1}{5}} \approx 0.871$, so $f(t) = 100(0.871)^t$.*

The function with the smallest daily growth factor is decaying the fastest, so magnesium-24 decays the fastest.

9. The growth of two different species of fish in a lake can be modeled by the functions shown below where t is time in months since January 2000. Assume these models will be valid for at least 5 years.

Fish A: $f(t) = 5000(1.3)^t$

Fish B: $g(t) = 10000(1.1)^t$

According to these models, explain why the fish population modeled by function f will eventually catch up to the fish population modeled by function g. Determine precisely when this will occur.

The fish population with the larger growth rate will eventually exceed the population with a smaller growth rate, so eventually there will be a larger population of Fish A.

Solve the equation $f(t) = g(t)$ for t to determine when the populations will be equal. After that point in time, the population of Fish A will exceed the population of Fish B.

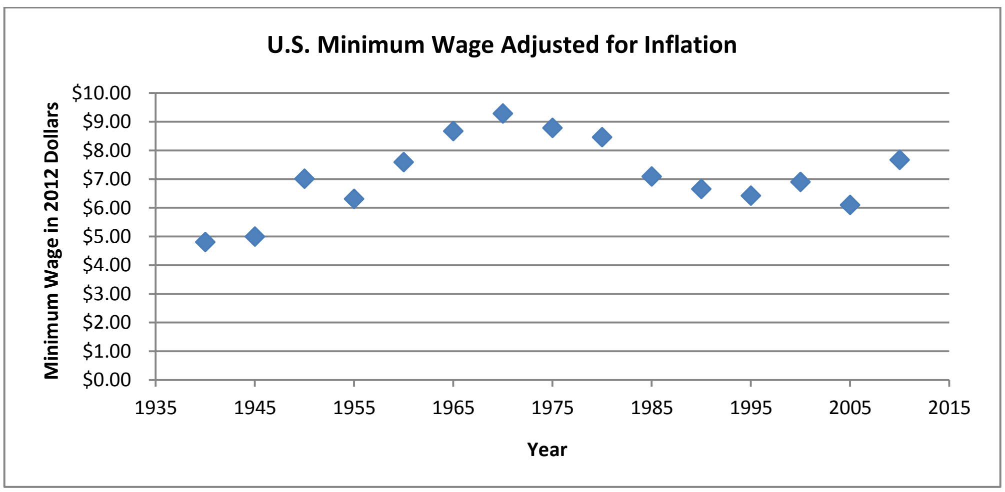
The solution is

$$5000(1.3)^t = 10000(1.1)^t$$
$$\frac{(1.3)^t}{(1.1)^t} = 2$$
$$\left(\frac{1.3}{1.1}\right)^t = 2$$
$$t = \frac{\log(2)}{\log\left(\frac{1.3}{1.1}\right)}$$
$$t \approx 4.15$$

During the fourth year, the population of Fish A will catch up to and then exceed the population of Fish B.

10. **When looking at U.S. minimum wage data, you can consider the nominal minimum wage, which is the amount paid in dollars for an hour of work in the given year. You can also consider the minimum wage adjusted for inflation. Below are a table showing the nominal minimum wage and a graph of the data when the minimum wage is adjusted for inflation. Do you think an exponential function would be an appropriate model for either situation? Explain your reasoning.**

Year	Nominal Minimum Wage
1940	$\$0.30$
1945	$\$0.40$
1950	$\$0.75$
1955	$\$0.75$
1960	$\$1.00$
1965	$\$1.25$
1970	$\$1.60$
1975	$\$2.10$
1980	$\$3.10$
1985	$\$3.35$
1990	$\$3.80$
1995	$\$4.25$
2000	$\$5.15$
2005	$\$5.15$
2010	$\$7.25$



Student solutions will vary. The inflation-adjusted minimum wage is clearly not exponential because it does not strictly increase or decrease. The other data when graphed does appear roughly exponential, and a good model would be $f(t) = 0.40(1.044)^t$.

11. **A dangerous bacterial compound forms in a closed environment but is immediately detected. An initial detection reading suggests the concentration of bacteria in the closed environment is one percent of the fatal exposure level. Two hours later, the concentration has increased to four percent of the fatal exposure level.**

   a. **Develop an exponential model that gives the percentage of fatal exposure level in terms of the number of hours passed.**

$$\begin{aligned} P(t) &= 1 \cdot \left(\frac{4}{1}\right)^{\frac{t}{2}} \\ &= 4^{\frac{t}{2}} \\ &= 2^t \end{aligned}$$

   b. **Doctors and toxicology professionals estimate that exposure to two-thirds of the bacteria's fatal concentration level will begin to cause sickness. Offer a time limit (to the nearest minute) for the inhabitants of the infected environment to evacuate in order to avoid sickness.**

$$\begin{aligned} 66.66 &= 2^t \\ \log(66.66) &= t \cdot \log(2) \\ t &= \frac{\log(66.66)}{\log(2)} \approx 6.0587 \end{aligned}$$

   Inhabitants should evacuate before 6 hours and 3 minutes.

   c. **A more conservative approach is to evacuate the infected environment before bacteria concentration levels reach 45% of the fatal level. Offer a time limit (to the nearest minute) for evacuation in this circumstance.**

$$\begin{aligned} 2^t &= 45 \\ t \cdot \log(2) &= \log(45) \\ t &= \frac{\log(45)}{\log(2)} \approx 5.492 \end{aligned}$$

   Inhabitants should evacuate within 5 hours and 30 minutes.

   d. **To the nearest minute, when will the infected environment reach 100% of the fatal level of bacteria concentration**

$$\begin{aligned} t \cdot \log(2) &= \log(100) \\ t &= \frac{2}{\log(2)} \approx 6.644 \end{aligned}$$

   The infected environment will reach 100% of the fatal level of bacteria in 6 hours and 39 minutes.

12. Data for the number of users at two different social media companies is given below. Assuming an exponential growth rate, which company is adding users at a faster annual rate? Explain how you know.

Social Media Company A	
Year	Number of Users (Millions)
2010	54
2012	185

Social Media Company B	
Year	Number of Users (Millions)
2009	360
2012	1056

Company A: The number of users (in millions) can be modeled by $A(t) = a\left(\frac{185}{54}\right)^{\frac{t}{2}}$ where a is the initial amount and t is time in years since 2010.

Company B: The number of users (in millions) can be modeled by $B(t) = b\left(\frac{1056}{360}\right)^{\frac{t}{3}}$ where b is the initial amount and t is time in years since 2009.

Rewriting the expressions, you can see that Company A's annual growth factor is $\left(\frac{185}{54}\right)^{\frac{1}{2}} \approx 1.851$, and Company B's annual growth factor is $\left(\frac{1056}{360}\right)^{\frac{1}{3}} \approx 1.432$. Thus, Company A is growing at the faster rate of 85.1% compared to Company B's 43.2%.

Lesson 28: Newton's Law of Cooling, Revisited

Student Outcomes

- Students apply knowledge of exponential and logarithmic functions and transformations of functions to a contextual situation.

Lesson Notes

Newton's law of cooling is a complex topic that appears in physics and calculus; the formula can be derived using differential equations. In Algebra I (Module 3), students completed a modeling lesson in which Newton's law of cooling was simplified to focus on the idea of applying transformations of functions to a contextual situation. In this lesson, students take another look at Newton's law of cooling, this time incorporating their knowledge of the number e and logarithms. Students now have the capability of finding the decay constant, k, for a contextual situation through the use of logarithms (**F-LE.A.4**). Students expand their understanding of exponential functions and transformations to build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential and relate these functions to the model (**F-BF.A.1.b**). The entire lesson highlights modeling with mathematics (MP.4) and also provides students with an opportunity to interpret scenarios using Newton's law of cooling when presented with functions represented in various ways (numerically, graphically, algebraically, or verbally) (**F-IF.C.9**).

Classwork

Opening (2 minutes)

Review the formula $T(t) = T_a + (T_0 - T_a) \cdot e^{-kt}$ that was first introduced in Algebra I. There is one difference in the current presentation of the formula; in Algebra I, the base was expressed as 2.718 because students had not yet learned about the number e. Allow students a minute to examine the given formula. Before they begin working, discuss each parameter in the formula as a class.

MP.2

- What does T_a represent? T_0? k? $T(t)$?
  - *The constant T_a represents the temperature surrounding the object, often called the ambient temperature. The initial temperature of the object is denoted by T_0. The constant k is called the decay constant. The temperature of the object after time t has elapsed is denoted by $T(t)$.*
- Is e one of the parameters in the formula?
  - *No; the number e is a constant that is approximately equal to 2.718.*
- Assuming that the temperature of the object is greater than the temperature of the environment, is this formula an example of exponential growth or decay?
  - *It is an example of decay, because the temperature is decreasing.*
- Why would it be decay when the base e is greater than 1? Shouldn't that be exponential growth?
  - *The base is raised to a negative exponent. If we rewrite the exponential expression using properties of exponents, we see that $e^{-kt} = \left(\frac{1}{e}\right)^{kt}$, and $\frac{1}{e} < 1$. In this form, we can clearly identify exponential decay.*

Newton's law of cooling is used to model the temperature of an object placed in an environment of a different temperature. The temperature of the object t hours after being placed in the new environment is modeled by the formula

$$T(t) = T_a + (T_0 - T_a) \cdot e^{-kt},$$

where:

$T(t)$ is the temperature of the object after a time of t hours has elapsed,

T_a is the ambient temperature (the temperature of the surroundings), assumed to be constant and not impacted by the cooling process,

T_0 is the initial temperature of the object, and

k is the decay constant.

Scaffolding:

Use the interactive demonstration on Wolfram Alpha that was used in Algebra I to assist in analyzing the formula.

http://demonstrations.wolfram.com/NewtonsLawOfCooling/

Mathematical Modeling Exercise 1 (15 minutes)

Have students work in groups on parts (a) and (b) of the exercise. Circulate the room and provide assistance as needed. Stop and debrief to ensure that students set up the equations correctly. Discuss the next scenario as a class before having students continue through the exercise.

Mathematical Modeling Exercise 1

A crime scene investigator is called to the scene of a crime where a dead body has been found. He arrives at the scene and measures the temperature of the dead body at 9:30 p.m. to be $78.3°\text{F}$. He checks the thermostat and determines that the temperature of the room has been kept at $74°\text{F}$. At 10:30 p.m., the investigator measures the temperature of the body again. It is now $76.8°\text{F}$. He assumes that the initial temperature of the body was $98.6°\text{F}$ (normal body temperature). Using this data, the crime scene investigator proceeds to calculate the time of death. According to the data he collected, what time did the person die?

a. Can we find the time of death using only the temperature measured at 9:30 p.m.? Explain.

No. There are two parameters that are unknown, k and t. We need to know the decay constant, k, in order to be able to find the elapsed time.

b. Set up a system of two equations using the data.

Let t_1 represent the elapsed time from the time of death until 9:30 when the first measurement was taken, and let t_2 represent the elapsed time between the time of death and 10:30 when the second measurement was taken. Then $t_2 = t_1 + 1$. We have the following equations:

$$T(t_1) = 74 + (98.6 - 74)e^{-kt_1}$$
$$T(t_2) = 74 + (98.6 - 74)e^{-kt_2}.$$

Substituting in our known value $T(t_1) = 78.3$ and $T(t_2) = 76.8$, we get the system:

$$78.3 = 74 + (98.6 - 74)e^{-kt_1}$$
$$76.8 = 74 + (98.6 - 74)e^{-k(t_1+1)}.$$

- Why do we need two equations to solve this problem?
  - *Because there are two unknown parameters.*
- What does t_1 represent in the equation? Why does the second equation contain $(t_1 + 1)$ instead of just t_1?
  - *The variable t_1 represents the elapsed time from time of death to 9:30 p.m. The second equation uses $(t_1 + 1)$ because the time of the second measurement is one hour later, so one additional hour has passed.*

MP.2 & MP.3

- Joanna set up her equations as follows:

$$78.3 = 74 + (98.6 - 74)e^{-k(t_2-1)}$$
$$76.8 = 74 + (98.6 - 74)e^{-kt_2}$$

- In her equations, what does t_2 represent?
  - *Elapsed time from time of death to 10:30 p.m.*
- Will she still find the same time of death? Explain why.

If students are unsure, have some groups work through the problem using one set of equations and some using the other. Re-address this question at the end.

  - *Yes, she will still find the same time of death. She will find a value of t that is one hour greater since she is measuring elapsed time to 10:30 rather than 9:30, but she will still get the same time of death.*

MP.1

- Now that we have this system of equations, how should we go about solving it?

Allow students to struggle with this for a few minutes. They may propose subtracting 74 from both sides or subtracting $98.6 - 74$.

$$4.3 = 24.6e^{-kt_1}$$
$$2.8 = 24.6e^{-k(t_1+1)}$$

- What do we need to do now?
  - *Combine the two equations in some way using the method of substitution or elimination.*
- What is our goal in doing this?
  - *We want to eliminate one of the variables.*
- Would it be helpful to subtract the two equations? If students say yes, have them try it.
  - *No. Subtracting one equation from the other did not eliminate a variable.*
- How else could we combine the equations?
  - *We could use the multiplication property of equality to divide 4.3 by 2.8 and $24.6e^{-kt_1}$ by $24.6e^{-k(t_1+1)}$.*

If nobody offers this suggestion, lead students to the idea by reminding them of the properties of exponents. If we divide the exponential expressions, we subtract the exponents and eliminate the variable t_1.

Have students continue the rest of the problem in groups.

c. Find the value of the decay constant, k.

$$\mathbf{4.3 = 24.6e^{-kt_1}}$$
$$\mathbf{2.8 = 24.6e^{-k(t_1+1)}}$$
$$\mathbf{\frac{4.3}{2.8} = e^{-kt_1+k(t_1+1)}}$$
$$\mathbf{\frac{4.3}{2.8} = e^{k}}$$
$$\mathbf{\ln\left(\frac{4.3}{2.8}\right) = \ln(e^{k})}$$
$$\mathbf{\ln\left(\frac{4.3}{2.8}\right) = k}$$
$$\mathbf{k \approx 0.429}$$

d. **What was the time of death?**

$$4.3 = 24.6e^{-0.429t_1}$$
$$\frac{4.3}{24.6} = e^{-0.429t_1}$$
$$\ln\left(\frac{4.3}{24.6}\right) = \ln\left(e^{-0.429t_1}\right)$$
$$\ln\left(\frac{4.3}{24.6}\right) = -0.429t_1$$
$$4.0656 = t_1$$

The person died approximately 4 hours before 9:30 p.m., so the time of death was approximately 5:30 p.m.

- Would we get the same time of death if we used the set of equations where t_2 represents time elapsed from death until 10:30 p.m.?
  - *Yes. We would find that $t_2 = 5$, so the death occurred 5 hours before 10:30 p.m., at 5:30 p.m.*

Mathematical Modeling Exercise 2 (10 minutes)

Allow students time to work in groups before discussing responses as a class. During the debrief, share and discuss work from different groups.

Mathematical Modeling Exercise 2

A pot of tea is heated to $90°C$. A cup of the tea is poured into a mug and taken outside where the temperature is $18°C$. After 2 minutes, the temperature of the cup of tea is approximately $65°C$.

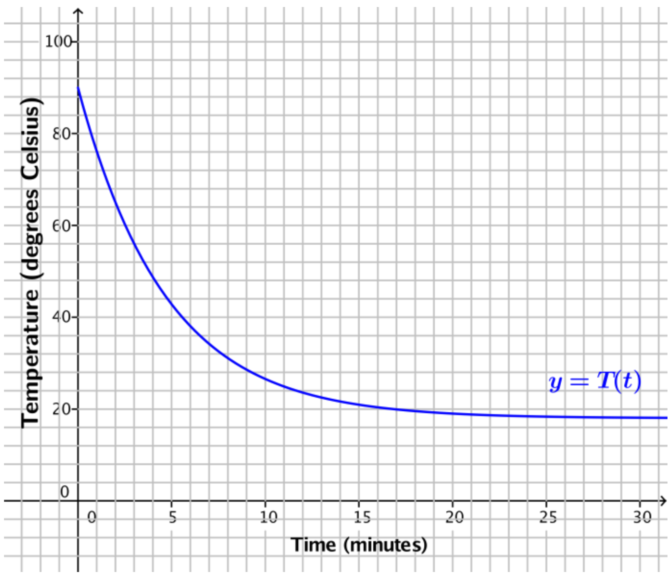
a. **Determine the value of the decay constant, k.**

$$T(2) = 18 + (90 - 18)e^{-k\cdot 2} = 65$$
$$72e^{-2k} = 47$$
$$e^{-2k} = \frac{47}{72}$$
$$-2k = \ln\left(\frac{47}{72}\right)$$
$$k \approx 0.2133$$

b. **Write a function for the temperature of the tea in the mug, T, in °C, as a function of time, t, in minutes.**

$T(t) = 18 + 72\,e^{-0.213t}$

c. Graph the function T.



Scaffolding:

Provide struggling students with a graphing calculator or other graphing utility so that they can better focus on the key concepts of the lesson.

d. Use the graph of T to describe how the temperature decreases over time.

Because the temperature is decreasing exponentially, the temperature drops rapidly at first and then slows down. After about 25 minutes, the temperature of the tea levels off.

e. Use properties of exponents to rewrite the temperature function in the form $T(t) = 18 + 72(1+r)^t$.

$$\begin{aligned} T(t) &= 18 + 72\, e^{-0.213\, t} \\ &= 18 + 72(e^{-0.213})^t \\ &\approx 18 + 72(0.8082)^t \\ &\approx 18 + 72(1 - 0.1918)^t \end{aligned}$$

f. In Lesson 26, we saw that the value of r represents the percent change of a quantity that is changing according to an exponential function of the form $f(t) = A(1+r)^t$. Describe what r represents in the context of the cooling tea.

The number r represents the percent change in the difference between the temperature of the tea and the temperature of the room. Because $r = -0.1918$, the temperature difference is decreasing by 19.18% each minute.

g. As more time elapses, what temperature does the tea approach? Explain using both the context of the problem and the graph of the function T.

The temperature of the tea approaches $18°\text{C}$. Within the context of the problem, this makes sense because that is the ambient temperature (the outside temperature), so when the tea reaches $18°\text{C}$ it stops cooling. Looking at the expression of the function T, we see that as $t \to \infty$, $(0.8082)^t \to 0$, so $T \to 18$.

Mathematical Modeling Exercise 3 (10 minutes)

Newton's law of cooling also applies when a cooler object is placed in an area with a warmer surrounding temperature. (In this case, we could call it Newton's law of heating.) Allow students time to work in groups before discussing responses as a class. During the debrief, share and discuss work from different groups.

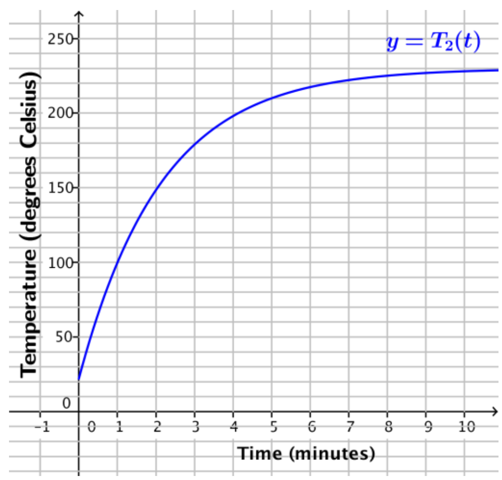
Mathematical Modeling Exercise 3

Two thermometers are sitting in a room that is $22°C$. When each thermometer reads $22°C$, the thermometers are placed in two different ovens. Data for the temperatures T_1 and T_2 of these thermometers (in $°C$) t minutes after being placed in the oven is provided below.

Thermometer 1:

t (minutes)	0	2	5	8	10	14
T_1 (°C)	22	75	132	173	175	176

Thermometer 2:



a. **Do the table and graph given for each thermometer support the statement that Newton's law of cooling also applies when the surrounding temperature is warmer? Explain.**

Yes. The graph shows a reflected exponential curve, which would indicate that a similar formula could be used. From both the table and the graph, it can be seen that the temperature increases rapidly at first and then levels off to the temperature of its surroundings; this coincides with what happens when an object is cooling (i.e., the temperature decreases rapidly and then levels off).

b. **Which thermometer was placed in a hotter oven? Explain.**

Thermometer 2 was placed in a hotter oven. The graph shows its temperature leveling off at approximately $230°C$, while the table indicates that thermometer 1 levels off at approximately $176°C$.

c. **Using a generic decay constant, k, without finding its value, write an equation for each thermometer expressing the temperature as a function of time.**

Thermometer 1: $T_1(t) = 176 + (22 - 176)e^{-kt}$

Thermometer 2: $T_2(t) = 230 + (22 - 230)e^{-kt}$

d. **How do the equations differ when the surrounding temperature is warmer than the object rather than cooler as in previous examples?**

In the case where we are placing a cool object into a warmer space, the coefficient in front of the exponential expression is negative rather than positive.

e. **How do the graphs differ when the surrounding temperature is warmer than the object rather than cooler as in previous examples?**

In the case where we are placing a cool object into a warmer space, the function increases rather than decreases.

Closing (3 minutes)

Use the closing to highlight how this lesson built on their experiences from Algebra I with exponential decay and transformations of functions as well as the content learned in this module, such as the number e and logarithms.

- For Exercise 2, describe the transformations required to graph T starting from the graph of the natural exponential function $f(t) = e^t$.
  - *The graph is reflected across the y-axis, stretched both vertically and horizontally, and translated up.*
- Why were logarithms useful in exploring Newton's law of cooling?
  - *It allowed us to find the decay constant or the amount of time elapsed, both of which involve solving an exponential equation.*
- How do you find the percent rate of change of the temperature difference from the Newton's law of cooling equation?
  - *Rewrite $T(t) = T_0 + (T_a - T_0)e^{-kt}$ as $T(t) = T_0 + (T_a - T_0)(e^{-k})^t$, then express e^{-k} as $e^{-k} = 1 - r$, for some number r. Then r represents the percent rate of change of the temperature difference.*

Exit Ticket (5 minutes)

Name ___________________________ Date ________________

Lesson 28: Newton's Law of Cooling, Revisited

Exit Ticket

A pizza, heated to a temperature of $400°\text{F}$, is taken out of an oven and placed in a $75°\text{F}$ room at time $t = 0$ minutes. The temperature of the pizza is changing such that its decay constant, k, is 0.325. At what time is the temperature of the pizza $150°\text{F}$ and, therefore, safe to eat? Give your answer in minutes.

Exit Ticket Sample Solutions

A pizza, heated to a temperature of $400°$ Fahrenheit, is taken out of an oven and placed in a $75°$F room at time $t = 0$ minutes. The temperature of the pizza is changing such that its decay constant, k, is 0.325. At what time is the temperature of the pizza $150°$F and, therefore, safe to eat? Give your answer in minutes.

$$T(t) = 75 + (400 - 75)e^{-0.325t} = 150$$
$$325e^{-0.325t} = 75$$
$$e^{-0.325t} = \frac{75}{325}$$
$$-0.325t = \ln\left(\frac{75}{325}\right)$$
$$t \approx 4.512$$

The pizza will reach $150°F$ after approximately $4\frac{1}{2}$ minutes.

Problem Set Sample Solutions

1. **Experiments with a covered cup of coffee show that the temperature (in degrees Fahrenheit) of the coffee can be modeled by the following equation:**

$$f(t) = 112e^{-0.08t} + 68,$$

where the time is measured in minutes after the coffee was poured into the cup.

a. **What is the temperature of the coffee at the beginning of the experiment?**

$180°$F

b. **What is the temperature of the room?**

$68°$F

c. **After how many minutes is the temperature of the coffee $140°$F? Give your answer to 3 decimal places.**

5.523 *minutes*

d. **What is the temperature of the coffee after a few hours have elapsed?**

The temperature is slightly above $68°$F.

e. **What is the percent rate of change of the difference between the temperature of the room and the temperature of the coffee?**

$$f(t) = 112(e^{-0.08t}) + 68$$
$$= 112(e^{-0.08})^t + 68$$
$$\approx 112(0.9231)^t + 68$$
$$\approx 112(1 - 0.0769)^t + 68$$

Thus, the percent rate of change of the temperature difference is a decrease of 7.69% each minute.

2. Suppose a frozen package of hamburger meat is removed from a freezer that is set at $0°\text{F}$ and placed in a refrigerator that is set at $38°\text{F}$. Six hours after being placed in the refrigerator, the temperature of the meat is $12°\text{F}$.

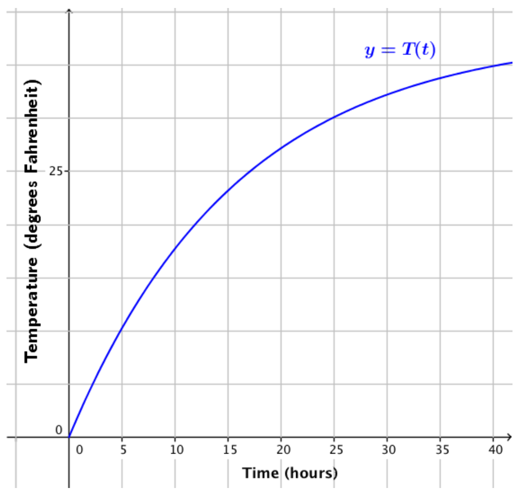
a. Determine the decay constant, k.

$k = 0.063$

b. Write a function for the temperature of the meat, T in Fahrenheit, as a function of time, t in hours.

$T(t) = 38 - 38e^{-0.063t}$

c. Graph the function T.



d. Describe the transformations required to graph the function T beginning with the graph of the natural exponential function $f(t) = e^t$.

The graph is stretched horizontally, reflected across the y-axis, stretched vertically, reflected across the x-axis, and translated up.

e. How long will it take the meat to thaw (reach a temperature above $32°\text{F}$)? Give answer to three decimal places.

29.299 *hours*

f. What is the percent rate of change of the difference between the temperature of the refrigerator and the temperature of the meat?

$$\begin{aligned} T(t) &= 38 - 38e^{-0.063t} \\ &\approx 38 - 38(0.9389)^t \\ &\approx 38 - 38(1 - 0.0611)^t \end{aligned}$$

So, the percent rate of change in the difference of temperature is 6.11%.

3. The table below shows the temperature of a pot of soup that was removed from the stove at time $t = 0$.

t (min)	0	10	20	30	40	50	60
T (° C)	100	34.183	22.514	20.446	20.079	20.014	20.002

a. What is the initial temperature of the soup?

$100°\text{C}$

b. What does the ambient temperature (room temperature) appear to be?

$20°\text{C}$

c. Use the temperature at $t = 10$ minutes to find the decay constant, k.

$k = 0.173$

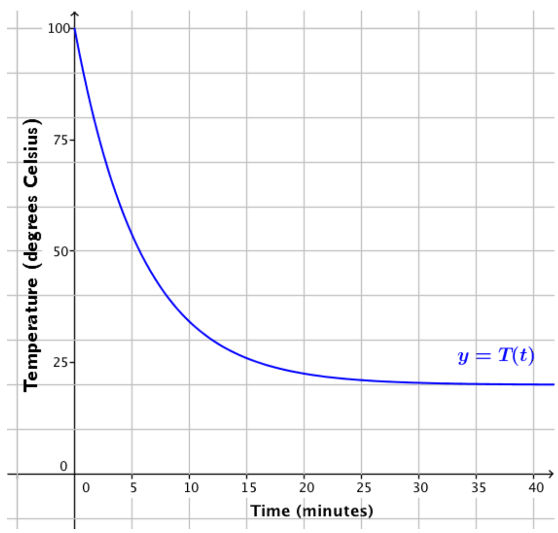
d. Confirm the value of k by using another data point from the table.

$T(40) = 20 + 80e^{-0.173 \cdot 40} \approx 20.079$

e. Write a function for the temperature of the soup (in Celsius) as a function of time in minutes.

$T(t) = 20 + 80e^{-0.173t}$

f. Graph the function T.



4. Match each verbal description with its correct graph and write a possible equation expressing temperature as a function of time.

   a. A pot of liquid is heated to a boil and then placed on a counter to cool.

      (iii), $T(t) = 75 + (212 - 75)e^{-kt}$ *(Equations will vary.)*

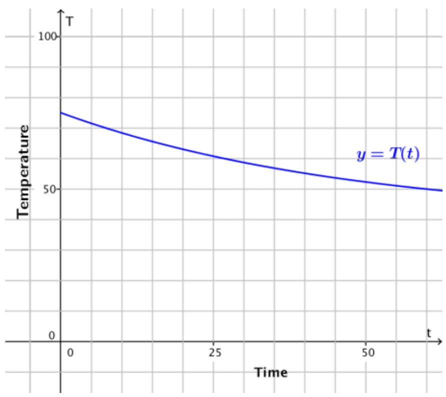
   b. A frozen dinner is placed in a preheated oven to cook.

      (ii), $T(t) = 400 + (32 - 400)e^{-kt}$ *(Equations will vary.)*

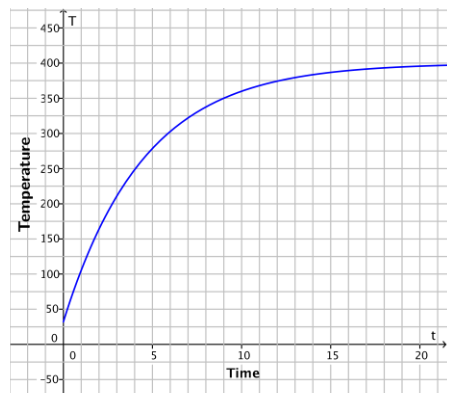
   c. A can of room-temperature soda is placed in a refrigerator.

      (i), $T(t) = 40 + (75 - 40)e^{-kt}$ *(Equations will vary.)*

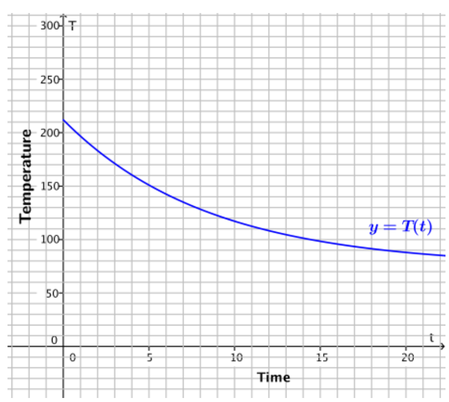
(i)



(ii)



(iii)



Lesson 29: The Mathematics Behind a Structured Savings Plan

Student Outcomes

- Students derive the sum of a finite geometric series formula.
- Students apply the sum of a finite geometric series formula to a structured savings plan.

Lesson Notes

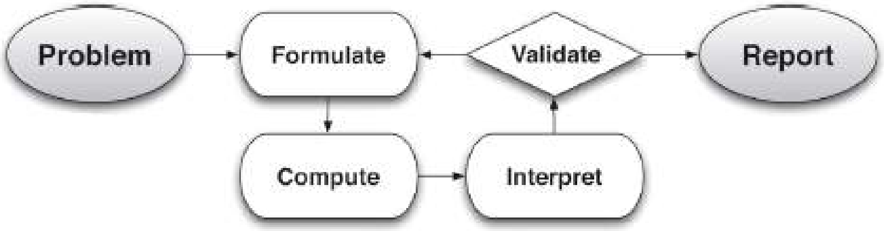
Module 3 ends with a series of lessons centered on finance. In the remaining lessons, students progress through the mathematics of a structured savings plan, buying a car, borrowing on credit cards, and buying a house. The module ends with an investigation of how to save over one million dollars in assets by the time the students are 40 years old. Throughout these lessons, students engage in various parts of the modeling cycle: formulating, computing, interpreting, validating, and re-formulating.

In Lesson 29, students derive the formula for the sum of a finite geometric series (**A-SSE.B.4**). Once established, students work with and develop fluency with summation notation, sometimes called sigma notation. Students are then presented with the problem of a structured savings plan (known as a sinking fund, but this terminology is avoided). Students use the modeling cycle to identify essential features of structured savings plans and develop a model from the formula for the sum of a finite geometric series. By the end of the lesson, students have both the formula for a sum of a finite geometric series (where the common ratio is not 1) and the formula for a structured savings plan. The structured savings plan is modified in the remaining lessons to apply to other types of loans, and additional formulas are developed based on that of the structured savings plan.

The formula $A_f = R\left(\frac{(1+i^n)-1}{i}\right)$ gives the future value A_f of a structured savings plan with recurring payment R, interest rate i per compounding period, and number of compounding periods n. In the context of loans, sometimes P is used to represent the payment instead of R, but R has been chosen because the formula is first presented as a structured savings plan and this notation avoids conflict with the common practice of using P for principal.

Recall that the definition of a sequence in high school is simply a function whose domain is the natural numbers or non-negative integers. Students have worked with arithmetic and geometric sequences in Module 3 of Algebra I as well as in Lesson 25 of this module. Sequences can be described both recursively and explicitly (**F-BF.A.2**). Although students primarily work with geometric sequences in Lessons 29 through 33, arithmetic sequences are reviewed in the problem sets of the lessons.

A copy of the modeling cycle flowchart is included below to assist with the modeling portions of these lessons. Whenever students consider a modeling problem, they need to first identify variables representing essential features in the situation, formulate a model to describe the relationships between the variables, analyze and perform operations on the relationships, interpret their results in terms of the original situation, validate their conclusions, and either improve the model or report on their conclusions and their reasoning. Both descriptive and analytic modeling are used. Suggestions are made for the teacher, but the full extent of modeling done in the first three lessons in this topic is left to the discretion of the teacher and can be adapted as time permits.



Note: Consider breaking this lesson up over two days.

Classwork

Opening Exercise (3 minutes)

This is a quick exercise designed to remind students of the future value function $F = P\left(1+\frac{r}{n}\right)^{nt}$ from Lesson 26.

> **Opening Exercise**
>
> **Suppose you invested $\$1,000$ in an account that paid an annual interest rate of 3% compounded monthly. How much would you have after 1 year?**
>
> ***Since $F = 1000\left(1+\frac{0.03}{12}\right)^{12}$, we have $F = (1.0025)^{12} = \$1,030.42$.***

After students have found the answer, have the following discussion *quickly*. It is important for setting up the notation for Topic E.

- To find the percent rate of change in the problem above, we took the annual interest rate of 3% compounded monthly (called the nominal APR) and divided it by 12. How do we express that in the future value formula from Lesson 26 using r and n?
  - *We can express the percent rate of change in the future value formula by using the expression $\frac{r}{n}$, where r is the unit rate associated to the nominal APR and n is the number of compounding periods in a year.*
- In today's lesson and in the next set of lessons, we are going to replace $\frac{r}{n}$ with something simpler: We use the letter i to represent the unit rate of the percent rate of change per time period (i.e., the interest rate per compounding period). What is the value of i in the problem above?
  - $i = 0.0025$
- Also, since we only need to make calculations based upon the number of time periods, we can use m to stand for the total number of time periods (i.e., total number of compounding periods). What is m in the problem above?
  - $m = 12$
- Thus, the formula we work with is $F = P(1+i)^m$ where P is the present value and F is the future value. This formula helps make our calculations more transparent, but we need to always remember to find the interest rate per compounding period first.

Discussion (7 minutes)

MP.4 & MP.8

This discussion sets up the reason for studying geometric series: Finding the future value of a structured savings plan. After this discussion, we develop the formula for the sum of a series and then return to this discussion to show how to use the formula to find the future value of a structured savings plan. This sum of a geometric series is brought up again and again throughout the remainder of the finance lessons to calculate information on structured savings plans, loans, annuities, and more. Look for ways you can use the discussion to help students formulate the problem.

- Let us consider the example of depositing $\$100$ at the end of every month into an account for 12 months. If there is no interest earned on these deposits, how much money do we have at the end of 12 months?
  - *We have* $12 \cdot \$100 = \$1{,}200$. *(Some students may be confused about the first and last payments. Have them imagine they start a job where they get paid at the end of each month, and they put* $\$100$ *from each paycheck into a special account. Then they would have* $\$0$ *in the account until the end of the first month at which time they deposit* $\$100$. *Similarly, they would have* $\$200$ *at the end of month* 2, $\$300$ *at the end of month* 3, *and so on.)*
- Now let us make an additional assumption: Suppose that our account is earning an annual interest rate of 3% compounded monthly. We would like to find how much will be in the account at the end of the year. The answer requires several calculations, not just one.
- What is the interest rate per month?
  - $i = 0.025$
- How much would the $\$100$ deposited at the end of month 1 and its interest be worth at the end of the year?
  - *Since the interest on the* $\$100$ *would be compounded* 11 *times, it and its interest would be worth* $\$100(1.025)^{11}$ *using the formula* $F = P(1+i)^m$. *(Some students may calculate the value to be* $\$131.21$, *but tell them to leave their answers in exponential form for now.)*
- How much would the $\$100$ deposited at the end of month 2 and its interest be worth at the end of the year?
  - $\$100(1.025)^{10}$
- Continue: Find how much the $\$100$ deposited at the end of the remaining months plus its compounded interest is worth at the end of the year by filling out the table:

Month Deposited	Amount at the End of the Year, in Dollars
1	$100(1.025)^{11}$
2	$100(1.025)^{10}$
3	$100(1.025)^{9}$
4	$100(1.025)^{8}$
5	$100(1.025)^{7}$
6	$100(1.025)^{6}$
7	$100(1.025)^{5}$
8	$100(1.025)^{4}$
9	$100(1.025)^{3}$
10	$100(1.025)^{2}$
11	$100(1.025)$
12	100

- Every deposit except for the final deposit earns interest. If we list the calculations from the final deposit to the first deposit, we get the following sequence:

$$100, 100(1.025)^1, 100(1.025)^2, 100(1.025)^3, \dots, 1000(1.025)^{10}, 1000(1.02)^{11}$$

- What do you notice about this sequence?
    - *The sequence is geometric with initial term* 100 *and common ratio* 1.025*.*
- Describe how to calculate the amount of money we will have at the end of 12 months.
    - *Sum the* 12 *terms in the geometric sequence to get the total amount in the account at the end of* 12 *months.*
- The answer,

$$100 + 100(1.025) + 100(1.025)^2 + \cdots + 100(1.025)^{11},$$

  is an example of a geometric series. In the next example, we find a formula to find the sum of this series quickly. For now, let's discuss the definition of series. When we add some of the terms of a sequence we get a series:

- **SERIES**: Let $a_1, a_2, a_3, a_4, \dots$ be a sequence of numbers. A sum of the form

$$a_1 + a_2 + a_3 + \cdots + a_n$$

  for some positive integer n is called a *series* (or *finite series*) and is denoted S_n.

  The a_i's are called the *terms* of the series. The number S_n that the series adds to is called the *sum* of the series.

- **GEOMETRIC SERIES:** A *geometric series* is a series whose terms form a geometric sequence.

  Since a geometric sequence is of the form $a, ar, ar^2, ar^3, \dots$, the general form of a finite geometric sequence is of the form

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1}.$$

- In the geometric series we generated from our structured savings plan, what is the first term a? What is the common ratio r? What is n?
    - $a = 100$, $r = 1.025$, *and* $n = 12$

Example 1 (10 minutes)

Work through the following example to establish the formula for finding the sum of a finite geometric series. In the example, we calculate the general form of a sum of a generic geometric series using letters a and r. We use letters instead of a numerical example for the following reasons:

- Using letters in this situation actually makes the computations clearer. Students need only keep track of two letters instead of several numbers (as in $3, 6, 12, 24, 48, 96, \dots$).

- Students have already investigated series several times before in different forms without realizing it. For example, throughout Topic A of Module 1 of Algebra II, they worked with the identity $(1-r)(1+r^2+r^3+\cdots+r^{n-1}) = 1-r^n$ for any real number r, and they even used the identity to find the sum $1+2+2^2+2^3+\cdots+2^{31}$. In fact, students already derived the geometric series sum formula in the Problem Set of Lesson 1 of this module.

However, depending upon your class's strengths, you might consider doing a "mirror" calculation. On the left half of your board/screen do the calculation below, and on the right half "mirror" a numerical example; that is, for every algebraic line you write on the left, write the corresponding line using numbers on the right. A good numerical series to use is $2+6+18+54+162$.

Example 1

Let $a, ar, ar^2, ar^3, ar^4, \ldots$ be a geometric sequence with first term a and common ratio r. Show that the sum S_n of the first n terms of the geometric series

$$S_n = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} \qquad (r \neq 1)$$

is given by the equation

$$S_n = a\left(\frac{1-r^n}{1-r}\right).$$

Give students 2–3 minutes to try the problem on their own (or in groups of two). It is completely okay if no one gets an answer—you are giving them structured time to persevere, test conjectures, and grapple with what they are expected to show. Give a hint as you walk around the class: What identity from Module 1 can we use? Or, just ask them to see if the formula works for $a=1$, $r=2$, $n=5$. If a student or group of students solves the problem, tell them to hold on to their solution for a couple of minutes. Then go through this discussion:

Scaffolding:

- For struggling students, do this calculation twice, first using a concrete sequence such as $a=3$ and $r=2$ before generalizing.

MP.7

- Multiply both sides of the equation $S_n = a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1}$ by r.
  - $r \cdot S_n = ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n$
- Compare the terms on the right-hand side of the old and new equations:

$$S_n = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}$$
$$r \cdot S_n = ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n$$

- What are the only terms on the right-hand side of the original equation and the new equation that are not found in both?
  - *The only terms that are not found in both are a and ar^n.*
- Therefore, when we subtract $S_n - rS_n$, all the common terms on the right-hand side of the equations subtract to zero leaving only $a - ar^n$ (after applying the associative and commutative properties repeatedly):

$$\begin{aligned} S_n - rS_n &= (a + ar + ar^2 + \cdots + ar^{n-1}) - (ar + ar^2 + \cdots + ar^{n-1} + ar^n) \\ &= a + (ar - ar) + (ar^2 - ar^2) + \cdots + (ar^{n-1} - ar^{n-1}) - ar^n \\ &= a - ar^n. \end{aligned}$$

- Isolate S_n in the equation $S_n - rS_n = a - ar^n$ to get the formula (when $r \neq 1$):

$$S_n(1-r) = a(1-r^n) \implies S_n = a\left(\frac{1-r^n}{1-r}\right).$$

Any student who used the identity equation $(1-r)(1+r+r^2+r^3+\cdots+r^{n-1}) = 1-r^n$ or a different verifiable method (like proof by induction) should be sent to the board to explain the solution to the rest of class as an alternative explanation to your explanation. If no one was able to show the formula, you can wrap up the explanation by linking the formula back to the work they have done with this identity: Divide both sides of the identity equation by $(1-r)$ to isolate the sum $1+r+r^2+\cdots+r^{n-1}$ and then for $r \neq 1$,

$$a+ar+ar^2+\cdots+ar^{n-1} = a(1+r+r^2+\cdots+r^{n-1}) = a\left(\frac{1-r^n}{1-r}\right).$$

Exercises 1–3 (5 minutes)

> *Scaffolding:*
> Ask advanced students to first find the sum
> $a+ar^2+ar^4+\cdots+ar^{2n-2}$
> and then the sum
> $ar+ar^3+ar^5+\cdots+ar^{2n-1}$.

Exercises 1–3

1. **Find the sum of the geometric series $3+6+12+24+48+96+192$.**

   The first term is 3, so $a=3$. The common ratio is $6/3=2$, so $r=2$. Since there are 7 terms, $n=7$.

   Thus, $S_7 = 3\cdot\frac{1-2^7}{1-2}$, or $S_7 = 381$.

2. **Find the sum of the geometric series $40+40(1.005)+40(1.005)^2+\cdots+40(1.005)^{11}$.**

   Reading directly off the sum, $a=40$, $r=1.005$, and $n=12$.

   Thus, $S_{12} = 40\cdot\left(\frac{1-1.005^{12}}{1-1.005}\right)$, so $S_{12}\approx 493.42$.

3. **Describe a situation that might lead to calculating the sum of the geometric series in Exercise 2.**

   Investing $\$40$ a month into an account with an annual interest rate of 6% compounded monthly (or investing $\$40$ a month into an account with an interest rate of 0.05% per month) can lead to the geometric series in Exercise 2.

Example 2 (2 minutes)

Let's now return to the Opening Exercise and answer the problem we encountered there.

Example 2

A $\$100$ deposit is made at the end of every month for 12 months in an account that earns interest at an annual interest rate of 3% compounded monthly. How much will be in the account immediately after the last payment?

Answer: The total amount is the sum $100+100(1.025)+100(1.025)^2+\cdots+100(1.025)^{11}$. This is a geometric series with $a=100$, $r=1.025$, and $n=12$. Using the formula for the sum of a geometric series,

$S_{12} = 100\left(\frac{1-1.025^{12}}{1-1.025}\right)$, so $S\approx 1379.56$. The account will have $\$1{,}379.56$ in it immediately after the last payment.

Point out to your students that \$1,379.56 is significantly more money than stuffing \$100 in your mattress every month for 12 months. A structured savings plan like this is one way people can build wealth over time. Structured savings plans are examples of *annuities*. An annuity is commonly thought of as a type of retirement plan, but in this lesson, we are using the term in a much simpler way to refer to any situation where money is transferred into an account in equal amounts on a regular, recurring basis.

Discussion (5 minutes)

Explain the following text to your students, and work with them to answer the questions below it.

Discussion

An *annuity* is a series of payments made at fixed intervals of time. Examples of annuities include structured savings plans, lease payments, loans, and monthly home mortgage payments. The term annuity sounds like it is only a yearly payment, but annuities often require payments monthly, quarterly, or semiannually. The *future amount of the annuity,* denoted A_f, is the sum of all the individual payments made plus all the interest generated from those payments over the specified period of time.

We can generalize the structured savings plan example above to get a generic formula for calculating the future value of an annuity A_f in terms of the recurring payment R, interest rate i, and number of payment periods n. In the example above, we had a recurring payment of $R = 100$, an interest rate per time period of $i = 0.025$, and 12 payments, so $n = 12$. To make things simpler, we always assume that the payments and the time period in which interest is compounded are at the same time. That is, we do not consider plans where deposits are made halfway through the month with interest compounded at the end of the month.

In the example, the amount A_f of the structured savings plan annuity was the sum of all payments plus the interest accrued for each payment:

$$A_f = R + R(1+i)^1 + R(1+i)^2 + \cdots + R(1+i)^{n-1}.$$

This, of course, is a geometric series with n terms, $a = R$, and $r = 1 + i$, which after substituting into the formula for a geometric series and rearranging is

$$A_f = R\left(\frac{(1+i)^n - 1}{i}\right).$$

- Explain how to get the formula above from the sum of a geometric series formula.
  - *Substitute R, $1 + i$, and n into the geometric series and rearrange:*

$$A_f = R\left(\frac{1-(1+i)^n}{1-(1+i)}\right) = R\left(\frac{1-(1+i)^n}{-i}\right) = R\left(\frac{(1+i)^n - 1}{i}\right).$$

- (Optional depending on time.) How much money would need to be invested every month into an account with an annual interest rate of 12% compounded monthly in order to have $\$3{,}000$ after 18 months?
  - $3000 = R\left(\frac{(1.01)^{18}-1}{0.01}\right)$. *Solving for R yields $R \approx 152.95$. For such a savings plan, $\$152.95$ would need to be invested monthly.*

Example 3 (5 minutes)

Example 3 develops summation notation, sometimes called sigma notation. Including summation notation at this point is natural to reduce the amount of writing in sums and to develop fluency before relying on summation notation in precalculus and calculus. Plus, it "wraps up" all the important components of describing a series (starting value, index, explicit formula, and ending value) into one convenient notation. We further explore summation notation in the problem sets of Lessons 29 through 33 to develop fluency with the notation. Summation notation is specifically used to represent series. Students may be interested to know that a similar notation exists for products; product notation makes use of a capital pi, Π, instead of a capital sigma but is otherwise identical in form to summation notation. For example, we can represent $n!$ by $\prod_{k=1}^{n} k = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n-1) \cdot n = n!$.

Mathematicians use notation to reduce the amount of writing and to prevent ambiguity in mathematical sentences. Mathematicians use a special symbol with sequences to indicate that we would like to sum up the sequence. This symbol is a capital sigma, Σ. The sum of a sequence is called a *series*.

- The first letter of summation is "S," and the Greek letter for "S" is sigma. Capital sigma looks like this: Σ.
- There is no rigid way to use Σ to represent a summation, but all notations generally follow the same rules. Let's discuss the most common way it used. Given a sequence $a_1, a_2, a_3, a_4, \ldots$, we can write the sum of the first n terms of the sequence using the expression:

$$\sum_{k=1}^{n} a_k.$$

- It is read, "The sum of a_k from $k = 1$ to $k = n$." The letter k is called the index of the summation. The notation acts like a for-next loop in computer programming: Replace k in the expression right after the sigma by the integers (in this case) $1, 2, 3, \ldots, n$, and add the resulting expressions together. Since

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_n,$$

the notation can be used to greatly simplify writing out the sum of a series.

- When the terms in a sequence are given by an explicit function, like the geometric sequence given by $a_k = 2^k$ for example, we often use the expression defining the explicit function instead of sequence notation. For example, the sum of the first five powers of 2 can be written as

$$\sum_{k=1}^{5} 2^k = 2^1 + 2^2 + 2^3 + 2^4 + 2^5.$$

Exercises 4–5 (3 minutes)

Exercises 4–5

4. Write the sum without using summation notation, and find the sum.

a. $\sum_{k=0}^{5} k$

$$\sum_{k=0}^{5} k = 0 + 1 + 2 + 3 + 4 + 5 = 15$$

b. $\sum_{j=5}^{7} j^2$

$$\sum_{j=5}^{7} j^2 = 5^2 + 6^2 + 7^2 = 25 + 36 + 49 = 110$$

c. $\sum_{i=2}^{4} \frac{1}{i}$

$$\sum_{i=2}^{4} \frac{1}{i} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{13}{12}$$

5. **Write each sum using summation notation. Do not evaluate the sum.**

a. $1^4 + 2^4 + 3^4 + 4^4 + 5^4 + 6^4 + 7^4 + 8^4 + 9^4$

$$\sum_{k=1}^{9} k^4$$

b. $1 + \cos(\pi) + \cos(2\pi) + \cos(3\pi) + \cos(4\pi) + \cos(5\pi)$

$$\sum_{k=0}^{5} \cos(k\pi)$$

c. $2 + 4 + 6 + \cdots + 1000$

$$\sum_{k=1}^{500} 2k$$

Closing (2 minutes)

Have students summarize the lesson in writing or with a partner. Circulate and informally assess understanding. Ensure that every student has the formula for the sum of a finite geometric series and the formula for the future value of a structured savings plan. Ask questions to prompt student memory of the definitions and formulas below.

Lesson Summary

- **Series: Let $a_1, a_2, a_3, a_4, \ldots$ be a sequence of numbers. A sum of the form**
$$a_1 + a_2 + a_3 + \cdots + a_n$$
for some positive integer n is called a *series* (or *finite series*) and is denoted S_n. The a_i's are called the *terms* of the series. The number S_n that the series adds to is called the *sum* of the series.
- **GEOMETRIC SERIES: A *geometric series* is a series whose terms form a geometric sequence.**
- **SUM OF A FINITE GEOMETRIC SERIES: The sum S_n of the first n terms of the geometric series $S_n = a + ar + \cdots + ar^{n-1}$ (when $r \neq 1$) is given by**
$$S_n = a\left(\frac{1 - r^n}{1 - r}\right).$$
- **The sum of a finite geometric series can be written in summation notation as**
$$\sum_{k=0}^{n-1} ar^k = a\left(\frac{1 - r^n}{1 - r}\right).$$
- **The generic formula for calculating the future value of an annuity A_f in terms of the recurring payment R, interest rate i, and number of periods n is given by**
$$A_f = R\left(\frac{(1 + i)^n - 1}{i}\right).$$

Exit Ticket (3 minutes)

Name ______________________________ Date ______________

Lesson 29: The Mathematics Behind a Structured Savings Plan

Exit Ticket

Martin attends a financial planning conference and creates a budget for himself, realizing that he can afford to put away $\$200$ every month in savings and that he should be able to keep this up for two years. If Martin has the choice between an account earning an interest rate of 2.3% yearly versus an account earning an annual interest rate of 2.125% compounded monthly, which account gives Martin the largest return in two years?

Exit Ticket Sample Solutions

MP.2

Martin attends a financial planning conference and creates a budget for himself, realizing that he can afford to put away $\$200$ every month in savings and that he should be able to keep this up for two years. If Martin has the choice between an account earning an interest rate of 2.3% yearly versus an account earning an annual interest rate of 2.125% compounded monthly, which account gives Martin the largest return in two years?

$$A_f = R\left(\frac{(1+i)^n - 1}{i}\right)$$

$$A_f = 2400\left(\frac{(1.023)^2 - 1}{0.023}\right) = 4855.20$$

$$A_f = 200\left(\frac{\left(1+\frac{0.0215}{12}\right)^{24} - 1}{\frac{0.0215}{12}}\right) \approx 4900.21$$

The account earning an interest rate of 2.125% compounded monthly returns more than the yearly account.

Problem Set Sample Solutions

The first problem in the Problem Set asks students to perform the research necessary to personalize Lesson 30. Problems 2–13 develop fluency with summation notation and summing a geometric series. In Problems 14–21, students apply the future value formula to financial scenarios, and the problem set ends with developing a formula for summing arithmetic series.

1. **A car loan is one of the first secured loans most Americans obtain. Research used car prices and specifications in your area to find a reasonable used car that you would like to own (under $\$10,000$). If possible, print out a picture of the car you selected.**

   a. **What is the year, make, and model of your vehicle?**

      Answers will vary. For instance, 2006 Pontiac G6 GT.

   b. **What is the selling price for your vehicle?**

      Answers will vary. For instance, $\$7,500$

c. **The following table gives the monthly cost per $\$1,000$ financed on a 5-year auto loan. Assume you have qualified for a loan with a 5% annual interest rate. What is the monthly cost of financing the vehicle you selected? (A formula is developed to find the monthly payment of a loan in Lesson 30.)**

Five-Year (60-month) Loan	
Interest Rate	Amount per $1000 Financed
1.0%	$\$17.09$
1.5%	$\$17.31$
2.0%	$\$17.53$
2.5%	$\$17.75$
3.0%	$\$17.97$
3.5%	$\$18.19$
4.0%	$\$18.41$
4.5%	$\$18.64$
5.0%	$\$18.87$
5.5%	$\$19.10$
6.0%	$\$19.33$
6.5%	$\$19.56$
7.0%	$\$19.80$
7.5%	$\$20.04$
8.0%	$\$20.28$
8.5%	$\$20.52$
9.0%	$\$20.76$

Answers will vary. At 5% interest, financing a $\$7,500$ car for 60 months costs $\$18.87 \cdot 7.5$, which is approximately $\$141.53$ per month.

d. **What is the gas mileage for your vehicle?**

Answers will vary. For instance, the gas mileage might be 29 miles per gallon.

e. **Suppose that you drive 120 miles per week and gas costs $\$4$ per gallon. How much does gas cost per month?**

Answers will vary but should be based on the answer to part (d). For instance, with gas mileage of 29 miles per gallon, the gas cost would be $\frac{120}{29} \cdot 4.3 \cdot \$4 \approx \$71.17$ per month.

2. **Write the sum without using summation notation, and find the sum.**

a. $\sum_{k=1}^{8} k$

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = 4 \cdot 9 = 36$$

b. $\sum_{k=-8}^{8} k$

$$-8 + -7 + -6 + -5 + -4 + -3 + -2 + -1 + 0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = 0$$

c. $\sum_{k=1}^{4} k^3$

$$1^3 + 2^3 + 3^3 + 4^3 = 1 + 8 + 27 + 64 = 100$$

d. $\sum_{m=0}^{6} 2m$

$$\begin{aligned} 2 \cdot 0 + 2 \cdot 1 + 2 \cdot 2 + 2 \cdot 3 + 2 \cdot 4 + 2 \cdot 5 + 2 \cdot 6 &= 0 + 2 + 4 + 6 + 8 + 10 + 12 \\ &= 3.5 \cdot 12 \\ &= 42 \end{aligned}$$

e. $\sum_{m=0}^{6} 2m + 1$

$$\begin{aligned} (2 \cdot 0 + 1) + (2 \cdot 1 + 1) + (2 \cdot 2 + 1) + (2 \cdot 3 + 1) + (2 \cdot 4 + 1) + (2 \cdot 5 + 1) + (2 \cdot 6 + 1) &= 42 + 7 \\ &= 49 \end{aligned}$$

f. $\sum_{k=2}^{5} \frac{1}{k}$

$$\begin{aligned} \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} &= \frac{30}{60} + \frac{20}{60} + \frac{15}{60} + \frac{12}{60} \\ &= \frac{77}{60} \end{aligned}$$

g. $\sum_{j=0}^{3} (-4)^{j-2}$

$$\begin{aligned} (-4)^{-2} + (-4)^{-1} + (-4)^{0} + (-4)^{1} &= \frac{1}{16} + -\frac{1}{4} + 1 + -4 \\ &= \frac{1}{16} - \frac{4}{16} + \frac{16}{16} - \frac{64}{16} \\ &= -\frac{51}{16} \end{aligned}$$

h. $\sum_{m=1}^{4} 16\left(\frac{3}{2}\right)^m$

$$\begin{aligned} \left(16\left(\frac{3}{2}\right)^1\right) + \left(16\left(\frac{3}{2}\right)^2\right) + \left(16\left(\frac{3}{2}\right)^3\right) + \left(16\left(\frac{3}{2}\right)^4\right) &= 16\left(\frac{3}{2} + \frac{9}{4} + \frac{27}{8} + \frac{81}{16}\right) \\ &= 16\left(\frac{24}{16} + \frac{36}{16} + \frac{54}{16} + \frac{81}{16}\right) \\ &= 24 + 36 + 54 + 81 \\ &= 195 \end{aligned}$$

i. $\sum_{j=0}^{3} \frac{105}{2j+1}$

$$\begin{aligned}\frac{105}{2\cdot 0+1}+\frac{105}{2\cdot 1+1}+\frac{105}{2\cdot 2+1}+\frac{105}{2\cdot 3+1} &= \frac{105}{1}+\frac{105}{3}+\frac{105}{5}+\frac{105}{7}\\ &= 105+35+21+15\\ &= 176\end{aligned}$$

j. $\sum_{p=1}^{3} p\cdot 3^p$

$$\begin{aligned}1\cdot 3^1+2\cdot 3^2+3\cdot 3^3 &= 3+18+81\\ &= 102\end{aligned}$$

k. $\sum_{j=1}^{6} 100$

$$100+100+100+100+100+100=600$$

l. $\sum_{k=0}^{4} \sin\left(\frac{k\pi}{2}\right)$

$$\begin{aligned}\sin\left(\frac{0\pi}{2}\right)+\sin\left(\frac{1\pi}{2}\right)+\sin\left(\frac{2\pi}{2}\right)+\sin\left(\frac{3\pi}{2}\right)+\sin\left(\frac{4\pi}{2}\right) &= 0+1+0+\ -1+0\\ &= 0\end{aligned}$$

m. $\sum_{k=1}^{9} \log\left(\frac{k}{k+1}\right)$

(Hint: You do not need a calculator to find the sum.)

$$\log\left(\frac{1}{2}\right)+\log\left(\frac{2}{3}\right)+\log\left(\frac{3}{4}\right)+\log\left(\frac{4}{5}\right)+\log\left(\frac{5}{6}\right)+\log\left(\frac{6}{7}\right)+\log\left(\frac{7}{8}\right)+\log\left(\frac{8}{9}\right)+\log\left(\frac{9}{10}\right)$$

$$\begin{aligned}&= \log(1)-\log(2)+\log(2)-\log(3)+\cdots-\log(10)\\ &= \log(1)-\log(10)\\ &= 0-1\\ &= -1\end{aligned}$$

3. Write the sum without using sigma notation. (You do not need to find the sum.)

a. $\sum_{k=0}^{4} \sqrt{k+3}$

$$\sqrt{0+3}+\sqrt{1+3}+\sqrt{2+3}+\sqrt{3+3}+\sqrt{4+3}=\sqrt{3}+\sqrt{4}+\sqrt{5}+\sqrt{6}+\sqrt{7}$$

b. $\sum_{i=0}^{8} x^i$

$$1+x^1+x^2+x^3+x^4+x^5+x^6+x^7+x^8$$

c. $\sum_{j=1}^{6} jx^{j-1}$

$$1x^{1-1} + 2x^{2-1} + 3x^{3-1} + 4x^{4-1} + 5x^{5-1} + 6x^{6-1} = 1 + 2x^1 + 3x^2 + 4x^3 + 5x^4 + 6x^5$$

d. $\sum_{k=0}^{9} (-1)^k x^k$

$$(-1)^0x^0 + (-1)^1x^1 + (-1)^2x^2 + (-1)^3x^3 + (-1)^4x^4 + (-1)^5x^5 + (-1)^6x^6 + (-1)^7x^7 + (-1)^8x^8 + (-1)^9x^9$$
$$= 1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - x^7 + x^8 - x^9$$

4. Write each sum using summation notation.

a. $1 + 2 + 3 + 4 + \cdots + 1000$

$$\sum_{k=1}^{1000} k$$

b. $2 + 4 + 6 + 8 + \cdots + 100$

$$\sum_{k=1}^{50} 2k$$

c. $1 + 3 + 5 + 7 + \cdots + 99$

$$\sum_{k=1}^{50} 2k - 1 \text{ or } \sum_{k=0}^{49} 2k + 1$$

d. $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots + \frac{99}{100}$

$$\sum_{k=1}^{99} \frac{k}{k+1} \text{ or } \sum_{k=2}^{100} \frac{k-1}{k}$$

e. $1^2 + 2^2 + 3^2 + 4^2 + \cdots + 10000^2$

$$\sum_{k=1}^{10000} k^2$$

f. $1 + x + x^2 + x^3 + \cdots + x^{200}$

$$\sum_{k=0}^{200} x^k$$

g. $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{49\cdot 50}$

$$\sum_{k=1}^{49} \frac{1}{k(k+1)} \text{ or } \sum_{k=2}^{50} \frac{1}{(k-1)k}$$

h. $1\ln(1) + 2\ln(2) + 3\ln(3) + \cdots + 10\ln(10)$

$$\sum_{k=1}^{10} k\ln(k)$$

5. **Use the geometric series formula to find the sum of the geometric series.**

a. $1 + 3 + 9 + \cdots + 2187$

$$\left(\frac{1-3^8}{1-3}\right) = \frac{6560}{2} = 3280$$

b. $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{512}$

$$\left(\frac{1-\left(\frac{1}{2}\right)^{10}}{1-\frac{1}{2}}\right) = \left(\frac{\frac{1023}{1024}}{\frac{1}{2}}\right) = \frac{1023}{1024}\cdot 2 = \frac{1023}{512}$$

c. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots - \frac{1}{512}$

$$\left(\frac{1-\left(-\frac{1}{2}\right)^{10}}{1+\frac{1}{2}}\right) = \frac{1023}{1024}\cdot\frac{2}{3} = \frac{341}{512}$$

d. $0.8 + 0.64 + 0.512 + \cdots + 0.32768$

$$0.8\left(\frac{1-0.8^5}{1-0.8}\right) = 0.8\cdot\frac{0.67232}{0.2} = 2.68928$$

e. $1 + \sqrt{3} + 3 + 3\sqrt{3} + \cdots + 243$

$$\left(\frac{1-\sqrt{3}^{11}}{1-\sqrt{3}}\right) \approx 573.5781477$$

f. $\sum_{k=0}^{5} 2^k$

$$\left(\frac{1-2^6}{1-2}\right) = 63$$

g. $\sum_{m=1}^{4} 5\left(\frac{3}{2}\right)^m$

$$\left(5\left(\frac{3}{2}\right)\right)\left(\frac{1-\left(\frac{3}{2}\right)^4}{1-\frac{3}{2}}\right) = \left(\frac{15}{2}\right)\left(\frac{\frac{81}{16}-1}{\frac{1}{2}}\right)$$
$$= \frac{15}{2}\left(\frac{65}{16}\right)2$$
$$= \frac{975}{16}$$
$$= 60.9375$$

h. $1 - x + x^2 - x^3 + \cdots + x^{30}$ in terms of x

$$\left(\frac{1-(-x)^{31}}{1-(-x)}\right) = \frac{x^{31}+1}{x+1}$$

i. $\sum_{m=0}^{11} 4^{\frac{m}{3}}$

$$\left(\frac{1-\left(4^{\frac{1}{3}}\right)^{12}}{1-4^{\frac{1}{3}}}\right) \approx 434.1156679$$

j. $\sum_{n=0}^{14} \left(\sqrt[5]{6}\right)^n$

$$\left(\frac{1-\left(\sqrt[5]{6}\right)^{15}}{1-\sqrt[5]{6}}\right) \approx 498.8756953$$

k. $\sum_{k=0}^{6} 2 \cdot \left(\sqrt{3}\right)^k$

$$2\left(\frac{1-\sqrt{3}^7}{1-\sqrt{3}}\right) \approx 125.033321$$

6. Let a_i represent the sequence of even natural numbers $\{2, 4, 6, 8, \ldots\}$ with $a_1 = 2$. Evaluate the following expressions.

a. $\sum_{i=1}^{5} a_i$

$$2 + 4 + 6 + 8 + 10 = 30$$

b. $\sum_{i=1}^{4} a_{2i}$

$$\begin{aligned} a_2 + a_4 + a_6 + a_8 &= 4 + 8 + 12 + 16 \\ &= 40 \end{aligned}$$

c. $\sum_{i=1}^{5} (a_i - 1)$

$$\begin{aligned} (2-1)+(4-1)+(6-1)+(8-1)+(10-1) &= 1+3+5+7+9 \\ &= 25 \end{aligned}$$

7. Let a_i represent the sequence of integers giving the yardage gained per rush in a high school football game $\{3, -2, 17, 4, -8, 19, 2, 3, 3, 4, 0, 1, -7\}$.

a. Evaluate $\sum_{i=1}^{13} a_i$. What does this sum represent in the context of the situation?

$$\begin{aligned} \sum_{i=1}^{13} a_i &= 3 + \ -2 + 17 + 4 + \ -8 + 19 + 2 + 3 + 3 + 4 + 0 + 1 + \ -7 \\ &= 56 + \ -17 \\ &= 39 \end{aligned}$$

This sum is the total rushing yards.

b. Evaluate $\frac{\sum_{i=1}^{13} a_i}{13}$. What does this expression represent in the context of the situation?

$$\frac{39}{13} = 3$$

The average yardage per rush is 3.

c. In general, if a_n describes any sequence of numbers, what does $\frac{\sum_{i=1}^{n} a_i}{n}$ represent?

The total divided by the number of numbers is the arithmetic mean or average of the set.

8. Let b_n represent the sequence given by the following recursive formula: $b_1 = 10$, $b_n = 5b_{n-1}$.

a. Write the first 4 terms of this sequence.

$$10, 50, 250, 1250$$

b. Expand the sum $\sum_{i=1}^{4} b_i$. Is it easier to add this series, or is it easier to use the formula for the sum of a finite geometric sequence? Explain your answer. Evaluate $\sum_{i=1}^{4} b_i$.

$$\sum_{i=1}^{4} b_i = 10 + 50 + 250 + 1250$$

Answers may vary based on personal opinion. Since this series consists of only four terms, it may be easier to simply add the terms together to find the sum. The sum is 1560.

c. Write an explicit form for b_n.

$b_n = 10 \cdot 5^{n-1}$, *where* n *is a positive integer.*

d. Evaluate $\sum_{i=1}^{10} b_i$.

$$(10) \cdot \left(\frac{1 - 5^{10}}{1 - 5}\right) = 10 \cdot \frac{9765624}{4} = 24414060$$

9. Consider the sequence given by $a_1 = 20$, $a_n = \frac{1}{2} \cdot a_{n-1}$.

a. Evaluate $\sum_{i=1}^{10} a_i$, $\sum_{i=1}^{100} a_i$, and $\sum_{i=1}^{1000} a_i$.

$$\sum_{i=1}^{10} a_i = 20\left(\frac{1 - \left(\frac{1}{2}\right)^{10}}{1 - \frac{1}{2}}\right) = 20\left(\frac{\frac{1023}{1024}}{\frac{1}{2}}\right) = 39.9609375$$

$$\sum_{i=1}^{100} a_i = 20\left(\frac{1 - \left(\frac{1}{2}\right)^{100}}{1 - \frac{1}{2}}\right) \approx 40$$

$$\sum_{i=1}^{1000} a_i = 20\left(\frac{1 - \left(\frac{1}{2}\right)^{1000}}{1 - \frac{1}{2}}\right) \approx 40$$

b. **What value does it appear this series is approaching as n continues to increase? Why might it seem like the series is bounded?**

The series is almost exactly 40. In the numerator we are subtracting a number that is incredibly small and gets even smaller the farther we go in the sequence. So, as $n \to \infty$, the sum approaches $\frac{20}{\frac{1}{2}} = 40$.

10. **The sum of a geometric series with 4 terms is 60, and the common ratio is $r = \frac{1}{2}$. Find the first term.**

$$60 = a\left(\frac{1-\left(\frac{1}{2}\right)^4}{1-\frac{1}{2}}\right)$$
$$60 = a\left(\frac{1-\frac{1}{16}}{\frac{1}{2}}\right)$$
$$60 = a\left(\frac{15}{16}\cdot 2\right)$$
$$60 = a\left(\frac{15}{8}\right)$$
$$a = 4\cdot 8 = 32$$

11. **The sum of the first 4 terms of a geometric series is 203, and the common ratio is 0.4. Find the first term.**

$$203 = a\left(\frac{1-0.4^4}{1-0.4}\right)$$
$$a = 203\left(\frac{0.6}{1-0.4^4}\right) = 125$$

12. **The third term in a geometric series is $\frac{27}{2}$, and the sixth term is $\frac{729}{16}$. Find the common ratio.**

$$ar^2 = \frac{27}{2}$$
$$ar^5 = \frac{729}{16}$$
$$r^3 = \frac{729}{16}\cdot\frac{2}{27} = \frac{27}{8}$$
$$r = \frac{3}{2}$$

13. **The second term in a geometric series is 10, and the seventh term is $10,240$. Find the sum of the first six terms.**

$$ar = 10$$
$$ar^6 = 10240$$
$$r^5 = 1024$$
$$r = 4$$
$$a = \frac{10}{4} = \frac{5}{2}$$
$$S_6 = \frac{5}{2}\left(\frac{1-4^6}{1-4}\right)$$
$$= \frac{5}{2}\left(\frac{4095}{3}\right)$$
$$= 3412.5$$

14. **Find the interest earned and the future value of an annuity with monthly payments of $\$200$ for two years into an account that pays 6% interest per year compounded monthly.**

$$A_f = 200\left(\frac{\left(1+\frac{0.06}{12}\right)^{24}-1}{\frac{0.06}{12}}\right) \approx 5086.39$$

The future value is $\$5,086.39$, and the interest earned is $\$286.39$.

15. **Find the interest earned and the future value of an annuity with annual payments of $\$1,200$ for 15 years into an account that pays 4% interest per year.**

$$A_f = 1200\left(\frac{(1+0.04)^{15}-1}{0.04}\right) \approx 24028.31$$

The future value is $\$24,028.31$, and the interest earned is $\$6,028.31$.

16. **Find the interest earned and the future value of an annuity with semiannual payments of $\$1,000$ for 20 years into an account that pays 7% interest per year compounded semiannually.**

$$A_f = 1000\left(\frac{\left(1+\frac{0.07}{2}\right)^{40}-1}{\frac{0.07}{2}}\right) \approx 84550.28$$

The future value is $\$84,550.28$, and the interest earned is $\$44,550.28$.

17. **Find the interest earned and the future value of an annuity with weekly payments of $\$100$ for three years into an account that pays 5% interest per year compounded weekly.**

$$A_f = 100\left(\frac{\left(1+\frac{0.05}{52}\right)^{156}-1}{\frac{0.05}{52}}\right) \approx 16822.05$$

The future value is $\$16,822.05$, and the interest earned is $\$1,222.05$.

18. **Find the interest earned and the future value of an annuity with quarterly payments of $\$500$ for 12 years into an account that pays 3% interest per year compounded quarterly.**

$$A_f = 500\left(\frac{\left(1+\frac{0.03}{4}\right)^{48}-1}{\frac{0.03}{4}}\right) \approx 28760.36$$

The future value is $\$28,760.36$, and the interest earned is $\$4,760.36$.

19. How much money should be invested every month with 8% interest per year compounded monthly in order to save up $\$10,000$ in 15 months?

$$10000 = R\left(\frac{\left(1+\frac{0.08}{12}\right)^{15}-1}{\frac{0.08}{12}}\right)$$

$$R = 10000\left(\frac{\frac{0.08}{12}}{\left(1+\frac{0.08}{12}\right)^{15}-1}\right)$$

$$\approx 636.11$$

Invest $\$636.11$ every month for 15 months at this interest rate to save up $\$10,000$.

20. How much money should be invested every year with 4% interest per year in order to save up $\$40,000$ in 18 years?

$$40000 = R\left(\frac{(1+0.04)^{18}-1}{0.04}\right)$$

$$R = 40000\left(\frac{0.04}{(1.04)^{18}-1}\right)$$

$$\approx 1559.733$$

Invest $\$1,559.73$ every year for 18 years at 4% interest per year to save up $\$40,000$.

21. Julian wants to save up to buy a car. He is told that a loan for a car costs $\$274$ a month for five years, but Julian does not need a car presently. He decides to invest in a structured savings plan for the next three years. Every month Julian invests $\$274$ at an annual interest rate of 2% compounded monthly.

a. How much will Julian have at the end of three years?

$$A_f = 274\left(\frac{\left(1+\frac{0.02}{12}\right)^{36}-1}{\frac{0.02}{12}}\right) \approx 10157.21$$

Julian will have $\$10,157.21$ at the end of the three years.

b. What are the benefits of investing in a structured savings plan instead of taking a loan out? What are the drawbacks?

The biggest benefit is that instead of paying interest on a loan, you earn interest on your savings. The drawbacks include that you have to wait to get what you want.

22. An *arithmetic series* is a series whose terms form an arithmetic sequence. For example, $2+4+6+\cdots+100$ is an arithmetic series since $2, 4, 6, 8, \ldots, 100$ is an arithmetic sequence with constant difference 2.

The most famous arithmetic series is $1+2+3+4+\cdots+n$ for some positive integer n. We studied this series in Algebra I and showed that its sum is $S_n = \frac{n(n+1)}{2}$. It can be shown that the general formula for the sum of an arithmetic series $a+(a+d)+(a+2d)+\cdots+[a+(n-1)d]$ is

$$S_n = \frac{n}{2}[2a+(n-1)d],$$

where a is the first term and d is the constant difference.

a. Use the general formula to show that the sum of $1+2+3+\cdots+n$ is $S_n = \frac{n(n+1)}{2}$.

$$S_n = \frac{n}{2}(2\cdot 1+(n-1)\cdot 1) = \frac{n}{2}(2+n-1) = \frac{n}{2}(n+1)$$

b. Use the general formula to find the sum of $2 + 4 + 6 + 8 + 10 + \cdots + 100$.

$$S_n = \frac{50}{2}(4 + (50 - 1)2) = 25(102) = 2550$$

23. The sum of the first five terms of an arithmetic series is 25, and the first term is 2. Find the constant difference.

$$25 = \frac{5}{2}(2 + a_5)$$
$$10 = 2 + a_5$$
$$a_5 = 8$$

$$8 = 2 + d(4)$$
$$6 = d(4)$$
$$d = \frac{3}{2}$$

24. The sum of the first nine terms of an arithmetic series is 135, and the first term is 17. Find the ninth term.

$$135 = \frac{9}{2}(17 + a_9)$$
$$30 = 17 + a_9$$
$$13 = a_9$$

$$13 = 17 + d(8)$$
$$-4 = d(8)$$
$$d = -\frac{1}{2}$$

25. The sum of the first and 100^{th} terms of an arithmetic series is 101. Find the sum of the first 100 terms.

$$S_{100} = \frac{100}{2}(101) = 5050$$

Lesson 30: Buying a Car

Student Outcomes

- Students use the sum of a finite geometric series formula to develop a formula to calculate a payment plan for a car loan and use that calculation to derive the present value of an annuity formula.

Lesson Notes

In this lesson, students explore the idea of getting a car loan. The lesson extends their knowledge on saving money from the last lesson to the mathematics behind borrowing it. The formula for the monthly payment on a loan is derived using the formula for the sum of a geometric series. Amortization tables are used to help students develop an understanding of borrowing money.

In this lesson, we derive the future amount of an annuity formula again in the context of purchasing a car and use it to understand the present value of an annuity formula:

$$A_p = R\left(\frac{1-(1+i)^{-n}}{i}\right).$$

It is helpful to think of the present value of an annuity A_p in the following way: Calculate the future amount of an annuity A_f (as in Lesson 29) to find out the total amount that would be in an account after making all of the payments. Then, use the compound interest formula $F = P(1+i)^n$ from Lesson 26 to compute how much would need to be invested today (i.e., A_p) in one single large deposit to equal the amount A_f in the future. More specifically, for an interest rate of i per time period with n payments each of amount R, then the present value can be computed (substituting A_p for P and A_f for F in the compound interest formula) to be

$$A_f = A_p(1+i)^n.$$

Using the future amount of an annuity formula and solving for A_p gives

$$A_p = R\left(\frac{(1+i)^n - 1}{i}\right)(1+i)^{-n},$$

which simplifies to the first formula above. The play between the sum of a geometric series (**A-SSE.B.4**) and the combination of functions to get the new function A_p (**F-BF.A.1b**) constitutes the entirety of the mathematical content of this lesson.

While the mathematics is fairly simple, the context—car loans and the amortization process—is also new to students. To help the car loan process make sense (and loans in general), we have students think about the following situation: Instead of paying the full price of a car immediately, a student asks the dealer to develop a loan payment plan in which the student pays the same amount each month. The car dealer agrees and does the following calculation to determine the amount R that the student should pay each month:

- The car dealer first imagines how much she would have if she took the amount of the loan (i.e., price of the car) and deposited it into an account for 60 months (5 years) at a certain interest rate per month.

- The car dealer then imagines taking the student's payments (R dollars) and depositing them into an account making the same interest rate per month. The final amount is calculated just like calculating the final amount of a structured savings plan from Lesson 29.
- The car dealer then reasons that, to be fair to her and her customer, the two final amounts should be the same—that is, the car dealer should have the same amount in each account at the end of 60 months either way. This sets up the equation above, which can then be solved for R.

This lesson is the first lesson where the concept of amortization appears. An example of amortization is the process of decreasing the amount owed on a loan over time, which decreases the amount of interest owed over time as well. This can be thought of as doing an annuity calculation like in Lesson 29 but run backward in time. Whenever possible, use online calculators such as http://www.bankrate.com/calculators/mortgages/amortization-calculator.aspx to generate amortization tables (i.e., tables that show the amount of the principal and interest for each payment). Students have filled in a few amortization tables in Lesson 26 as an application of interest, but the concept was not presented in its entirety.

Classwork

Opening Exercise (2 minutes)

The following problem is similar to homework students did in the previous lesson; however, the savings terms are very similar to those found in car loans.

Opening Exercise

Write a sum to represent the future amount of a structured savings plan (i.e., annuity) if you deposit $\$250$ into an account each month for 5 years that pays 3.6% interest per year, compounded monthly. Find the future amount of your plan at the end of 5 years.

$$250(1.003)^{59} + 250(1.003)^{58} + \cdots + 250(1.003) + 250 = 250\left(\frac{(1.003)^{60} - 1}{0.003}\right) \approx 16407.90.$$

The amount in the account after 5 years will be $\$16,407.90$.

Example (15 minutes)

Many people take out a loan to purchase a car and then repay the loan on a monthly basis. Announce that we will figure out how banks determine the monthly payment for a loan in today's class.

- When deciding to get a car loan, there are many things to consider. What do we know that goes into getting a loan for a vehicle?
  - *Look for the following: down payment, a monthly payment, interest rates on the loan, number of years of the loan. Explain any of these terms that students may not know.*

For car loans, a down payment is not always required, but a typical down payment is 15% of the total cost of the vehicle. Let's assume throughout this example that no down payment is required.

This example provides a sequence of problems to work through with your students that guides students through the process for finding the recurring monthly payment for a car loan described in the teacher notes. After the example, students are given more information on buying a car and calculate the monthly payment for a car that they researched on the Internet as part of their homework in Lesson 29.

Example

Jack wanted to buy a $\$9,000$ 2-door sports coupe but could not pay the full price of the car all at once. He asked the car dealer if she could give him a loan where he paid a monthly payment. She told him she could give him a loan for the price of the car at an annual interest rate of 3.6% compounded monthly for 60 months (5 years).

The problems below exhibit how Jack's car dealer used the information above to figure out how much his monthly payment of R dollars per month should be.

a. **First, the car dealer imagined how much she would have in an account if she deposited $\$9,000$ into the account and left it there for 60 months at an annual interest rate of 3.6% compounded monthly. Use the compound interest formula $F = P(1+i)^n$ to calculate how much she would have in that account after 5 years. This is the amount she would have in the account after 5 years if Jack gave her $\$9,000$ for the car, and she immediately deposited it.**

   $F = 9000(1+0.003)^{60} = 9000(1.003)^{60} \approx 10772.05$. *At the end of 60 months, she would have $\$10,772.05$ in the account.*

b. **Next, she figured out how much would be in an account after 5 years if she took each of Jack's payments of R dollars and deposited it into a bank that earned 3.6% per year (compounded monthly). Write a sum to represent the future amount of money that would be in the annuity after 5 years in terms of R, and use the sum of a geometric series formula to rewrite that sum as an algebraic expression.**

   This is like the structured savings plan in Lesson 29. The future amount of money in the account after 5 years can be represented as

   $$R(1.003)^{59} + R(1.003)^{58} + \cdots + R(1.003) + R.$$

   Applying the sum of a geometric series formula

   $$S_n = a\left(\frac{1-r^n}{1-r}\right)$$

   to the geometric series above using $a = R$, $r = 1.003$, and $n = 60$, one gets

   $$S_n = R\left(\frac{1-(1.003)^{60}}{1-1.003}\right) = R\left(\frac{(1.003)^{60}-1}{0.003}\right).$$

At this point, we have re-derived the future amount of an annuity formula. Point this out to your students! Help them to see the connection between what they are doing in this context with what they did in Lesson 29. The future value formula is

$$A_f = R\left(\frac{(1+i)^n - 1}{i}\right).$$

c. **The car dealer then reasoned that, to be fair to her and Jack, the two final amounts in both accounts should be the same—that is, she should have the same amount in each account at the end of 60 months either way. Write an equation in the variable R that represents this equality.**

   $$9000(1.003)^{60} = R\left(\frac{(1.003)^{60}-1}{0.003}\right)$$

d. **She then solved her equation to get the amount R that Jack would have to pay monthly. Solve the equation in part (c) to find out how much Jack needed to pay each month.**

Solving for R in the equation above, we get

$$R = 9000 \cdot (1.003)^{60}\left(\frac{0.003}{(1.003)^{60} - 1}\right) \approx 164.13.$$

Thus, Jack will need to make regular payments of $\$164.13$ a month for 60 months.

Ask students questions to see if they understand what the $\$164.13$ means. For example, if Jack decides not to buy the car and instead deposited $\$164.13$ a month into an account earning 3.6% interest compounded monthly, how much does he have at the end of 60 months? Students should be able to answer $\$10{,}772.05$, the final amount of the annuity that the car dealer calculated in part (a) or (b). Your goal is to help them see that both ways of calculating the future amount should be equal.

Discussion (10 minutes)

In this discussion, students are lead to the present value of an annuity formula using the calculations they just did in the example (**F-BF.A.1b**).

MP.8

- Let's do the calculations in part (a) of the example again but this time using A_p for the loan amount (the present value of an annuity), i for the interest rate per time period, n to be the number of time periods. As in part (a), what is the future value of A_p if it is deposited in an account with an interest rate of i per time period for n compounding periods?
  - $F = A_p(1+i)^n$
- As in part (b) of the example above, what is the future value of an annuity A_f in terms of the recurring payment R, interest rate i, and number of periods n?
  - $A_f = R\left(\frac{1-(1+i)^n - 1}{i}\right)$
- If we assume (as in the example above) that both methods produce the same future value, we can equate $F = A_f$ and write the following equation:

$$A_p(1+i)^n = R\left(\frac{(1+i)^n - 1}{i}\right).$$

- Where did we use this formula in the example above?
  - *The equation derived in part (c).*
- We can now solve this equation for R as we did in the example, but it is more common in finance to solve for A_p by multiplying both sides by $(1+i)^{-n}$:

$$A_p = R\left(\frac{(1+i)^n - 1}{i}\right)(1+i)^{-n},$$

and then distributing it through the binomial to get the *present value of an annuity* formula:

$$A_p = R\left(\frac{1-(1+i)^{-n}}{i}\right).$$

- When a bank (or a car dealer) makes a loan that is to be repaid with recurring payments R, then the payments form an annuity whose present value A_p is the amount of the loan. Thus, we can use this formula to find the payment amount R given the size of the loan A_p (as in Example 1), or we can find the size of the loan A_p if we know the size of the payments R.

Exercise (3 minutes)

Exercise

A college student wants to buy a car and can afford to pay $\$200$ per month. If she plans to take out a loan at 6% interest per year with a recurring payment of $\$200$ per month for four years, what price car can she buy?

$$A_p = 200\left(\frac{1-(1.005)^{-48}}{0.005}\right) \approx 8516.06$$

She can afford to take out a $\$8,516.06$ loan. If she has no money for a down payment, she can afford a car that is about $\$8,500$.

You might want to point out to your students that the present value formula can always be easily and quickly derived from the future amount of annuity formula $A_f = R\left(\frac{1-(1+i)^n}{i}\right)$ and the compound interest formula $A_f = A_p(1+i)^n$ (using the variables A_f and A_p instead of F and P).

Mathematical Modeling Exercise (8 minutes)

The customization and open-endedness of this challenge depends upon how successful students were in researching the price of a potential car in the Problem Set to Lesson 29. For students who did not find a car, you can have them use the list provided below. After the challenge, there are some suggestions for ways to introduce other modeling elements into the challenge. Use the suggestions as you see fit. The solutions throughout this section are based on the 2007 two-door small coupe.

MP.2 & MP.4

Mathematical Modeling Exercise

In the Problem Set of Lesson 29, you researched the price of a car that you might like to own. In this exercise, we determine how much a car payment would be for that price for different loan options.

If you did not find a suitable car, select a car and selling price from the list below:

Car	Selling Price
2005 Pickup Truck	$\$9,000$
2007 Two-Door Small Coupe	$\$7,500$
2003 Two-Door Luxury Coupe	$\$10,000$
2006 Small SUV	$\$8,000$
2008 Four-Door Sedan	$\$8,500$

a. **When you buy a car, you must pay sales tax and licensing and other fees. Assume that sales tax is 6% of the selling price and estimated license/title/fees are 2% of the selling price. If you put a $\$1,000$ down payment on your car, how much money do you need to borrow to pay for the car and taxes and other fees?**

Answers will vary. For the 2007 two-door small coupe:
$7500 + 7500(0.06) + 7500(0.02) - 1000 = 7100$. *You would have to borrow $\$7,100$.*

Scaffolding:

For English language learners, provide a visual image of each vehicle type along with a specific make and model.

- Pickup Truck
- 2-Door Small Coupe
- 2-Door Luxury Coupe
- Small SUV
- 4-Door Sedan

b. **Using the loan amount you computed above, calculate the monthly payment for the different loan options shown below:**

Loan 1	**36-month loan at 2%**
Loan 2	**48-month loan at 3%**
Loan 3	**60-month loan at 5%**

Answers will vary. For the 2007 two-door small coupe:

Loan 1: $7100 = R\left(\frac{1-\left(1+\frac{0.02}{12}\right)^{-36}}{\frac{0.02}{12}}\right)$*; therefore,* $R \approx 203.36$*. The monthly payment would be* $\$203.36$*.*

Loan 2: $7100 = R\left(\frac{1-\left(1+\frac{0.03}{12}\right)^{-48}}{\frac{0.03}{12}}\right)$*; therefore,* $R \approx 157.15$*. The monthly payment would be* $\$157.15$*.*

Loan 3: $7100 = R\left(\frac{1-\left(1+\frac{0.05}{12}\right)^{-60}}{\frac{0.05}{12}}\right)$*; therefore,* $R \approx 133.99$*. The monthly payment would be* $\$133.99$*.*

c. **Which plan, if any, keeps the monthly payment under $\$175$? Of the plans under $\$175$ per month, why might you choose a plan with fewer months even though it costs more per month?**

Answers will vary. Loan 2 and Loan 3 are both under $\$175$ a month. When the monthly payments are close (like Loan 2 and Loan 3), the fewer payments you make with Loan 2 means you pay less overall for that loan.

If a student found a dealer that offered a loan for the car they were researching, encourage them to do the calculations above for terms of that loan. (Call it loan option 4.)

Further Modeling Resources

If students are interested in the actual details of purchasing and budgeting for a car, consider having them research vehicle fees, including the sales tax in your state (or county), inspection fees (if any), license and title fees, and car insurance rates.

Furthermore, you can ask:

- What sort of extra costs go into buying a car?
  - *Answers may vary, but students should be able identify sales tax. Other fees include license and title. An inspection is often required shortly after purchase. Individual states may have additional fees.*
- What extra costs go into maintaining a car?
  - *Insurance, repairs, maintenance like oil changes, car washing/detailing, etc. For used and new cars, use an online calculator to estimate car insurance and maintenance costs as well as likely depreciation and interest costs for a loan.*

Closing (2 minutes)

Close this lesson by asking students to summarize in writing or with a partner what they know so far about borrowing money to buy a car.

- Based on the work you did in this lesson, summarize what you know so far about borrowing money to buy a car.
  - *Making the loan term longer does make the monthly payment go down but causes the total interest paid to go up. Interest rates, down payment, and total length of the loan all affect the monthly payment. In the end, the amount of the loan you get depends on what you can afford to pay per month based on your budget.*

> **Lesson Summary**
>
> **The total cost of car ownership includes many different costs in addition to the selling price, such as sales tax, insurance, fees, maintenance, interest on loans, gasoline, etc.**
>
> **The present value of an annuity formula can be used to calculate monthly loan payments given a total amount borrowed, the length of the loan, and the interest rate. The present value A_p (i.e., loan amount) of an annuity consisting of n recurring equal payments of size R and interest rate i per time period is**
>
> $$A_p = R\left(\frac{1-(1+i)^{-n}}{i}\right).$$
>
> **Amortization tables and online loan calculators can also help you plan for buying a car.**
>
> **The amount of your monthly payment depends on the interest rate, the down payment, and the length of the loan.**

Exit Ticket (5 minutes)

Name ______________________________ Date ______________

Lesson 30: Buying a Car

Exit Ticket

Fran wants to purchase a new boat. She starts looking for a boat around $\$6{,}000$. Fran creates a budget and thinks that she can afford $\$250$ every month for 2 years. Her bank charges her 5% interest per year, compounded monthly.

1. What is the actual monthly payment for Fran's loan?

2. If Fran can only pay $\$250$ per month, what is the most expensive boat she can buy without a down payment?

Exit Ticket Sample Solutions

Fran wants to purchase a new boat. She starts looking for a boat around $\$6,000$. Fran creates a budget and thinks that she can afford $\$250$ every month for 2 years. Her bank charges her 5% interest per year, compounded monthly.

1. **What is the actual monthly payment for Fran's loan?**

$$6000 = R\left(\frac{1-\left(1+\frac{0.05}{12}\right)^{-24}}{\frac{0.05}{12}}\right)$$

$$R = 6000\left(\frac{\left(\frac{0.05}{12}\right)}{1-\left(1+\frac{0.05}{12}\right)^{-24}}\right)$$

$$R \approx 263.23$$

The actual monthly payment for Fran's loan would be $\$263.23$.

2. **If Fran can only pay $\$250$ per month, what is the most expensive boat she can buy without a down payment?**

$$P = 250\left(\frac{1-\left(1+\frac{0.05}{12}\right)^{-24}}{\frac{0.05}{12}}\right)$$

$$P \approx 5698.47$$

Fran can afford a boat that costs about $\$5,700$ if she does not have a down payment.

Problem Set Sample Solutions

1. **Benji is 24 years old and plans to drive his new car about 200 miles per week. He has qualified for first-time buyer financing, which is a 60-month loan with 0% down at an interest rate of 4%. Use the information below to estimate the monthly cost of each vehicle.**

CAR A: 2010 **Pickup Truck for $\$12,000$, 22 miles per gallon**

CAR B: 2006 **Luxury Coupe for $\$11,000$, 25 miles per gallon**

Gasoline: $\$4.00$ per gallon **New vehicle fees: $\$80$** **Sales Tax: 4.25%**

Maintenance Costs:

(2010 model year or newer): 10% of purchase price annually

(2009 model year or older): 20% of purchase price annually

Insurance:

Average Rate Ages 25–29	$\$100$ per month
If you are male If you are female	Add $\$10$ per month Subtract $\$10$ per month
Type of Car Pickup Truck Small Two-Door Coupe or Four-Door Sedan Luxury Two- or Four-Door Coupe	 Subtract $\$10$ per month Subtract $\$10$ per month Add $\$15$ per month
Ages 18–25	Double the monthly cost

a. How much money will Benji have to borrow to purchase each car?

Benji would have to borrow $\$12,000$ for the truck and $\$11,000$ for the coupe.

b. What is the monthly payment for each car?

$$12000 = R\left(\frac{1-\left(1+\frac{0.04}{12}\right)^{-60}}{\frac{0.04}{12}}\right)$$

$$R = 12000\left(\frac{\left(\frac{0.04}{12}\right)}{1-\left(1+\frac{0.04}{12}\right)^{-60}}\right)$$

$$R \approx 221.00$$

The truck would cost $\$221.00$ every month.

$$11000 = R\left(\frac{1-\left(1+\frac{0.04}{12}\right)^{-60}}{\frac{0.04}{12}}\right)$$

$$R = 11000\left(\frac{\left(\frac{0.04}{12}\right)}{1-\left(1+\frac{0.04}{12}\right)^{-60}}\right)$$

$$R \approx 202.58$$

The coupe would cost $\$202.58$ every month.

c. What are the annual maintenance costs and insurance costs for each car?

Truck: $10\% \cdot 12000 = 1200$ for the maintenance. Insurance will vary based on the gender of student. Male students will be $\$200$ per month or $\$2,400$ per year, while female students will be $\$160$ per month or $\$1,920$ per year.

Car: $20\% \cdot 11000 = 2200$ for maintenance. Male students will cost $\$250$ per month or $\$3,000$ per year, while female students will cost $\$210$ per month or $\$2,520$ per year.

d. Which car should Benji purchase? Explain your choice.

Answers will vary depending on personal preference and experience, as well as financial backgrounds. Answers should be supported using the mathematics of parts (a), (b), and (c).

2. Use the total initial cost of buying your car from the lesson to calculate the monthly payment for the following loan options.

Option	Number of Months	Down Payment	Interest Rate	Monthly Payment
Option A	48 months	$\$0$	2.5%	$\$175.31$
Option B	60 months	$\$500$	3.0%	$\$134.77$
Option C	60 months	$\$0$	4.0%	$\$147.33$
Option D	36 months	$\$1,000$	0.9%	$\$197.15$

Answers will vary. Suggested answers assume an $\$8,000$ car.

a. **For each option, what is the total amount of money you will pay for your vehicle over the life of the loan?**

Option A: $\$175.31 \cdot 48 = \$8414.88$

Option B: $\$500 + \$134.77 \cdot 60 = \$8586.20$

Option C: $\$147.33 \cdot 60 = \$8839.80$

Option D: $\$1000 + \$197.15 \cdot 36 = \$8097.40$

b. **Which option would you choose? Justify your reasoning.**

Answers will vary. Option B is the cheapest per month but requires a down payment. Of the plans without down payments, Option A saves the most money in the end, but Option C is cheaper per month. Option D saves the most money long term but requires the largest down payment and the largest monthly payment.

3. **Many lending institutions allow you to pay additional money toward the principal of your loan every month. The table below shows the monthly payment for an $\$8,000$ loan using Option A above if you pay an additional $\$25$ per month.**

Month/ Year	Payment	Principal Paid	Interest Paid	Total Interest	Balance
Aug. 2014	$ 200.31	$ 183.65	$ 16.67	$ 16.67	$ 7,816.35
Sept. 2014	$ 200.31	$ 184.03	$ 16.28	$ 32.95	$ 7,632.33
Oct. 2014	$ 200.31	$ 184.41	$ 15.90	$ 48.85	$ 7,447.91
Nov. 2014	$ 200.31	$ 184.80	$ 15.52	$ 64.37	$ 7,263.12
Dec. 2014	$ 200.31	$ 185.18	$ 15.13	$ 79.50	$ 7,077.94
Jan. 2015	$ 200.31	$ 185.57	$ 14.75	$ 94.25	$ 6,892.37
Feb. 2015	$ 200.31	$ 185.95	$ 14.36	$ 108.60	$ 6,706.42
Mar. 2015	$ 200.31	$ 186.34	$ 13.97	$ 122.58	$ 6,520.08
April 2015	$ 200.31	$ 186.73	$ 13.58	$ 136.16	$ 6,333.35
May 2015	$ 200.31	$ 187.12	$ 13.19	$ 149.35	$ 6,146.23
June 2015	$ 200.31	$ 187.51	$ 12.80	$ 162.16	$ 5,958.72
July 2015	$ 200.31	$ 187.90	$ 12.41	$ 174.57	$ 5,770.83
Aug. 2015	$ 200.31	$ 188.29	$ 12.02	$ 186.60	$ 5,582.54
Sept. 2015	$ 200.31	$ 188.68	$ 11.63	$ 198.23	$ 5,393.85
Oct. 2015	$ 200.31	$ 189.08	$ 11.24	$ 209.46	$ 5,204.78
Nov. 2015	$ 200.31	$ 189.47	$ 10.84	$ 220.31	$ 5,015.31
Dec. 2015	$ 200.31	$ 189.86	$ 10.45	$ 230.75	$ 4,825.45

Note: The months from January 2016 to December 2016 are not shown.

Jan. 2017	\$ 200.31	\$ 195.07	\$ 5.24	\$ 330.29	\$ 2,320.92
Feb. 2017	\$ 200.31	\$ 195.48	\$ 4.84	\$ 335.12	\$ 2,125.44
Mar. 2017	\$ 200.31	\$ 195.88	\$ 4.43	\$ 339.55	\$ 1,929.56
April 2017	\$ 200.31	\$ 196.29	\$ 4.02	\$ 343.57	\$ 1,733.27
May 2017	\$ 200.31	\$ 196.70	\$ 3.61	\$ 347.18	\$ 1,536.57
June 2017	\$ 200.31	\$ 197.11	\$ 3.20	\$ 350.38	\$ 1,339.45
July 2017	\$ 200.31	\$ 197.52	\$ 2.79	\$ 353.17	\$ 1,141.93
Aug. 2017	\$ 200.31	\$ 197.93	\$ 2.38	\$ 355.55	\$ 944.00
Sept. 2017	\$ 200.31	\$ 198.35	\$ 1.97	\$ 357.52	\$ 745.65
Oct. 2017	\$ 200.31	\$ 198.76	\$ 1.55	\$ 359.07	\$ 546.90
Nov. 2017	\$ 200.31	\$ 199.17	\$ 1.14	\$ 360.21	\$ 347.72
Dec. 2017	\$ 200.31	\$ 199.59	\$ 0.72	\$ 360.94	\$ 148.13
Jan. 2018	\$ 148.44	\$ 148.13	\$ 0.31	\$ 361.25	\$ 0.00

How much money would you save over the life of an $\$8,000$ loan using Option A if you paid an extra $\$25$ per month compared to the same loan without the extra payment toward the principal?

Using Option A without paying extra toward the principal each month is a monthly payment of $\$175.31$. The total amount you will pay is $\$8,414.88$. If you pay the extra $\$25$ per month, you make 41 payments of $\$200.31$ and a final payment of $\$148.44$ for a total amount of $\$8,361.15$. You would save $\$53.73$.

4. **Suppose you can afford only $\$200$ a month in car payments, and your best loan option is a 60-month loan at 3%. How much money could you spend on a car? That is, calculate the present value of the loan with these conditions.**

$$P = 200\left(\frac{1-\left(1+\frac{0.03}{12}\right)^{-60}}{\frac{0.03}{12}}\right)$$

$$P \approx 11130.47$$

You can afford a loan of about $\$11,000$. If there is no down payment, then the car would need to cost about $\$11,000$.

5. **Would it make sense for you to pay an additional amount per month toward your car loan? Use an online loan calculator to support your reasoning.**

While pre-paying on a loan can save you money for a relatively short-term loan like a vehicle loan, there is usually not a significant cost savings. Most students will probably elect to pocket the extra monthly costs and pay slightly more over the life of the loan. One option is paying off a loan early. That can save you more money and can be explored online as an extension question for advanced learners.

6. **What is the sum of each series?**

   a. $900 + 900(1.01)^1 + 900(1.01)^2 + \cdots + 900(1.01)^{59}$

$$900\left(\frac{1-(1.01)^{60}}{1-1.01}\right) \approx 73502.703$$

b. $\sum_{n=0}^{47} 15000\left(1+\frac{0.04}{12}\right)^n$

$$\sum_{n=0}^{47} 15000\left(1+\frac{0.04}{12}\right)^n = 15000\left(\frac{1-\left(1+\frac{0.04}{12}\right)^{48}}{1-\left(1+\frac{0.04}{12}\right)}\right)$$
$$= 15000\left(\frac{\left(1+\frac{0.04}{12}\right)^{48}-1}{\frac{0.04}{12}}\right)$$
$$\approx 779394.015$$

7. Gerald wants to borrow $\$12,000$ in order to buy an engagement ring. He wants to repay the loan by making monthly installments for two years. If the interest rate on this loan is $9\frac{1}{2}\%$ per year, compounded monthly, what is the amount of each payment?

$$12000 = R\left(\frac{1-\left(1+\frac{0.095}{12}\right)^{-24}}{\frac{0.095}{12}}\right)$$
$$R = 12000\left(\frac{\left(\frac{0.095}{12}\right)}{1-\left(1+\frac{0.95}{12}\right)^{-24}}\right)$$
$$R \approx 550.97$$

Gerald will need to pay $\$550.97$ each month.

8. Ivan plans to surprise his family with a new pool using his Christmas bonus of $\$4,200$ as a down payment. If the price of the pool is $\$9,500$ and Ivan can finance it at an interest rate of $2\frac{7}{8}\%$ per year, compounded quarterly, how long is the term of the loan if his payments are $\$285.45$ per quarter?

$$5300 = 285.45\left(\frac{1-\left(1+\frac{0.02875}{4}\right)^{-n}}{\frac{0.02875}{4}}\right)$$
$$\frac{5300}{285.45}\cdot\frac{0.02875}{4} = 1-\left(1+\frac{0.02875}{4}\right)^{-n}$$
$$\left(1+\frac{0.02875}{4}\right)^{-n} = 1-\frac{5300}{285.45}\cdot\frac{0.02875}{4}$$
$$-n\cdot\ln\left(1+\frac{0.02875}{4}\right) = \ln\left(1-\frac{5300}{285.45}\cdot\frac{0.02875}{4}\right)$$
$$n = -\frac{\ln\left(1-\frac{5300}{285.45}\cdot\frac{0.02875}{4}\right)}{\ln\left(1+\frac{0.02875}{4}\right)}$$
$$n \approx 20$$

It will take Ivan 20 quarters, or five years, to pay off the pool at this rate.

9. Jenny wants to buy a car by making payments of $\$120$ per month for three years. The dealer tells her that she needs to put a down payment of $\$3,000$ on the car in order to get a loan with those terms at a 9% interest rate per year, compounded monthly. How much is the car that Jenny wants to buy?

$$P - 3000 = 120\left(\frac{1-\left(1+\frac{0.09}{12}\right)^{-36}}{\frac{0.09}{12}}\right)$$

$$P \approx 3\,773.62 + 3000$$

The car Jenny wants to buy is about $\$6,773.62$.

10. Kelsey wants to refinish the floors in her house and estimates that it will cost $\$39,000$ to do so. She plans to finance the entire amount at $3\frac{1}{4}\%$ interest per year, compounded monthly for 10 years. How much is her monthly payment?

$$39000 = R\left(\frac{1-\left(1+\frac{0.0325}{12}\right)^{-120}}{\frac{0.0325}{12}}\right)$$

$$R = 39000\left(\frac{\left(\frac{0.0325}{12}\right)}{1-\left(1+\frac{0.0325}{12}\right)^{-120}}\right)$$

$$R \approx 381.10$$

Kelsey will have to pay $\$381.10$ *every month.*

11. Lawrence coaches little league baseball and needs to purchase all new equipment for his team. He has $\$489$ in donations, and the team's sponsor will take out a loan at $4\frac{1}{2}\%$ interest per year, compounded monthly for one year, paying up to $\$95$ per month. What is the most that Lawrence can purchase using the donations and loan?

$$P - 489 = 95\left(\frac{1-\left(1+\frac{0.045}{12}\right)^{-12}}{\frac{0.045}{12}}\right)$$

$$P \approx 489 + 1112.69$$

The team will have access to $\$1,601.69$.

Lesson 31: Credit Cards

Student Outcomes

- Students compare payment strategies for a decreasing credit card balance.
- Students apply the sum of a finite geometric series formula to a decreasing balance on a credit card.

Lesson Notes

This lesson develops the necessary tools and terminology to analyze the mathematics behind credit cards and other unsecured loans. Credit cards can provide flexibility to budgets, but they must be carefully managed to avoid the pitfalls of bad credit. For young adults, credit card interest rates can be expected to be between 19.99% and 29.99% per year (29.99% is currently the maximum allowable interest rate by federal law). Adults with established credit can be offered interest rates around 8% to 14%. The credit limit for a first credit card is typically around $\$500$, but these limits quickly increase with a history of timely payments.

In this modeling lesson, students explore the mathematics behind calculating the monthly balance on a single credit card purchase and recognize that the decreasing balance can be modeled by the sum of a finite geometric series (**A-SSE.B.4**). We are intentionally keeping the use of rotating credit such as credit cards simple in this lesson. The students make one charge of $\$1{,}500$ on this hypothetical credit card and pay down the balance without making any additional charges. With this simple example, we can realistically ignore the fact that the interest on a credit card is charged based on the average daily balance of the account; in our example, the daily balance only changes once per month when the payment is made.

The students need to recall the following definitions from Lesson 29:

- **SERIES:** Let $a_1, a_2, a_3, a_4, \ldots$ be a sequence of numbers. A sum of the form

$$a_1 + a_2 + a_3 + \cdots + a_n$$

for some positive integer n is called a *series* (or *finite series*) and is denoted S_n. The a_i's are called the *terms* of the series. The number S_n that the series adds to is called the *sum* of the series.

- **GEOMETRIC SERIES:** A *geometric series* is a series whose terms form a geometric sequence.

The sum S_n of the first n terms of the finite geometric series $S_n = a + ar + \cdots + ar^{n-1}$ (when $r \neq 1$) is given by

$$S_n = a\left(\frac{1-r^n}{1-r}\right).$$

The sum formula of a geometric series can be written in summation notation as

$$\sum_{k=0}^{n-1} ar^k = a\left(\frac{1-r^n}{1-r}\right).$$

Classwork

Opening (3 minutes)

Assign students to small groups, and keep them in the same groups throughout this lesson. In the first mathematical modeling exercise, all groups work on the same problem, but in the second mathematical modeling exercise, the groups are assigned one of three different payment schemes to investigate.

- In the previous lesson, you investigated the mathematics needed for a car loan. What if you have decided to buy a car, but you have not saved up enough money for the down payment? If you are buying through a dealership, it is possible to put the down payment onto a credit card. For today's lesson, we investigate the finances of charging $\$1,500$ onto a credit card for the down payment on a car. We investigate different payment plans and how much you end up paying in total using each plan.
- The annual interest rates on a credit card for people who have not used credit in the past tend to be much higher than for adults with established good credit, ranging between 14.99% and 29.99%, which is the maximum interest rate allowed by law. Throughout this lesson, we use a 19.99% annual interest rate, and we explore problems with other interest rates in the Problem Set.
- One of the differences between a credit card and a loan is that you can pay as much as you want toward your credit card balance, as long as it is at least the amount of the "minimum payment," which is determined by the lender. In many cases, the minimum payment is the sum of the interest that has accrued over the month and 1% of the outstanding balance, or $\$25$, whichever is greater.
- Another difference between a credit card and a loan is that a loan has a fixed term of repayment—you pay it off over an agreed-upon length of time such as five years—and that there is no fixed term of repayment for a credit card. You can pay it off as quickly as you like by making large payments, or you can pay less and owe money for a longer period of time. In the mathematical modeling exercise, we investigate the scenario of paying a fixed monthly payment of various sizes toward a credit card balance of $\$1,500$.

Mathematical Modeling Exercise (25 minutes)

In this exercise, students model the repayment of a single charge of $\$1,500$ to a credit card that charges 19.99% annual interest. Before beginning the Mathematical Modeling Exercise, assign students to small groups, and assign groups to be either part of the 50-team, 100-team, or 150-team. The groups in each of the three teams investigate how long it takes to pay down the $\$1,500$ balance making fixed payments of either $\$50$, $\$100$, or $\$150$ each month.

As you circulate the room while students are working, take note of groups that are working well together on this set of problems. Select at least one group on each team to present their work at the end of the exercise period.

Mathematical Modeling Exercise

You have charged $\$1,500$ for the down payment on your car to a credit card that charges 19.99% annual interest, and you plan to pay a fixed amount toward this debt each month until it is paid off. We denote the balance owed after the n^{th} payment has been made as b_n.

a. **What is the monthly interest rate, i? Approximate i to 5 decimal places.**

$$i = \frac{0.1999}{12} \approx 0.01666$$

Scaffolding:

For struggling students, use an interest rate of 24.00% so that $i = 0.02$ and $r = 1.02$.

b. You have been assigned to either the 50-team, the 100-team, or the 150-team, where the number indicates the size of the monthly payment R you make toward your debt. What is your value of R?

Students will answer 50, 100, or 150 as appropriate.

c. Remember that you can make any size payment toward a credit card debt, as long as it is at least as large as the minimum payment specified by the lender. Your lender calculates the minimum payment as the sum of 1% of the outstanding balance and the total interest that has accrued over the month, or $\$25$, whichever is greater. Under these stipulations, what is the minimum payment? Is your monthly payment R at least as large as the minimum payment?

The minimum payment is $0.01(\$1500) + 0.01666(\$1500) = \$39.99$. All given values of R are greater than the minimum payment.

d. Complete the following table to show 6 months of payments.

Month, n	Interest Due (in dollars)	Payment, R (in dollars)	Paid to Principal (in dollars)	Balance, b_n (in dollars)
0				$1,500.00$
1	24.99	50	25.01	$1,474.99$
2	24.57	50	25.43	$1,449.56$
3	24.15	50	25.85	$1,423.71$
4	23.72	50	26.28	$1,397.43$
5	23.28	50	26.72	$1,370.71$
6	22.83	50	27.17	$1,343.54$

Month, n	Interest Due (in dollars)	Payment, R (in dollars)	Paid to Principal (in dollars)	Balance, b_n (in dollars)
0				$1,500.00$
1	24.99	100	75.01	$1,424.99$
2	23.74	100	76.26	$1,348.73$
3	22.47	100	77.53	$1,271.20$
4	21.18	100	78.82	$1,192.38$
5	19.86	100	80.14	$1,112.24$
6	18.53	100	81.47	$1,030.77$

Month, n	Interest Due (in dollars)	Payment, R (in dollars)	Paid to Principal (in dollars)	Balance, b_n (in dollars)
0				$1,500.00$
1	24.99	150	125.01	$1,374.99$
2	22.91	150	127.09	$1,247.90$
3	20.79	150	129.21	$1,118.69$
4	18.64	150	131.36	987.33
5	16.45	150	133.55	853.78
6	14.22	150	135.78	718.00

e. Write a recursive formula for the balance b_n in month n in terms of the balance b_{n-1}.

To calculate the new balance, b_n, we compound interest for one month on the previous balance b_{n-1} and then subtract the payment R:

$$b_n = b_{n-1}(1+i) - R, \text{ with } b_0 = 1500.$$

f. **Write an explicit formula for the balance b_n in month n, leaving the *expression* $1+i$ in symbolic form.**

We have the following formulas:

$$\begin{aligned} b_1 &= b_0(1+i) - R \\ b_2 &= b_1(1+i) - R \\ &= [b_0(1+i) - R](1+i) - R \\ &= b_0(1+i)^2 - R(1+i) - R \\ b_3 &= b_2(1+i) - R \\ &= [b_0(1+i)^2 - R(1+i) - R](1+i) - R \\ &= b_0(1+i)^3 - R(1+i)^2 - R(1+i) - R \\ &\vdots \\ b_n &= b_0(1+i)^n - R(1+i)^{n-1} - R(1+i)^{n-2} - \cdots - R(1+i) - R \end{aligned}$$

g. **Rewrite your formula in part (f) using r to represent the quantity $(1+i)$.**

$$\begin{aligned} b_n &= b_0r^n - Rr^{n-1} - Rr^{n-2} - \cdots - Rr - R \\ &= b_0r^n - R(1 + r + r^2 + \cdots + r^{n-1}) \end{aligned}$$

h. **What can you say about your formula in part (g)? What term do we use to describe r in this formula?**

The formula in part (g) contains the sum of a finite geometric series with common ratio r.

i. **Write your formula from part (g) in summation notation using Σ.**

$$\begin{aligned} b_n &= b_0r^n - R(1 + r + r^2 + \cdots + r^{n-1}) \\ &= b_0r^n - R\sum_{k=0}^{n-1} r^k \end{aligned}$$

Scaffolding:

Ask advanced learners to develop a generic formula for the balance b_n in terms of the payment amount R and the growth factor r.

j. **Apply the appropriate formula from Lesson 29 to rewrite your formula from part (g).**

Using the sum of a finite geometric series formula,

$$\begin{aligned} b_n &= b_0r^n - R(1 + r + r^2 + \cdots + r^{n-1}) \\ &= b_0r^n - R\left(\frac{1-r^n}{1-r}\right) \end{aligned}$$

k. **Find the month when your balance is paid off.**

The balance is paid off when $b_n = 0$. (The final payment is less than a full payment so that the debt is not overpaid.)

Students will likely do this calculation with the values of r, b_0, and R substituted in.

$$b_0r^n - R\left(\frac{1-r^n}{1-r}\right) = 0$$
$$b_0r^n = R\left(\frac{1-r^n}{1-r}\right)$$
$$(1-r)(b_0r^n) = R(1-r^n)$$
$$(1-r)(b_0r^n) + Rr^n = R$$
$$r^n(b_0(1-r)+R) = R$$
$$r^n = \frac{R}{(b_0(1-r)+R)}$$
$$n\log(r) = \log\left(\frac{R}{(b_0(1-r)+R)}\right)$$
$$n = \frac{\log\left(\frac{R}{(b_0(1-r)+R)}\right)}{\log(r)}$$

If $R = 50$, then $n \approx 41.925$. The debt is paid off in 42 months.

If $R = 100$, then $n \approx 17.49$. The debt is paid off in 18 months.

If $R = 150$, then $n \approx 11.0296$. The debt is paid off in 12 months.

l. **Calculate the total amount paid over the life of the debt. How much was paid solely to interest?**

For $R = 50$: The debt is paid in 41 payments of $\$50$, and the last payment is the amount b_{41} with interest:

$$\begin{aligned}50(41) + (1+i)b_{41} &= 2050 + r\left(b_0r^n - R\left(\frac{1-r^n}{1-r}\right)\right)\\ &\approx 2050 + r(45.61)\\ &\approx 2096.37.\end{aligned}$$

The total amount paid using monthly payments of $\$50$ is $\$2,096.37$. Of this amount, $\$596.37$ is interest.

For $R = 100$: The debt is paid in 17 payments of $\$100$, and the last payment is the amount b_{17} with interest.

$$\begin{aligned}100(17) + (1+i)b_{17} &= 1700 + r\left(b_0r^{17} - R\left(\frac{1-r^{17}}{1-r}\right)\right)\\ &\approx 1700 + r(40.52)\\ &\approx 1740.52\end{aligned}$$

The total amount paid using monthly payments of $\$100$ is $\$1,740.52$. Of this amount, $\$240.52$ is interest.

For $R = 150$: The debt is paid in 11 payments of $\$150$, and the last payment is the amount b_{11} with interest.

$$\begin{aligned}150(11) + (1+i)b_{11} &= 1700 + r\left(b_0r^n - R\left(\frac{1-r^n}{1-r}\right)\right)\\ &\approx 1650 + r(4.49)\\ &\approx 1654.49\end{aligned}$$

The total amount paid using monthly payments of $\$150$ is $\$1,654.49$. Of this amount, $\$154.49$ is interest.

Discussion (9 minutes)

Have students from each team present their solutions to parts (k) and (l) to the class. After the three teams have made their presentations, lead students through the following discussion, which should help them to make sense of the different results that arise from the different payment values R.

- What happens to the number of payments as you increase the amount R of the recurring monthly payment?
  - *As the amount R of the payment increases, the number of payments decreases.*
- What happens to the total amount of interest paid as you increase the amount R of the recurring monthly payment?
  - *As the amount R of the payment increases, the number of payments decreases.*
- What is the largest possible amount of the payment R? In that case, how many payments are made?
  - *The largest possible payment would be to pay the entire balance in one payment:* $(1+i)\$1500 = \$1524.99$.

Ask students about the formulas that they developed in the Mathematical Modeling Exercise to calculate the balance of the debt in month n. Students may use different notations, but they should have come up with a formula similar to $b_n = b_0 r^n - R\left(\frac{1-r^n}{1-r}\right)$. Depending on what notation the students used, you may need to draw the parallel from this formula to the present value of an annuity formula developed in Lesson 30. If we substitute $b_n = 0$ as the future value of the annuity when it is paid off in n payments, and $A_p = b_0$ as the present value/initial value of the annuity, then we have

$$b_n = b_0 r^n - R\left(\frac{1-r^n}{1-r}\right)$$

$$0 = A_p r^n - R\left(\frac{1-r^n}{1-r}\right)$$

$$A_p r^n = R\left(\frac{1-r^n}{1-r}\right)$$

$$A_p(1+i)^n = R\left(\frac{1-(1+i)^n}{1-(1+i)}\right)$$

$$A_p(1+i)^n = R\left(\frac{1-(1+i)^n}{-i}\right)$$

$$A_p = R\left(\frac{(1+i)^n-1}{i}\right)\cdot(1+i)^{-n}$$

$$A_p = R\left(\frac{1-(1+i)^{-n}}{i}\right).$$

Closing (3 minutes)

Ask students to summarize the main points of the lesson either in writing or with a partner. Some highlights that should be included are listed below.

- Calculating the balance from a single purchase on a credit card requires that we sum a finite geometric series.
- We have a formula from Lesson 29 that calculates the sum of a finite geometric series:
$$\sum_{k=0}^{n-1} ar^k = a\left(\frac{1-r^n}{1-r}\right).$$
- When you have incurred a credit card debt, you need to decide how to pay it off.
    - *If you choose to make a lower payment each month, then both the time required to pay off the debt and the total interest paid over the life of the debt increases.*
    - *If you choose to make a higher payment each month, then both the time required to pay off the debt and the total interest paid over the life of the debt decreases.*

Exit Ticket (5 minutes)

Name ____ Date ________________

Lesson 31: Credit Cards

Exit Ticket

Suppose that you currently have one credit card with a balance of $\$10{,}000$ at an annual rate of 24.00% interest. You have stopped adding any additional charges to this card and are determined to pay off the balance. You have worked out the formula $b_n = b_0 r^n - R(1 + r + r^2 + \cdots + r^{n-1})$, where b_0 is the initial balance, b_n is the balance after you have made n payments, $r = 1 + i$, where i is the monthly interest rate, and R is the amount you are planning to pay each month.

a. What is the monthly interest rate i? What is the growth rate, r?

b. Explain why we can rewrite the given formula as $b_n = b_0 r^n - R\left(\frac{1-r^n}{1-r}\right)$.

c. How long does it take to pay off this debt if you can afford to pay a constant $\$250$ per month? Give the answer in years and months.

Exit Ticket Sample Solutions

Suppose that you currently have one credit card with a balance of $\$10,000$ at an annual rate of 24.00% interest. You have stopped adding any additional charges to this card and are determined to pay off the balance. You have worked out the formula $b_n = b_0 r^n - R(1 + r + r^2 + \cdots + r^{n-1})$, where b_0 is the initial balance, b_n is the balance after you have made n payments, $r = 1 + i$, where i is the monthly interest rate, and R is the amount you are planning to pay each month.

a. **What is the monthly interest rate i? What is the growth rate, r?**

The monthly interest rate i is given by $i = \frac{0.24}{12} = 0.02$, and $r = 1 + i = 1.02$.

b. **Explain why we can rewrite the given formula as $b_n = b_0 r^n - R\left(\frac{1-r^n}{1-r}\right)$.**

Using summation notation and the sum formula for a finite geometric series, we have

$$\begin{aligned} 1 + r + r^2 + \cdots + r^{n-1} &= \sum_{k=0}^{n-1} r^k \\ &= \frac{1 - r^n}{1 - r}. \end{aligned}$$

Then the formula becomes

$$\begin{aligned} b_n &= b_0 r^n - R(1 + r + r^2 + \cdots + r^{n-1}) \\ &= b_0 r^n - R\left(\frac{1 - r^n}{1 - r}\right). \end{aligned}$$

c. **How long does it take to pay off this debt if you can afford to pay a constant $\$250$ per month? Give the answer in years and months.**

When the debt is paid off, $b_n \le 0$. Then $b_0 r^n - R\left(\frac{1-r^n}{1-r}\right) = 0$, and $b_0 r^n = R\left(\frac{1-r^n}{1-r}\right)$. Since $b_0 = 10000$, $R = 250$, and $r = 1.02$, we have

$$\begin{aligned} 10000(1.02)^n &\le 250\left(\frac{1 - 1.02^n}{1 - 1.02}\right) \\ 10000(1.02)^n &\le -12500(1 - 1.02^n) \\ 10000(1.02)^n &\le 12500(1.02^n - 1) \\ (1.02)^n &\le 1.25(1.02)^n - 1.25 \\ 1.25 &\le 0.25(1.02)^n \\ 5 &\le 1.02^n \\ \log(5) &\le n\log(1.02) \\ n &\ge \frac{\log(5)}{\log(1.02)} \\ n &\ge 81.27 \end{aligned}$$

It takes 82 months to pay off this debt, which means it takes 6 years and 10 months.

Problem Set Sample Solutions

Problems 1–4 ask students to compare credit card scenarios with the same initial debt and the same monthly payments but different interest rates. Problems 5, 6, and 7 require students to compare properties of functions given by different representations, which aligns with **F-IF.C.9** and **F-LE.B.5**.

The final two problems in this Problem Set require students to do some online research in preparation for Lesson 32, in which they select a career and model the purchase of a house. Have some printouts of real-estate listings ready to hand to students who have not brought their own to class. Feel free to add some additional constraints to the criteria for selecting a house to purchase. The career data in Problem 9 can be found at http://themint.org/teens/starting-salaries.html. For additional jobs and more information, please visit the U.S. Bureau of Labor Statistics at http://www.bls.gov/ooh and http://www.bls.gov/ooh/about/teachers-guide.htm. The salary for the “entry-level full-time” position is based on the projected federal minimum wage in 2016 of $\$10.10$ per hour and a 2,000-hour work year.

1. **Suppose that you have a $\$2,000$ balance on a credit card with a 29.99% annual interest rate, compounded monthly, and you can afford to pay $\$150$ per month toward this debt.**

   a. **Find the amount of time it takes to pay off this debt. Give your answer in months and years.**

$$2000\left(1+\frac{0.2999}{12}\right)^n - 150\left(\frac{1-\left(1+\frac{0.2999}{12}\right)^n}{-\frac{0.2999}{12}}\right) = 0$$

$$2000\left(1+\frac{0.2999}{12}\right)^n = 150\left(\frac{\left(1+\frac{0.2999}{12}\right)^n - 1}{\frac{0.2999}{12}}\right)$$

$$\frac{2999}{9000}\left(1+\frac{0.2999}{12}\right)^n = \left(1+\frac{0.2999}{12}\right)^n - 1$$

$$\left(1+\frac{0.2999}{12}\right)^n\left(\frac{2999}{9000}-1\right) = -1$$

$$\left(1+\frac{0.2999}{12}\right)^n\left(1-\frac{2999}{9000}\right) = 1$$

$$n\cdot\log\left(1+\frac{0.2999}{12}\right) + \log\left(\frac{6001}{9000}\right) = \log(1)$$

$$n\cdot\log\left(1+\frac{0.2999}{12}\right) = -\log\left(\frac{6001}{9000}\right)$$

$$n = -\frac{\log\left(\frac{6001}{9000}\right)}{\log\left(1+\frac{0.2999}{12}\right)}$$

$$n \approx 16.419$$

   So it takes 1 year and 5 months to pay off the debt.

   b. **Calculate the total amount paid over the life of the debt.**

   $16.419 \cdot \$150 = \$2462.85$

   c. **How much money was paid entirely to the interest on this debt?**

   $\$462.85$

2. Suppose that you have a $\$2,000$ balance on a credit card with a 14.99% annual interest rate, and you can afford to pay $\$150$ per month toward this debt.

a. Find the amount of time it takes to pay off this debt. Give your answer in months and years.

$$2000\left(1+\frac{0.1499}{12}\right)^n - 150\left(\frac{1-\left(1+\frac{0.1499}{12}\right)^n}{-\frac{0.1499}{12}}\right) = 0$$

$$2000\left(1+\frac{0.1499}{12}\right)^n = 150\left(\frac{\left(1+\frac{0.1499}{12}\right)^n - 1}{\frac{0.1499}{12}}\right)$$

$$\frac{1499}{9000}\left(1+\frac{0.1499}{12}\right)^n = \left(1+\frac{0.1499}{12}\right)^n - 1$$

$$\left(1+\frac{0.1499}{12}\right)^n\left(\frac{1499}{9000}-1\right) = -1$$

$$\left(1+\frac{0.1499}{12}\right)^n\left(1-\frac{1499}{9000}\right) = 1$$

$$n\cdot\log\left(1+\frac{0.1499}{12}\right) + \log\left(\frac{7501}{9000}\right) = \log(1)$$

$$n\cdot\log\left(1+\frac{0.1499}{12}\right) = -\log\left(\frac{7501}{9000}\right)$$

$$n = -\frac{\log\left(\frac{7501}{9000}\right)}{\log\left(1+\frac{0.1499}{12}\right)}$$

$$n \approx 14.676$$

The loan is paid off in 1 year and 3 months.

b. Calculate the total amount paid over the life of the debt.

$14.676\cdot\$150 = \$2,201.40$

c. How much money was paid entirely to the interest on this debt?

$\$201.40$

3. **Suppose that you have a $\$2,000$ balance on a credit card with a 7.99% annual interest rate, and you can afford to pay $\$150$ per month toward this debt.**

   a. **Find the amount of time it takes to pay off this debt. Give your answer in months and years.**

$$2000\left(1+\frac{0.0799}{12}\right)^n - 150\left(\frac{1-\left(1+\frac{0.0799}{12}\right)^n}{-\frac{0.0799}{12}}\right) = 0$$

$$2000\left(1+\frac{0.0799}{12}\right)^n = 150\left(\frac{\left(1+\frac{0.0799}{12}\right)^n - 1}{\frac{0.0799}{12}}\right)$$

$$\frac{799}{9000}\left(1+\frac{0.0799}{12}\right)^n = \left(1+\frac{0.0799}{12}\right)^n - 1$$

$$\left(1+\frac{0.0799}{12}\right)^n\left(\frac{799}{9000}-1\right) = -1$$

$$\left(1+\frac{0.0799}{12}\right)^n\left(1-\frac{799}{9000}\right) = 1$$

$$n\cdot\log\left(1+\frac{0.0799}{12}\right) + \log\left(\frac{8201}{9000}\right) = \log(1)$$

$$n\cdot\log\left(1+\frac{0.0799}{12}\right) = -\log\left(\frac{8201}{9000}\right)$$

$$n = -\frac{\log\left(\frac{8201}{9000}\right)}{\log\left(1+\frac{0.0799}{12}\right)}$$

$$n \approx 14.009$$

   The loan is paid off in 1 year and 3 months.

   b. **Calculate the total amount paid over the life of the debt.**

   $14.009 \cdot \$150 = \$2101.35$

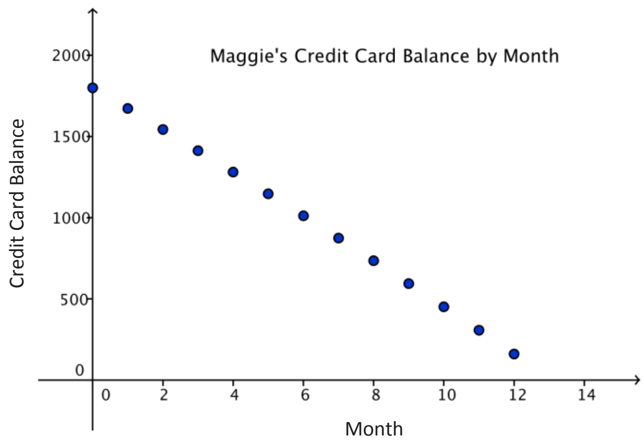
   c. **How much money was paid entirely to the interest on this debt?**

   $\$101.35$

4. **Summarize the results of Problems 1, 2, and 3.**

   Answers will vary but should include the fact that the total interest paid in each case dropped by about half with every problem. Lower interest rates meant that the loan was paid off more quickly and that less was paid in total.

5. Brendan owes $\$1,500$ on a credit card with an interest rate of 12%. He is making payments of $\$100$ every month to pay this debt off. Maggie is also making regular payments to a debt owed on a credit card, and she created the following graph of her projected balance over the next 12 months.



a. Who has the higher initial balance? Explain how you know.

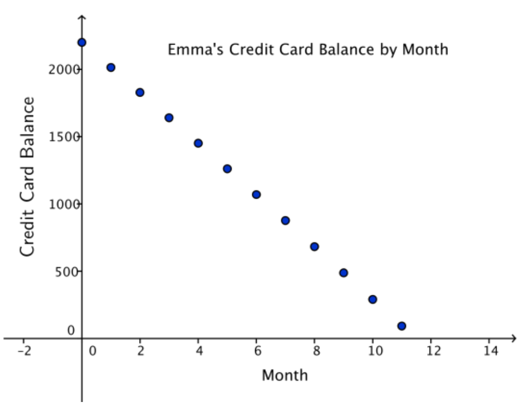
Reading from the graph, Maggie's initial balance is between $\$1,700$ and $\$1,800$, and we are given that Brendan's initial balance is $\$1,500$, so Maggie has the larger initial balance.

b. Who will pay their debt off first? Explain how you know.

From the graph, it appears that Maggie will pay off her debt between months 12 and 14. Brendan's balance in month n can be modeled by the function $b_n = 1500(1.01)^n - 100\left(\frac{1.01^n - 1}{0.01}\right)$, which is equal to zero when $n \approx 16.3$. Thus, Brendan's debt will be paid in month 17, so Maggie's debt will be paid off first.

6. Alan and Emma are both making $\$200$ monthly payments toward balances on credit cards. Alan has prepared a table to represent his projected balances, and Emma has prepared a graph.

Alan's Credit Card Balance			
Month, n	Interest	Payment	Balance, b_n
0	———	———	$2,000.00$
1	41.65	200	$1,841.65$
2	38.35	200	$1,680.00$
3	34.99	200	$1,514.99$
4	31.55	200	$1,346.54$
5	28.04	200	$1,174.58$
6	24.46	200	999.04
7	20.81	200	819.85
8	17.07	200	636.92
9	13.26	200	450.18
10	9.37	200	259.55
11	5.41	200	64.96



a. What is the annual interest rate on Alan's debt? Explain how you know.

One month's interest on the balance of $\$2,000$ was $\$41.65$, so $41.65 = i(2000)$. Then the monthly interest rate is $i = 0.020825$, and the annual rate is $12i = 0.2499$, so the annual rate on Alan's debt is 24.99%.

b. **Who has the higher initial balance? Explain how you know.**

From the table, we can see that Alan's initial balance is $\$2,000$, while Emma's initial balance is the y-intercept of the graph, which is above $\$2,000$. Thus, Emma's initial balance is higher.

c. **Who will pay their debt off first? Explain how you know.**

Both Alan and Emma will pay their debts off in month 12 because both of their balances in month 11 are under $\$100$.

d. **What do your answers to parts (a), (b), and (c) tell you about the interest rate for Emma's debt?**

Because Emma had the higher initial balance, and they made the same number of payments, Emma must have a lower interest rate on her credit card than Alan does. In fact, since the graph decreases apparently linearly, this implies that Emma has an interest rate of 0%.

7. **Both Gary and Helena are paying regular monthly payments to a credit card balance. The balance on Gary's credit card debt can be modeled by the recursive formula $g_n = g_{n-1}(1.01666) - 200$ with $g_0 = 2500$, and the balance on Helena's credit card debt can be modeled by the explicit formula $h_n = 2000(1.01666)^n - 250\left(\frac{1.01666^n - 1}{0.01666}\right)$ for $n \geq 0$.**

   a. **Who has the higher initial balance? Explain how you know.**

   Gary has the higher initial balance. Helena's initial balance is $\$2,000$, and Gary's is $\$2,500$.

   b. **Who has the higher monthly payment? Explain how you know.**

   Helena has the higher monthly payment. She is paying $\$250$ every month while Gary is paying $\$200$.

   c. **Who will pay their debt off first? Explain how you know.**

   Helena will pay her debt off first since she starts at a lower balance and is paying more per month. Additionally, they appear to have the same interest rates.

8. **In the next lesson, we will apply the mathematics we have learned to the purchase of a house. In preparation for that task, you need to come to class prepared with an idea of the type of house you would like to buy.**

   a. **Research the median housing price in the county where you live or where you wish to relocate.**

   Answers will vary.

   b. **Find the range of prices that are within 25% of the median price from part (a). That is, if the price from part (a) was P, then your range is $0.75P$ to $1.25P$.**

   Answers will vary.

   c. **Look at online real estate websites, and find a house located in your selected county that falls into the price range specified in part (b). You will be modeling the purchase of this house in Lesson 32, so bring a printout of the real estate listing to class with you.**

   Answers will vary.

9. **Select a career that interests you from the following list of careers. If the career you are interested in is not on this list, check with your teacher to obtain permission to perform some independent research. Once it has been selected, use the career to answer questions in Lesson 32 and Lesson 33.**

Occupation	Median Starting Salary	Education Required
Entry-level full-time (wait staff, office clerk, lawn care worker, etc.)	$\$20,200$	High school diploma or GED
Accountant	$\$54,630$	4-year college degree
Athletic Trainer	$\$36,560$	4-year college degree
Chemical Engineer	$\$78,860$	4-year college degree
Computer Scientist	$\$93,950$	4-year college degree or more
Database Administrator	$\$64,600$	4-year college degree
Dentist	$\$136,960$	Graduate degree
Desktop Publisher	$\$34,130$	4-year college degree
Electrical Engineer	$\$75,930$	4-year college degree
Graphic Designer	$\$39,900$	2- or 4-year college degree
HR Employment Specialist	$\$42,420$	4-year college degree
HR Compensation Manager	$\$66,530$	4-year college degree
Industrial Designer	$\$54,560$	4-year college degree or more
Industrial Engineer	$\$68,620$	4-year college degree
Landscape Architect	$\$55,140$	4-year college degree
Lawyer	$\$102,470$	Law degree
Occupational Therapist	$\$60,470$	Master's degree
Optometrist	$\$91,040$	Master's degree
Physical Therapist	$\$66,200$	Master's degree
Physician—Anesthesiology	$\$259,948$	Medical degree
Physician—Family Practice	$\$137,119$	Medical degree
Physician's Assistant	$\$74,980$	2 years college plus 2-year program
Radiology Technician	$\$47,170$	2-year degree
Registered Nurse	$\$57,280$	2- or 4-year college degree plus
Social Worker—Hospital	$\$48,420$	Master's degree
Teacher—Special Education	$\$47,650$	Master's degree
Veterinarian	$\$71,990$	Veterinary degree

Lesson 32: Buying a House

Student Outcomes

- Students model the scenario of buying a house.
- Students recognize that a mortgage is mathematically equivalent to car loans studied in Lesson 30 and apply the present value of annuity formula to a new situation.

Lesson Notes

In the Problem Set of Lesson 31, students selected both a future career and a home that they would like to purchase. In this lesson, the students investigate the question of whether or not they can afford the home that they have selected on the salary of a novice working in the career that they have chosen. We do not develop the standard formulas for mortgage payments, but rather students use the concepts from prior lessons on buying a car and paying off a credit card balance to decide for themselves how to model mortgage payments (MP.4). Have students work in pairs or small groups through this lesson, but each student should be working through their own scenario with their own house and their own career. That is, students decide together how to approach the problem, but they each work with their own numbers.

If you teach in a region where the cost of living is particularly high, the median starting salaries given in the list in Problem 9 of Lesson 31 may need to be appropriately adjusted upward in order to make any home purchase feasible in this exercise. Use your professional judgment to make these adjustments.

The students have the necessary mathematical tools to model the payments on a mortgage, but they may not realize it. Allow them to struggle, to debate, and to persevere with the task of deciding how to model this situation (MP.1). It will eventually become apparent that the process of buying a house is only slightly more complicated mathematically than the process of buying a car and that the present value of an annuity formula developed in Lesson 30 applies in this situation (**A-SSE.B.4**). The formula

$$A_p = R\left(\frac{1-(1+i)^{-n}}{i}\right)$$

can be solved for the monthly payment R:

$$R = \frac{A_p \cdot i}{1-(1+i)^{-n}}$$

and this formula can be used to answer many of the questions in this lesson. Students may apply the formulas immediately, or they may investigate the balance on the mortgage without using the formulas, which should lead them to develop these formulas on their own. Be sure to ask students to explain their thinking in order to accurately assess their understanding of the mathematics.

Classwork

Opening (3 minutes)

- As part of your homework last night, you have selected a potential career that interests you, and you have selected a house that you would like to purchase.

Call on a few students to ask them to share the careers that they have selected, the starting salary, and the price of the home they have chosen.

- Today you answer the following question: Can you afford the house that you have chosen? There are a few constraints that you need to keep in mind.
  - The total monthly payment for the house cannot exceed 30% of your monthly salary.
  - Your payment includes the payment of the loan for the house and payments into an account called an *escrow account*, which is used to pay for taxes and insurance on your home.
  - Mortgages are usually offered with 30, 20, or 15-year repayment options. Start with a 30-year mortgage.
  - You need to make a down payment on the house, meaning that you pay a certain percentage of the price up front and borrow the rest. You make a 10% down payment for this exercise.

> *Scaffolding:*
>
> For struggling students, illustrate the concepts of mortgage, escrow, and down payments using a concrete example with sample values.

Mathematical Modeling Exercise (25 minutes)

Students may immediately recognize that the previous formulas from Lessons 30 and 31 can be applied to a mortgage, or they may investigate the balance on the mortgage without using the formulas. Both approaches are presented in the sample responses below.

> **Mathematical Modeling Exercise**
>
> **Now that you have studied the mathematics of structured savings plans, buying a car, and paying down a credit card debt, it's time to think about the mathematics behind the purchase of a house. In the problem set in Lesson 31, you selected a future career and a home to purchase. The question of the day is this: Can you buy the house you have chosen on the salary of the career you have chosen? You need to adhere to the following constraints:**
>
> - **Mortgages are loans that are usually offered with 30-, 20-, or 15-year repayment options. Start with a 30-year mortgage.**
> - **The annual interest rate for the mortgage will be 5%.**
> - **Your payment includes the payment of the loan for the house and payments into an account called an *escrow account*, which is used to pay for taxes and insurance on your home. We approximate the annual payment to escrow as 1.2% of the home's selling price.**
> - **The bank can only approve a mortgage if the total monthly payment for the house, including the payment to the escrow account, does not exceed 30% of your monthly salary.**
> - **You have saved up enough money to put a 10% down payment on this house.**

> *Scaffolding:*
>
> Struggling students may need to be presented with a set of carefully structured questions:
>
> 1. What is the monthly salary for the career you chose?
> 2. What is 30% of your monthly salary?
> 3. How much money needs to be paid into the escrow account each year?
> 4. How much money needs to be paid into the escrow account each month?
> 5. What is the most expensive house that the bank will allow you to purchase?
> 6. Is a mortgage like a car loan?
> 7. What is the formula we used to model a car loan?
> 8. Which of the values A_p, n, i, and R do we know?
> 9. Can you rewrite that formula to isolate the R?
> 10. What is the monthly payment according to the formula?
> 11. Will the bank allow you to purchase the house that you have chosen?

1. **Will the bank approve a 30-year mortgage on the house that you have chosen?**

I chose the career of a graphic designer, with a starting salary of $\$39,900$. My monthly salary is $\frac{\$39900}{12} = \$3325$.

Thirty percent of my $\$3,325$ monthly salary is $\$997.50$.

I found a home that is suitable for $\$190,000$.

Since $0.012(190000) = 2280$, I need to contribute $\$2,280$ to escrow for the year, which means I need to pay $\$190$ to escrow each month.

I will make a $\$19,000$ down payment, meaning that I need a mortgage for $\$171,000$.

APPROACH 1: We can think of the total owed on the house in two different ways.

- *If we had placed the original loan amount $A_p = 171000$ in a savings account earning 5% annual interest, then the future amount in 30 years would be $A_f = A_p(1+i)^{360}$.*
- *If we deposit a payment of R into an account monthly and let the money in the account accumulate and earn interest for 30 years, then the future value is*

$$\begin{aligned} A_f &= R + R(1+i) + R(1+i)^2 + \cdots + R(1+i)^{359} \\ &= R\sum_{k=0}^{359}(1+i)^k \\ &= R\left(\frac{1-(1+i)^{360}}{1-(1+i)}\right) \\ &= R\left(\frac{(1+i)^{360}-1}{i}\right) \end{aligned}$$

Setting these two expressions for A_f equal to each other, we have

$$A_p(1+i)^{360} = R\left(\frac{(1+i)^{360}-1}{i}\right),$$

so

$$R = \frac{A_p \cdot i \cdot (1+i)^{360}}{(1+i)^{360}-1},$$

which can also be expressed as

$$R = \frac{A_p \cdot i}{1-(1+i)^{-360}}.$$

This is the formula for the present value of an annuity, but rewritten to isolate R.

Then using my values of A_p, i and n we have

$$R = \frac{171000(0.004167)}{1-(1.004167)^{-360}}$$
$$R = 918.01.$$

Then, the monthly payment on the house I chose would be $R + 190 = 1\,108.01$. The bank will not lend me the money to buy this house because $\$1,108.01$ is higher than $\$997.50$.

APPROACH 2: From Lesson 30, we know that the present value of an annuity formula is $A_p = R\left(\frac{1-(1+i)^{-n}}{i}\right)$, where i is the monthly interest rate, R is the monthly payment, and n is the number of months in the term. In my example, $i = \frac{0.05}{12} \approx 0.004167$, R is unknown, $n = 12 \cdot 30 = 360$, and $A_p = 171000$. We can solve the above formula for R, then we can substitute the known values of the variables and calculate the resulting payment R.

$$A_p = R\left(\frac{1-(1+i)^{-n}}{i}\right)$$
$$A_p \cdot i = R(1-(1+i)^{-n})$$

$$R = \frac{A_p \cdot i}{1-(1+i)^{-n}}$$

Then using my values of A_p, i and n we have

$$R = \frac{171\,000(0.004167)}{1-(1.004167)^{-360}}$$
$$R = 918.01.$$

Then, the monthly payment on the house I chose would be $R + 190 = 1108.01$. The bank will not lend me the money to buy this house because $\$1,108.01$ is higher than $\$997.50$.

2. **Answer either (a) or (b) as appropriate.**
   a. **If the bank approves a 30-year mortgage, do you meet the criteria for a 20-year mortgage? If you could get a mortgage for any number of years that you want, what is the shortest term for which you would qualify?**

      (This scenario did not happen in this example.)

   b. **If the bank does not approve a 30-year mortgage, what is the maximum price of a house that fits your budget?**

      The maximum that the bank allows for my monthly payment is 30% of my monthly salary, which is $\$997.50$. This includes the payment to the loan and to escrow. If the total price of the house is H dollars, then I will make a down payment of $0.1H$ and finance $0.9H$. Using the present value of an annuity formula, we have

$$0.9H = R\left(\frac{1-(1+i)^{-n}}{i}\right)$$
$$0.9H = R\left(\frac{1-(1.004167)^{-360}}{0.004167}\right)$$
$$0.9H = R(186.282)$$

      However, R represents just the payment to the loan and not the payment to the escrow account. We know that the escrow portion is one-twelfth of 1.2% of the house value. If we denote the total amount paid for the loan and escrow by P, then $P = R + 0.001H$, so $R = P - 0.001H$. We know that the largest value for P is $P = 997.50$, so then

$$0.9H = R(186.282)$$
$$0.9H = (997.50 - 0.001H)(186.282)$$
$$0.9H = 185816 - 0.186282H$$
$$1.086282H = 185816$$
$$H = 171056.87$$

      Then, I can only afford a house that is priced at or below $\$171,056.87$.

Scaffolding:

Mortgage rates can be as low as 3.0%, and in the 1990's rates were often as high as 10%. Ask early finishers to compute the maximum price of a house that they can afford first with an annual interest rate of 5%, then with an annual interest rate of 3%, and then with an annual interest rate of 10%.

Discussion (9 minutes)

As time permits, ask students to present their results to the class and to explain their thinking. Select students who were approved for their mortgage and those who were not approved to make presentations. Be sure that students who did not immediately recognize that the present value of an annuity formula applies to a mortgage understand that this method is valid. Then, debrief the modeling exercise with the following questions:

- If the bank did not approve the loan, what are your options?
  - *I could wait to purchase the house and save up a larger down payment, I could get a higher-paying job, or I could look for a more reasonably priced house.*
- What would happen if the annual interest rate on the mortgage increased to 8%?
  - *If the annual interest rate on the mortgage increased to 8%, then the monthly payments would increase dramatically since the loan term is always fixed.*
- Why does the bank limit the amount of the mortgage to 30% of your income?
  - *The bank wants to ensure that you will pay back the loan and that you will not overextend your finances.*

Closing (3 minutes)

Ask students to summarize the lesson with a partner or in writing by responding to the following questions:

- Which formula from the previous lessons was useful to calculate the monthly payment on the mortgage? Why did that formula apply to this situation?
- How is a mortgage like a car loan? How is it different?
- How is paying a mortgage like paying a credit card balance? How is it different?

Exit Ticket (5 minutes)

Name ____ Date ____________________

Lesson 32: Buying a House

Exit Ticket

1. Recall the present value of an annuity formula, where A_p is the present value, R is the monthly payment, i is the monthly interest rate, and n is the number of monthly payments:

$$A_p = R\left(\frac{1-(1+i)^{-n}}{i}\right).$$

Rewrite this formula to isolate R.

2. Suppose that you want to buy a house that costs $\$175{,}000$. You can make a 10% down payment, and 1.2% of the house's value is paid into the escrow account each month.

   a. Find the monthly payment for a 30-year mortgage on this house.

   b. Find the monthly payment for a 15-year mortgage on this house.

Exit Ticket Sample Solutions

1. Recall the present value of an annuity formula, where A_p is the present value, R is the monthly payment, i is the monthly interest rate, and n is the number of monthly payments:

$$A_p = R\left(\frac{1-(1+i)^{-n}}{i}\right).$$

Rewrite this formula to isolate R.

$$R = \frac{A_p}{\frac{1-(1+i)^{-n}}{i}}$$

$$R = \frac{A_p \cdot i}{1-(1+i)^{-n}}$$

2. Suppose that you want to buy a house that costs $\$175,000$. You can make a 10% down payment, and 1.2% of the house's value is paid into the escrow account each month.

a. Find the total monthly payment for a 30-year mortgage at 4.25% interest on this house.

We have $A_p = 0.9(175000) = 157500$, and the monthly escrow payment is $\frac{1}{12}(0.012)(\$175000) = \$175$. The monthly interest rate i is given by $i = \frac{0.045}{12} = 0.00375$, and $n = 12 \cdot 30 = 360$. Then the formula from Problem 1 gives

$$\begin{aligned} R &= \frac{A_p \cdot i}{1-(1+i)^{-n}} \\ &= \frac{(157500)(0.00375)}{1-(1.00375)^{-360}} \\ &\approx 798.03 \end{aligned}$$

Thus, the payment to the loan is $\$798.03$ each month. Then the total monthly payment is $\$798.03 + \$175 = \$973.03$.

b. Find the total monthly payment for a 15-year mortgage at 3.75% interest on this house.

We have $A_p = 0.9(175000) = 157500$, and the monthly escrow payment is $\frac{1}{12}(0.012)(\$175000) = \$175$. The monthly interest rate i is given by $i = \frac{0.0375}{12} = 0.003125$, and $n = 12 \cdot 15 = 180$. Then the formula from Problem 1 gives

$$\begin{aligned} R &= \frac{A_p \cdot i}{1-(1+i)^{-n}} \\ &= \frac{(157500)(0.003125)}{1-(1.003\,125)^{-180}} \\ &\approx 1145.38 \end{aligned}$$

Thus, the payment to the loan is $\$1,145.38$ each month. Then the total monthly payment is $\$1,145.38 + \$175 = \$1,320.38$.

Problem Set Sample Solutions

The results of Exercise 1 are needed for the modeling exercise in Lesson 33, in which students make a plan to save up $\$1{,}000{,}000$ in assets in 15 years, including paying off their home in that time.

1. **Use the house you selected to purchase in the Problem Set from Lesson 31 for this problem.**

   a. **What was the selling price of this house?**

   Student responses will vary. The sample response will continue to use a house that sold for $\$190,000$.

   b. **Calculate the total monthly payment, R, for a 15-year mortgage at 5% annual interest, paying 10% as a down payment and an annual escrow payment that is 1.2% of the full price of the house.**

   Using the payment formula with $A_p = 0.9(190000) = 171000$, $i = \frac{0.05}{12} \approx 0.004167$, *and* $n = 15 \cdot 12 = 180$, *we have*

$$\begin{aligned} R &= \frac{A_p \cdot i}{1-(1+i)^{-n}} \\ &= \frac{(171000)(0.004167)}{1-(1.004167)^{-180}} \\ &\approx 1352.29 \end{aligned}$$

   The escrow payment is $\frac{1}{12}(0.012)(\$190000) = \$190.00$. *The total monthly payment is* $\$1,352.29 + \$190.00 = \$1,542.29$.

2. **In the summer of 2014, the average listing price for homes for sale in the Hollywood Hills was $\$2,663,995$.**

   a. **Suppose you want to buy a home at that price with a 30-year mortgage at 5.25% annual interest, paying 10% as a down payment and with an annual escrow payment that is 1.2% of the full price of the home. What is your total monthly payment on this house?**

   Using the payment formula with $A_p = 0.9(2663995) = 2397595.50$, $i = \frac{0.0525}{12} \approx 0.004375$, *and* $n = 360$, *we have*

$$\begin{aligned} R &= \frac{A_p \cdot i}{1-(1+i)^{-n}} \\ &= \frac{(2397595.50)(0.004375)}{1-(1.004375)^{-360}} \\ &= 13239.61 \end{aligned}$$

   The escrow payment is $\frac{1}{12}(0.012)(\$2663995) = \$2664.00$. *The total monthly payment is* $\$13,239.61 + \$2,664 = \$15,903.61$.

   b. **How much is paid in interest over the life of the loan?**

   The total amount paid is $\$13239.61(360) = \$4766259.60$, *and the purchase price was* $\$2,663,995$. *The amount of interest is the difference:* $\$4,766,259.60 - \$2,663,995 = \$2,102,264.60$.

3. Suppose that you would like to buy a home priced at $\$200,000$. You plan to make a payment of 10% of the purchase price and pay 1.2% of the purchase price into an escrow account annually.

   a. Compute the total monthly payment and the total interest paid over the life of the loan for a 30-year mortgage at 4.8% annual interest.

   Using the payment formula with $A_p = 0.9(200000) = 180000$, $i = \frac{0.048}{12} = 0.004$, and $n = 360$, we have

$$\begin{aligned} R &= \frac{A_p \cdot i}{1-(1+i)^{-n}} \\ &= \frac{(180000)(0.004)}{1-(1.004)^{-360}} \\ &\approx 994.40 \end{aligned}$$

   The escrow payment is $\frac{1}{12}(0.012)(\$200000) = \$200.00$. The total monthly payment is $\$994.40 + \$200 = \$1,194.40$.

   The total amount of interest is the difference between the total amount paid, $360(\$994.40)$, which is $\$357,984$, and the selling price $\$200,000$, so the total interest paid is $\$157,984$.

   b. Compute the total monthly payment and the total interest paid over the life of the loan for a 20-year mortgage at 4.8% annual interest.

   Using the payment formula with $A_p = 0.9(200000) = 180000$, $i = \frac{0.048}{12} = 0.004$, and $n = 240$, we have

$$\begin{aligned} R &= \frac{A_p \cdot i}{1-(1+i)^{-n}} \\ &= \frac{(180000)(0.004)}{1-(1.004)^{-240}} \\ &\approx 1168.12 \end{aligned}$$

   The escrow payment is $\frac{1}{12}(0.012)(\$200000) = \$200.00$. The total monthly payment is $\$1,168.12 + \$200 = \$1,368.12$.

   The total amount of interest is the difference between the total amount paid, $240(\$1168.12)$, which is $\$280,348.80$, and the selling price, $\$200,000$, so the total interest paid is $\$80,348.80$.

   c. Compute the total monthly payment and the total interest paid over the life of the loan for a 15-year mortgage at 4.8% annual interest.

   Using the payment formula with $A_p = 0.9(200000) = 180000$, $i = \frac{0.048}{12} = 0.004$, and $n = 180$, we have

$$\begin{aligned} R &= \frac{A_p \cdot i}{1-(1+i)^{-n}} \\ &= \frac{(180000)(0.004)}{1-(1.004)^{-180}} \\ &\approx 1404.75 \end{aligned}$$

   The escrow payment is $\frac{1}{12}(0.012)(\$200000) = \$200.00$. The total monthly payment is $\$1,404.75 + \$200 = \$1,604.75$.

   The total amount of interest is the difference between the total amount paid, $180(\$1404.75)$, which is $\$252,855$, and the selling price, $\$200,000$, so the total interest paid is $\$52,855$.

4. Suppose that you would like to buy a home priced at $\$180,000$. You qualify for a 30-year mortgage at 4.5% annual interest and pay 1.2% of the purchase price into an escrow account annually.

a. Calculate the total monthly payment and the total interest paid over the life of the loan if you make a 3% down payment.

With a three percent down payment, you need to borrow $A_p = 0.97(\$180000) = \$174600$. *We have* $i = \frac{0.045}{12} = 0.00375$, *and* $n = 360$, *so*

$$\begin{aligned} R &= \frac{A_p \cdot i}{1-(1+i)^{-n}} \\ &= \frac{(174600)(0.00375)}{1-(1.00375)^{-360}} \\ &\approx 884.67 \end{aligned}$$

The escrow payment is $\frac{1}{12}(0.012)(\$180000) = \$180.00$. *The total monthly payment is* $\$884.67 + \$180 = \$1,064.67$.

The total amount of interest is the difference between the total amount paid, $360(\$884.67)$, *which is* $\$318,481.20$, *and the selling price,* $\$180,000$, *so the total interest paid is* $\$138,481.20$.

b. Calculate the total monthly payment and the total interest paid over the life of the loan if you make a 10% down payment.

With a ten percent down payment, you need to borrow $A_p = 0.9(\$180000) = \$162000$. *We have* $i = \frac{0.045}{12} = 0.00375$, *and* $n = 360$, *so*

$$\begin{aligned} R &= \frac{A_p \cdot i}{1-(1+i)^{-n}} \\ &= \frac{(162000)(0.00375)}{1-(1.00375)^{-360}} \\ &\approx 820.83 \end{aligned}$$

The escrow payment is $\frac{1}{12}(0.012)(\$180000) = \$180.00$. *The total monthly payment is* $\$820.83 + \$180 = \$1,000.83$.

The total amount of interest is the difference between the total amount paid, $360(\$820.83)$, *which is* $\$295,498.80$, *and the selling price,* $\$180,000$, *so the total interest paid is* $\$115,498.80$.

c. Calculate the total monthly payment and the total interest paid over the life of the loan if you make a 20% down payment.

With a twenty percent down payment, you need to borrow $A_p = 0.8(\$180000) = \$144000$. *We have* $i = \frac{0.045}{12} = 0.00375$, *and* $n = 360$, *so*

$$\begin{aligned} R &= \frac{A_p \cdot i}{1-(1+i)^{-n}} \\ &= \frac{(144000)(0.00375)}{1-(1.00375)^{-360}} \\ &\approx 729.63 \end{aligned}$$

The escrow payment is $\frac{1}{12}(0.012)(\$180000) = \$180.00$. *The total monthly payment is* $\$729.63 + \$180 = \$909.63$.

The total amount of interest is the difference between the total amount paid, $360(\$729.63)$, *which is* $\$262,666.80$, *and the selling price,* $\$180,000$, *so the total interest paid is* $\$82,666.80$.

d. Summarize the results of parts (a), (b), and (c) in the chart below.

Percent Down Payment	Amount of Down Payment	Total Interest Paid
3%	$\$5,400$	$\$138,481.20$
10%	$\$18,000$	$\$115,498.80$
20%	$\$36,000$	$\$82,666.80$

5. The following amortization table shows the amount of payments to principal and interest on a $\$100,000$ mortgage at the beginning and the end of a 30-year loan. These payments do not include payments to the escrow account.

Month/ Year	Payment	Principal Paid	Interest Paid	Total Interest	Balance
Sept. 2014	$ 477.42	$ 144.08	$ 333.33	$ 333.33	$ 99,855.92
Oct. 2014	$ 477.42	$ 144.56	$ 332.85	$ 666.19	$ 99,711.36
Nov. 2014	$ 477.42	$ 145.04	$ 332.37	$ 998.56	$ 99,566.31
Dec. 2014	$ 477.42	$ 145.53	$ 331.89	$ 1,330.45	$ 99,420.78
Jan. 2015	$ 477.42	$ 146.01	$ 331.40	$ 1,661.85	$ 99,274.77
Mar. 2044	$ 477.42	$ 467.98	$ 9.44	$ 71,845.82	$ 2,363.39
April 2044	$ 477.42	$ 469.54	$ 7.88	$ 71,853.70	$ 1,893.85
May 2044	$ 477.42	$ 471.10	$ 6.31	$ 71,860.01	$ 1,422.75
June 2044	$ 477.42	$ 472.67	$ 4.74	$ 71,864.75	$ 950.08
July 2044	$ 477.42	$ 474.25	$ 3.17	$ 71,867.92	$ 475.83
Aug. 2044	$ 477.42	$ 475.83	$ 1.59	$ 71,869.51	$ 0.00

a. What is the annual interest rate for this loan? Explain how you know.

Since the first interest payment is $i \cdot \$100000 = \$333.33$, the monthly interest rate is $i = 0.0033333$, and the annual interest rate is then $12i \approx 0.0399996$, so $i = 4.0\%$.

b. Describe the changes in the amount of principal paid each month as the month n gets closer to 360.

As n gets closer to 360, the amount of the payment that is allocated to principal increases to nearly the amount of the entire payment.

c. Describe the changes in the amount of interest paid each month as the month n gets closer to 360.

As n gets closer to 360, the amount of the payment that is allocated to interest decreases to nearly zero.

6. **Suppose you want to buy a $\$200,000$ home with a 30-year mortgage at 4.5% annual interest paying 10% down with an annual escrow payment that is 1.2% of the price of the home.**

   a. **Disregarding the payment to escrow, how much do you pay toward the loan on the house each month?**

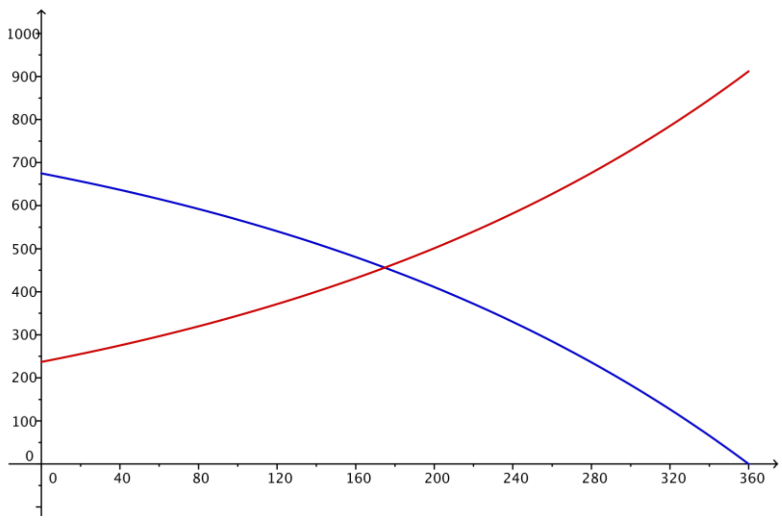
$$\begin{aligned} R &= \frac{A_p \cdot i}{1-(1+i)^{-n}} \\ &= \frac{(180000)(0.00375)}{1-(1.00375)^{-360}} \\ &\approx 912.03 \end{aligned}$$

The amount paid toward the loan on the house each month is $\$912.03$.

   b. **What is the total monthly payment on this house?**

The monthly escrow payment is $\frac{1}{12}(0.012)(200000) = 200$, so the total monthly payment is $\$1,112.03$.

   c. **The graph below depicts the amount of your payment from part (b) that goes to the interest on the loan and the amount that goes to the principal on the loan. Explain how you can tell which graph is which.**



The amount paid to interest starts high and decreases, while the amount paid to principal starts low and then increases over the life of the loan. Thus, the blue curve that starts around 675 and decreases represents the amount paid to interest, and the red curve that starts around 240 and increases represents the amount paid to principal.

7. **Student loans are very similar to both car loans and mortgages. The same techniques used for car loans and mortgages can be used for student loans. The difference between student loans and other types of loans is that usually students are not required to pay anything until 6 months after they stop being full-time students.**

   a. **An unsubsidized student loan accumulates interest while a student remains in school. Sal borrows $\$9,000$ his first term in school at an interest rate of 5.95% per year compounded monthly and never makes a payment. How much will he owe $4\frac{1}{2}$ years later? How much of that amount is due to compounded interest?**

This is a compound interest problem without amortization since Sal does not make any payments.

$$9000 \cdot \left(1+\frac{0.0595}{12}\right)^{54} \approx 11755.40$$

Sal will owe $\$11,755.40$ at the end of $4\frac{1}{2}$ years. Since he borrowed $\$9,000$, he owes $\$2,755.40$ in interest.

b. **If Sal pays the interest on his student loan every month while he is in school, how much money has he paid?**

Since Sal pays the interest on his loan every month, the principal never grows. Every month, the interest is calculated by

$$9000 \cdot \frac{0.0595}{12} \approx \$44.63.$$

If Sal pays $\$44.63$ *every month for* $4\frac{1}{2}$ *years, he will have paid* $\$2,410.02$.

c. **Explain why the answer to part (a) is different than the answer to part (b).**

If Sal pays the interest each month, as in part (b), then no interest ever compounds. If he skips the interest payments while he is in school, then the compounding process charges interest on top of interest, increasing the total amount of interest owed on the loan.

8. **Consider the sequence** $a_0 = 10000$, $a_n = a_{n-1} \cdot \frac{1}{10}$ **for** $n \geq 1$.

a. **Write the explicit form for the** n^{th} **term of the sequence.**

$$a_1 = 10000\left(\frac{1}{10}\right) = 1000$$
$$a_2 = \frac{1}{10}(a_1) = 10000\left(\frac{1}{10}\right)^2$$
$$a_3 = \frac{1}{10}(a_2) = 10000\left(\frac{1}{10}\right)^3$$
$$\vdots$$
$$a_n = 10000\left(\frac{1}{10}\right)^n$$

b. **Evaluate** $\sum_{k=0}^{4} a_k$.

$$\sum_{k=0}^{4} a_k = 10000 + 1000 + 100 + 10 + 1 = 11111$$

c. **Evaluate** $\sum_{k=0}^{6} a_k$.

$$\sum_{k=0}^{6} a_k = 10000 + 1000 + 100 + 10 + 1 + 0.1 + 0.01 = 11111.11$$

d. **Evaluate** $\sum_{k=0}^{8} a_k$ **using the sum of a geometric series formula.**

$$\sum_{k=0}^{8} a_k = 10000\frac{(1-r^9)}{1-r}$$
$$= 10000\frac{\left(1-\left(\frac{1}{10}\right)^9\right)}{1-\frac{1}{10}}$$
$$= 11,111.1111$$

e. **Evaluate $\sum_{k=0}^{10} a_k$ using the sum of a geometric series formula.**

$$\begin{aligned}\sum_{k=0}^{10} a_k &= 10000\frac{(1-r^{11})}{1-r}\\ &= 10000\frac{\left(1-\left(\frac{1}{10}\right)^{11}\right)}{1-\frac{1}{10}}\\ &= 11111.111\,111\end{aligned}$$

f. **Describe the value of $\sum_{k=0}^{n} a_k$ for any value of $n \geq 4$.**

The value of $\sum_{k=0}^{n} a_k$ for any $n \geq 4$ is $11111.\underbrace{111\ldots1}_{n-4\text{ ones}}$.

Lesson 33: The Million Dollar Problem

Student Outcomes

- Students use geometric series to calculate how much money should be saved each month to have 1 million in assets within a specified amount of time.

Lesson Notes

Amortization calculators and other online calculators are not advanced enough to easily develop a savings plan that results in earning $\$1$ million in assets by the time a student reaches the age of 40. Students continue their exploration of the formula for the future value of a structured savings plan from Lesson 29 (**A-SSE.B.4**).

We continue to use the future value of an annuity formula $A_f = R\left(\frac{(1+i)^n - 1}{i}\right)$ developed in Lesson 29. The formula is discussed more extensively in Lesson 29, but it is always good to remember: The finance formulas in these lessons are direct applications of the sum for a geometric series and the compound interest formula. Throughout these lessons, we have re-derived these formulas in different contexts for two reasons: firstly, so that students recognize the usefulness of geometric series, and secondly, so that students can make the realization that the types of financial activities (savings plans, car loans, credit cards, etc.) initially appear to be different, but in the end all require the same calculation. The goal is for students to continue to build the formulas from geometric series until they are proficient with the meaning and uses of the formulas.

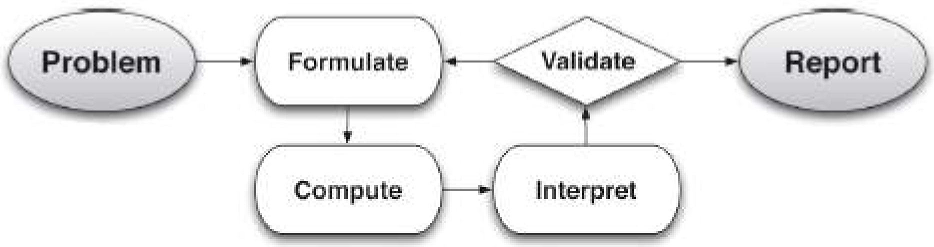
In the future value of an annuity formula stated above, the amount of money that somebody wants to have in the future is A_f. The amount deposited (which is generically called the payment) per compounding period is R, and the interest per compounding period is i. The total number of compounding periods is n. The amount of money it will take the person to save this amount A_f is the monthly payment times the number of payments, nR. When the annuity represents a loan, the monthly payment times the number of payments is called the total cost of a loan.

For loans, we rewrite the basic compound interest formula as $A_f = A_p(1 + i)^n$, set it equal to the formula above from Lesson 29, and solve for the annuity's principal A_p. In this context, the present value of the annuity is the same as the principal of the loan (i.e., the loan amount). The present value of an annuity can be thought of as the lump sum amount one would need to invest now in order to earn the future value of an annuity through compound interest alone.

In this modeling lesson, you have the option of including the appreciation (or depreciation) rate of the property value. Look online for sources such as an interactive map of appreciation rates that can provide the statewide average rate of appreciation (or depreciation) in your region. We use the rate of 2.95% in the sample answers throughout the lesson. The example house used in this lesson continues to be the $\$190{,}000$ house from Lesson 32.

Some fundamental budgeting is included to provide the framework for the house purchase. The material students developed in Lesson 31 is used to provide flexibility to budgets and to discuss paying off debts in the context of budgets. Students develop and combine functions for the appreciation of their home, the balance in a savings account, and the value of their car to answer the question of the lesson: How can I accumulate $\$1{,}000{,}000$ in assets (**F-BF.A.1b**)?

A copy of the modeling cycle flowchart is included below to assist with the modeling portions of these lessons. Whenever students consider a modeling problem, they need first to identify variables representing essential features in the situation, formulate a model to describe the relationships between the variables, analyze and perform operations on the relationships, interpret their results in terms of the original situation, validate their conclusions, and either improve the model or report their conclusions and their reasoning. The exercises provided in this lesson suggest and provide a road map for you to structure the lesson around the modeling cycle flowchart, but they are only a road map: How much you use the exercises is left to your discretion and as time permits. For example, you may wish to start the class with just the opening question on saving $\$1$ million in 15 years and let the students decide how to move through the modeling cycle on their own without using the exercise questions as prompts. Regardless, each student's report should take into consideration the ideas discussed in the exercises.



Classwork

Opening (4 minutes)

- Now that you are in your mid-twenties, own a car, a house, and have a career, the question remains: What savings plan would you need to generate $\$1{,}000{,}000$ in assets over the next 15 years?
- What assets do we have that we can include?
  - *House, car, and savings*
- Over the long run, property values generally appreciate, but most cars depreciate. Assume the used car you bought back in Lesson 30 is depreciating and not an asset. But the house you bought does hold value (called equity), and that equity is an asset. Let's focus specifically on the value of your house from Lesson 32. What formula can we use to calculate the value of your house in 15 years?
  - *The formula is $F = P(1 + r)^{15}$, where r is the appreciation rate per year (this can be researched on the internet to find your local appreciation rate).*

Inform students what appreciation rate they should use for their house.

- After finding this value, the problem then becomes, "How much do you need to deposit monthly to add up to $\$1{,}000{,}000$ after 15 years?" What type of problem does this sound like?
  - *This sounds like a structured savings plan like we studied in Lesson 29.*

Opening Exercise (15 minutes)

Take the time for students to calculate the estimated value of their home and record the results. Have students plot their appreciation curve over 15 years and compare both the appreciated values of their homes and the graphs they produce with each other (**F-IF.C.7e**, **F-IF.C.8**, **F-IF.C.9**).

Opening Exercise

In Problem 1 of the Problem Set of Lesson 32, you calculated the monthly payment for a 15-year mortgage at a 5% annual interest rate for the house you chose. You need that monthly payment to answer these questions.

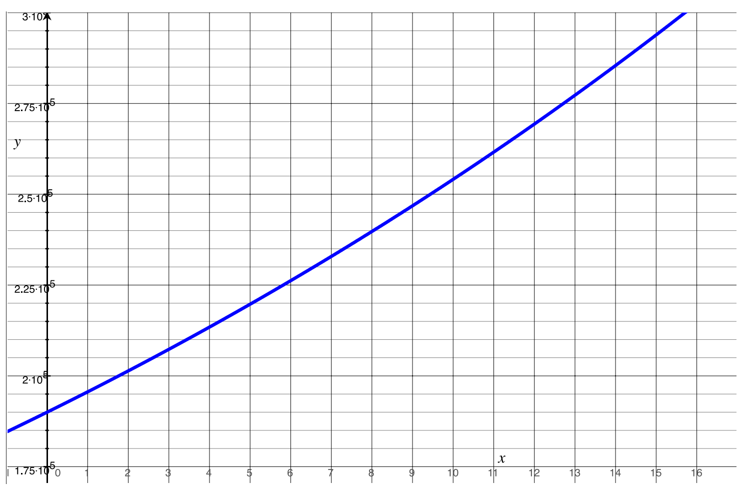
a. **About how much do you expect your home to be worth in 15 years?**

Answers will vary, but should follow similar steps. For example:

Step 1: $F = 190000(1.0295)^{15}$
Step 2: $F \approx 293866$

My home will be worth about $\$294,000$ if it continues to appreciate at an average rate of 2.95% every year.

b. **For $0 \leq x \leq 15$, plot the graph of the function $f(x) = P(1+r)^x$ where r is the appreciation rate and P is the initial value of your home.**



c. **Compare the image of the graph you plotted in part (b) with a partner, and write your observations of the differences and similarities. What do you think is causing the differences that you see in the graphs? Share your observations with another group to see if your conclusions are correct.**

MP.3

Answers will vary. Although the growth rate is the same for all students in the class, depending on the initial value of the home, different homes will appreciate more quickly than others. For instance, a house that is valued at $\$100,000$ will increase to $\$154,666.15$ in 15 years. This increase is about $\$49,000$ less than the increase for a home with a $\$190,000$ initial value. The differences are caused by the differences in initial value. Since the rate of increase is a percentage and the home's value increases exponentially, the increase is markedly different for more expensive homes and will be even more significant as years increase.

Your friend Julia bought a home at the same time as you but chose to finance the loan over 30 years. Julia also was able to avoid a down payment and financed the entire value of her home. This allowed her to purchase a more expensive home, but 15 years later she still has not paid off the loan. Consider the following amortization table representing Julia's mortgage, and answer the following questions by comparing the table with your graph.

Payment #	Beginning Balance	Payment on Interest	Payment on Principal
1	$\$145,000$	$\$543.75$	$\$190.94$
⋮	⋮	⋮	⋮
178	$\$96,784.14$	$\$362.94$	$\$371.75$
179	$\$96,412.38$	$\$361.55$	$\$373.15$
180	$\$96,039.23$	$\$360.15$	$\$374.55$

d. In Julia's neighborhood, her home has grown in value at around 2.95% per year. Considering how much she still owes the bank, how much of her home does she own after 15 years (the equity in her home)? Express your answer in dollars to the nearest thousand and as a percent of the value of her home.

Julia's home is worth $145000(1.0295)^{15} \approx 224265.91$*, which is about* $\$224,000$*. She owes* $\$96,039.23$*, which still leaves* $\$224,000 - \$96,000 = \$128,000$*. This means she owns about* $\$128,000$ *of her home or* $\frac{128}{224} = \frac{4}{7} \approx 57\%$ *of her home.*

e. Reasoning from your graph in part (b) and the table above, if both you and Julia sell your homes in 15 years at the homes' appreciated values, who would have more equity?

Answers will vary. Any student whose home is worth more than $\$83,000$ *initially will have more equity than Julia after* 15 *years, which should be clear from the graph.*

f. How much more do you need to save over 15 years to have assets over $\$1,000,000$?

Answers will vary. For example:

$$\$1,000,000 - \$294,000 = \$706,000$$

I will need to save $\$706,000$ *in the next* 15 *years.*

Mathematical Modeling Exercises (17 minutes)

Have students work individually or in pairs to figure out the monthly payment they need to save up to $\$1$ million in assets over 15 years. Although 7% compounded quarterly is used as the interest rate on the account, if time permits or with more advanced students, multiple interest rates may be given to different groups of students and the differences between the accounts analyzed and discussed.

The question of what type of account is being used for savings is left to the discretion of the teacher and may be omitted from the discussion. Possibilities include bonds, CDs, and stocks. Bonds and CDs are relatively secure and safe investments but have maximum interest rates around 2–3% annually. The stock market may seem like a risky place to invest, but mutual funds based upon stocks over the long run can provide relatively stable growth. For a list of stock market annual growth rates as well as a compound annual growth rate (CAGR) calculator, please visit http://www.moneychimp.com/features/market_cagr.htm. The data suggests that the CAGR is around 6.86% per year adjusted for inflation, which we have rounded to 7% to give the most optimistic calculations—an annual interest rate of 7% compounded quarterly will double about every 10 years.

Throughout these exercises the modeling cycle should be emphasized so that students use the process correctly. If necessary, draw on the board the modeling flowchart included at the beginning of the lesson to keep students on task.

Mathematical Modeling Exercises

Assume you can earn 7% interest annually, compounded monthly, in an investment account. Develop a savings plan so that you will have $\$1$ million in assets in 15 years (including the equity in your paid-off house).

1. Use your answer to Opening Exercise, part (g) as the future value of your savings plan.

a. How much will you have to save every month to save up $\$1$ million in assets?

Answers will vary. For example, since $i = \frac{0.07}{12} \approx 0.00583$, $A_f = 706000$, and $n = 180$,

$$706\,000 = R\left(\frac{(1+0.00583)^{180}-1}{0.00583}\right)$$
$$R = 706000 \cdot \frac{0.00583}{(1.00583)^{180}-1}$$
$$R \approx 2228.17$$

The monthly payment to save $\$706,000$ in 15 years at 7% interest compounded quarterly would be $\$2,228.17$.

You should not expect students to answer this problem as easily as the answer above implies. Walk around the room encouraging students to try a simpler problem first—maybe one where they earn \$5,000 after making four payments. Also, the answer above is the shortest answer possible. Many of your students may need to write out a geometric series inductively to get what the deposits and interest earned will look like. Above all, these last few modeling lessons are meant to let students figure out the solution on their own, so please give them the time to do so. Challenge students who get the answer quickly with the following questions: How is the formula derived? What does it mean?

b. Recall the monthly payment to pay off your home in 15 years (from Problem 1 of the Problem Set of Lesson 32). How much are the two together? What percentage of your monthly income is this for the profession you chose?

The monthly payment on a 15-year loan was about $\$1,352$ (a 5% annual interest loan on $\$171,000$ for a $\$190,000$ house with $\$19,000$ down for 15 years). The savings payment coupled with the monthly mortgage comes to about $\$3,579$.

Answers will vary on the percentage of monthly income.

It is very likely that the total amount of the two may exceed 50% of the monthly income. If so, you can lead your students to recalculate a more reasonable scenario like taking 20 years to generate \$1 million in assets. Have them come up with the plan.

2. Write a report supported by the calculations you did above on how to save $\$1$ million (or more) in your lifetime.

Answers will vary.

You may wish to assign this as homework so students can type up their plan, make a slide presentation, blog about it, write it in their journal, etc.

Closing (4 minutes)

Debrief students on their understanding of the mathematics of finance. Suggested questions are listed below with likely responses. Have students answer on their own or with a partner in writing.

- What formula made all of our work with structured savings plans, credit cards, and loans possible?
  - *The formula for the sum of a finite geometric sequence is* $S_n = a \cdot \left(\frac{1-r^n}{1-r}\right)$.
- What does each part of the formula for the sum of a finite geometric sequence represent?
  - *The* n^{th} *partial sum is* S_n, a *is the first term,* r *is the common ratio, and* n *is the number of terms.*
- How does this translate to the formula for the future value of a structured savings plan?
  - *Structured savings plans are geometric series with initial terms* R *standing for recurring payment,* $1+i$ *is the common ratio, and* n *is the total number of payments. The sum of all the payments and the interest they earn is the future value of the structured savings plan,* A_f. *We get* $A_f = R \cdot \left(\frac{1-(1+i)^n}{1-(1+i)}\right)$ *which simplifies to* $A_f = R \cdot \left(\frac{(1+i)^n - 1}{i}\right)$.

The next question is included as a reminder to students to reconnect the work they did in Lessons 30, 31, and 32 with Lesson 33.

- For loans and credit cards, we set the future value of a savings plan equal to the future value of a compound interest account to find the present value, or balance of the loan. State the formula for the present value of a loan, and identify its parts.
  - *The present value of a loan is derived from* $A_p(1+i)^n = R \cdot \left(\frac{(1+i)^n - 1}{i}\right)$ *which simplifies to* $A_p = R \cdot \left(\frac{1-(1+i)^{-n}}{i}\right)$. *The present value or balance of the loan is* A_p, R *is the recurring payment,* i *is the interest rate, and* n *is the number of payments.*

Exit Ticket (5 minutes)

Name ______________________________ Date ______________

Lesson 33: The Million Dollar Problem

Exit Ticket

1. At age 25, you begin planning for retirement at 65. Knowing that you have 40 years to save up for retirement and expecting an interest rate of 4% per year compounded monthly throughout the 40 years, how much do you need to deposit every month to save up $\$2$ million for retirement?

2. Currently, your savings for each month is capped at $\$400$. If you start investing all of this into a savings plan earning 1% interest annually, compounded monthly, then how long will it take to save $\$160,000$? (Hint: Use logarithms.)

Exit Ticket Sample Solutions

1. At age 25, you begin planning for retirement at 65. Knowing that you have 40 years to save up for retirement and expecting an interest rate of 4% per year, compounded monthly, throughout the 40 years, how much do you need to deposit every month to save up $\$2$ million for retirement?

$$A_f = R\left(\frac{(1+i)^n - 1}{i}\right)$$

$$2 \times 10^6 = R\left(\frac{\left(1+\frac{0.04}{12}\right)^{12\cdot 40} - 1}{\frac{0.04}{12}}\right)$$

$$R = 2 \times 10^6 \cdot \left(\frac{0.04}{12}\right) \div \left(\left(1+\frac{0.04}{12}\right)^{480} - 1\right)$$

$$R \approx 1692.10$$

You need to deposit $\$1,692.10$ every month for 40 years to save $\$2$ million at 4% interest.

2. Currently, your savings for each month is capped at $\$400$. If you start investing all of this into a savings plan earning 1% interest annually, compounded monthly, then how long should it take to save $\$160,000$? (Hint: Use logarithms.)

$$160000 = 400\left(\frac{\left(1+\frac{0.01}{12}\right)^{12t} - 1}{\frac{0.01}{12}}\right)$$

$$400 = \frac{\left(1+\frac{0.01}{12}\right)^{12t} - 1}{\frac{0.01}{12}}$$

$$400 \cdot \left(\frac{0.01}{12}\right) = \left(1+\frac{0.01}{12}\right)^{12t} - 1$$

$$\frac{4}{12} + 1 = \left(1+\frac{0.01}{12}\right)^{12t}$$

$$\frac{4}{3} = \left(1+\frac{0.01}{12}\right)^{12t}$$

$$\ln\left(\frac{4}{3}\right) = \ln\left(\left(1+\frac{0.01}{12}\right)^{12t}\right)$$

$$\ln\left(\frac{4}{3}\right) = 12t \cdot \ln\left(1+\frac{0.01}{12}\right)$$

$$t = \frac{\ln\left(\frac{4}{3}\right)}{12\ln\left(1+\frac{0.01}{12}\right)}$$

$$\approx 28.7802$$

It would take 28 years and 10 months to save up $\$160,000$ with only $\$400$ deposited every month.

Problem Set Sample Solutions

1. Consider the following scenario: You would like to save up $\$50,000$ after 10 years and plan to set up a structured savings plan to make monthly payments at 4.125% interest annually, compounded monthly.

   a. What lump sum amount would you need to invest at this interest rate in order to have $\$50,000$ after 10 years?

$$50000 = P\left(1+\frac{0.04125}{12}\right)^{120}$$

$$P = 50000 \div \left(1+\frac{0.04125}{12}\right)^{120}$$

$$P \approx 33123.08$$

   You would need to deposit $\$33,123.08$ now to save up to $\$50,000$.

   b. Use an online amortization calculator to find the monthly payment necessary to take a loan for the amount in part (a) at this interest rate and for this time period.

   $\$337.33$

   c. Use $A_f = R\left(\frac{(1+i)^n-1}{i}\right)$ to solve for R.

$$50000 = R\left(\frac{\left(1+\frac{0.04125}{12}\right)^{120}-1}{\frac{0.04125}{12}}\right)$$

$$R \approx 337.33$$

   The monthly payment would be $\$337.33$.

   d. Compare your answers to part (b) and part (c). What do you notice? Why did this happen?

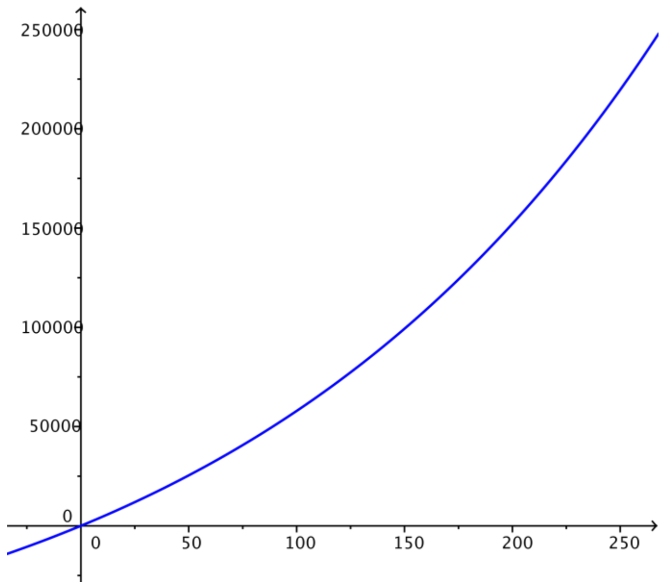
   The answers are the same. The present value of an annuity is the cost of a loan and can be found by setting the loan equal to the compound interest formula, which is what we did originally. Once we had the cost of a loan, the amortization calculator was able to find the monthly payment. In part (c) we used the future value of an annuity to find the same quantity.

2. For structured savings plans, the future value of the savings plan as a function of the number of payments made at that point is an interesting function to examine. Consider a structured savings plan with a recurring payment of $\$450$ made monthly and an annual interest rate of 5.875% compounded monthly.

   a. State the formula for the future value of this structured savings plan as a function of the number of payments made. Use f for the function name.

$$f(x) = 450\left(\frac{\left(1+\frac{0.05875}{12}\right)^{x}-1}{\left(\frac{0.05875}{12}\right)}\right)$$

b. **Graph the function you wrote in part (a) for $0 \le x \le 216$.**



c. **State any trends that you notice for this function.**

The function appears to be exponential in nature. It is increasing at an increasing rate.

d. **What is the approximate value of the function f for $x = 216$?**

$$f(216) = 450\left(\frac{\left(1+\frac{0.05875}{12}\right)^{216} - 1}{\left(\frac{0.05875}{12}\right)}\right)$$
$$\approx 172041.21$$

e. **What is the domain of f? Explain.**

Since the compounding is monthly, the domain of f is normally considered to be a positive integer (i.e., the number of periods).

f. **If the domain of the function is restricted to natural numbers, is the function a geometric sequence? Why or why not?**

No, the function is not a geometric sequence, since it does not have a common ratio. For instance, from $x = 1$ to $x = 2$, the value of the ratio is approximately 2.004896, but from $x = 2$ to $x = 3$, the value of the ratio is 1.50367.

g. Recall that the n^{th} partial sums of a geometric sequence can be represented with S_n. It is true that $f(x) = S_x$ for positive integers x, since it is a geometric sequence; that is, $S_x = \sum_{i=1}^{x} ar^i$. State the geometric sequence whose sums of the first x terms are represented by this function.

The geometric sequence has first term 450 and common ratio $\left(1 + \frac{0.05875}{12}\right)$. It can be written as $a_n = 450 \cdot \left(1 + \frac{0.05875}{12}\right)^{n-1}$.

h. April has been following this structured savings plan for 18 years. April says that taking out the money and starting over should not affect the total money earned because the interest rate does not change. Explain why April is incorrect in her reasoning.

The function is increasing exponentially, so the larger the balance, the more it grows. If the money is taken out, then the growth would be reset back to the beginning, although you would have that money.

3. Henry plans to have $\$195,000$ in property in 14 years and would like to save up to $\$1$ million by depositing $\$3,068.95$ each month at 6% interest per year, compounded monthly. Tina's structured savings plan over the same time span is described in the following table:

Deposit #	Amount Saved
30	$\$110,574.77$
31	$\$114,466.39$
32	$\$118,371.79$
33	$\$122,291.02$
34	$\$126,224.14$
⋮	⋮
167	$\$795,266.92$
168	$\$801,583.49$

a. Who has the higher interest rate? Who pays more every month?

From the table it looks like Tina pays more every month, but Henry has the higher interest rate. Henry would have about $\$99,000$ after 30 payments, but Tina has more at that point. After 168 payments, Henry will save up $\$805,000$, while Tina has only saved $\$801,583$. Over long periods of time, a higher interest rate will eventually beat larger payments.

b. At the end of 14 years, who has more money from their structured savings plan? Does this agree with what you expected? Why or why not?

Henry has more, but just barely. The larger payment Tina was making was not enough to stay ahead of Henry for 14 years, but he looks to have just passed her recently.

c. At the end of 40 years, who has more money from their structured savings plan?

Henry has extended his lead significantly by this point. Once he overtakes Tina, his savings continues to grow at a faster rate than Tina's savings.

4. Edgar and Paul are two brothers that each get an inheritance of $\$150,000$. Both plan to save up over $\$1,000,000$ in 25 years. Edgar takes his inheritance and deposits the money into an investment account earning 8% interest annually, compounded monthly, payable at the end of 25 years. Paul spends his inheritance but uses a structured savings plan that is represented by the sequence $b_n = 1275 + b_{n-1} \cdot \left(1 + \frac{0.0775}{12}\right)$ with $b_0 = 1275$ in order to save up more than $\$1,000,000$.

a. Which of the two has more money at the end of 25 years?

Let $E(x)$ represent Edgar's savings and $P(x)$ represent Paul's savings.

Then $E(x) = 150000\left(1 + \frac{0.08}{12}\right)^{300} \approx 1101026.40.$

$$\begin{aligned} P(x) &= \sum_{i=0}^{299} 1275\left(1 + \frac{0.0775}{12}\right)^{i} \\ &= 1275\left(\frac{1 - \left(1 + \frac{0.0775}{12}\right)^{300}}{1 - \left(1 + \frac{0.0775}{12}\right)}\right) \\ &= 1275\left(\frac{\left(1 + \frac{0.0775}{12}\right)^{300} - 1}{\frac{0.0775}{12}}\right) \\ &\approx 1164432.17 \end{aligned}$$

Paul has $\$63,405.78$ more than Edgar at the end of 25 years.

b. What are the pros and cons of both brothers' plans? Which would you rather do? Why?

Answers will vary between the two plans and may include a combination of both.

Edgar only has to make a single payment, and he inherited the money, so it does not come out of his normal budget. He does not have to worry about the account again, but he makes less money than Paul overall and cannot access the money until the end of the 25 years. Edgar also pays much less than Paul does.

Paul ends up paying $\$382,500$ in order to save up his million, but he does this slowly over the 25 years, so he does not have a huge pinch at any point in time. Paul ends up saving more money in the long run.