Mathematics Curriculum

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1: Polynomial, Rational, and Radical Relationships





Algebra II • Module 1

Polynomial, Rational, and Radical Relationships

OVERVIEW

In this module, students draw on their foundation of the analogies between polynomial arithmetic and baseten computation, focusing on properties of operations, particularly the distributive property (A-SSE.B.2, A-APR.A.1). Students connect multiplication of polynomials with multiplication of multi-digit integers and division of polynomials with long division of integers (A-APR.A.1, A-APR.D.6). Students identify zeros of polynomials, including complex zeros of quadratic polynomials, and make connections between zeros of polynomials and solutions of polynomial equations (A-APR.B.3). Students explore the role of factoring, as both an aid to the algebra and to the graphing of polynomials (A-SSE.2, A-APR.B.2, A-APR.B.3, F-IF.C.7c). Students continue to build upon the reasoning process of solving equations as they solve polynomial, rational, and radical equations, as well as linear and non-linear systems of equations (A-REI.A.1, A-REI.A.2, A-REI.C.6, A-REI.C.7). The module culminates with the fundamental theorem of algebra as the ultimate result in factoring. Students pursue connections to applications in prime numbers in encryption theory, Pythagorean triples, and modeling problems.

An additional theme of this module is that the arithmetic of rational expressions is governed by the same rules as the arithmetic of rational numbers. Students use appropriate tools to analyze the key features of a graph or table of a polynomial function and relate those features back to the two quantities that the function is modeling in the problem (**F-IF.C.7c**).

Focus Standards

Reason quantitatively and use units to solve problems.

N-Q.A.2² Define appropriate quantities for the purpose of descriptive modeling.*

Perform arithmetic operations with complex numbers.

N-CN.A.1 Know there is a complex number *i* such that $i^2 = -1$, and every complex number has the form a + bi with *a* and *b* real.

²This standard is assessed in Algebra II by ensuring that some modeling tasks (involving Algebra II content or securely held content from previous grades and courses) require the student to create a quantity of interest in the situation being described (i.e., this is not provided in the task). For example, in a situation involving periodic phenomena, the student might autonomously decide that amplitude is a key variable in a situation and then choose to work with peak amplitude.



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N-CN.A.2 Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.

Use complex numbers in polynomial identities and equations.

N-CN.C.7 Solve quadratic equations with real coefficients that have complex solutions.

Interpret the structure of expressions.

A-SSE.A.2³ Use the structure of an expression to identify ways to rewrite it. For example, see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$.

Understand the relationship between zeros and factors of polynomials.

- **A-APR.B.2**⁴ Know and apply the Remainder Theorem: For a polynomial p(x) and a number a, the remainder on division by x a is p(a), so p(a) = 0 if and only if (x a) is a factor of p(x).
- **A-APR.B.3**⁵ Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.

Use polynomial identities to solve problems.

A-APR.C.4 Prove polynomial identities and use them to describe numerical relationships. For example, the polynomial identity $(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2$ can be used to generate *Pythagorean triples.*

Rewrite rational expressions.

A-APR.D.6⁶ Rewrite simple rational expressions in different forms; write a(x)/b(x) in the form q(x) + r(x)/b(x), where a(x), b(x), q(x), and r(x) are polynomials with the degree of r(x) less than the degree of b(x), using inspection, long division, or, for the more complicated examples, a computer algebra system.

Understand solving equations as a process of reasoning and explain the reasoning.

A-REI.A.1⁷ Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method.

⁴Include problems that involve interpreting the remainder theorem from graphs and in problems that require long division. ⁵ In Algebra II, tasks include quadratic, cubic, and quadratic polynomials and polynomials for which factors are not provided. For example, find the zeros of $(x^2 - 1)(x^2 + 1)$.

⁷In Algebra II, tasks are limited to simple rational or radical equations.



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³In Algebra II, tasks are limited to polynomial, rational, or exponential expressions. Examples: see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$. In the equation $x^2 + 2x + 1 + y^2 = 9$, see an opportunity to rewrite the first three terms as $(x + 1)^2$, thus recognizing the equation of a circle with radius 3 and center (-1, 0). See $(x^2 + 4)/(x^2 + 3)$ as $((x^2 + 3) + 1)/(x^2 + 3)$, thus recognizing an opportunity to write it as $1 + 1/(x^2 + 3)$.

⁶Include rewriting rational expressions that are in the form of a complex fraction.

A-REI.A.2 Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.

Solve equations and inequalities in one variable.

- **A-REI.B.4**⁸ Solve quadratic equations in one variable.
 - b. Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as $a \pm bi$ for real numbers a and b.

Solve systems of equations.

- **A-REI.C.6**⁹ Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables.
- **A-REI.C.7** Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically. For example, find the points of intersection between the line y = -3x and the circle $x^2 + y^2 = 3$.

Analyze functions using different representations.

- **F-IF.C.7** Graph functions expressed symbolically and show key features of the graph (by hand in simple cases and using technology for more complicated cases).*
 - c. Graph polynomial functions, identifying zeros when suitable factorizations are available and showing end behavior.

Translate between the geometric description and the equation for a conic section.

G-GPE.A.2 Derive the equation of a parabola given a focus and directrix.

Extension Standards

The (+) standards below are provided as an extension to Module 1 of the Algebra II course to provide coherence to the curriculum. They are used to introduce themes and concepts that are fully covered in the Precalculus course.

Use complex numbers in polynomial identities and equations.

N-CN.C.8 (+) Extend polynomial identities to the complex numbers. For example, rewrite $x^2 + 4$ as (x + 2i)(x - 2i).

⁹In Algebra II, tasks are limited to 3×3 systems.



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⁸In Algebra II, in the case of equations having roots with nonzero imaginary parts, students write the solutions as $a \pm bi$, where a and b are real numbers.

N-CN.C.9 (+) Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials.

Rewrite rational expressions.

A-APR.C.7 (+) Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression; add, subtract, multiply, and divide rational expressions.

Foundational Standards

Use properties of rational and irrational numbers.

N-RN.B.3 Explain why the sum or product of two rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational.

Reason quantitatively and use units to solve problems.

N-Q.A.1 Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays.*

Interpret the structure of expressions.

- A-SSE.A.1 Interpret expressions that represent a quantity in terms of its context.*
 - a. Interpret parts of an expression, such as terms, factors, and coefficients.
 - b. Interpret complicated expressions by viewing one or more of their parts as a single entity. For example, interpret $P(1 + r)^n$ as the product of P and a factor not depending on P.

Write expressions in equivalent forms to solve problems.

- **A-SSE.B.3** Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.*
 - a. Factor a quadratic expression to reveal the zeros of the function it defines.

Perform arithmetic operations on polynomials.

A-APR.A.1 Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.



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Create equations that describe numbers or relationships.

- A-CED.A.1 Create equations and inequalities in one variable and use them to solve problems. Include equations arising from linear and quadratic functions, and simple rational and exponential functions.*
- A-CED.A.2 Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.*
- A-CED.A.3 Represent constraints by equations or inequalities and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context. For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.*
- A-CED.A.4 Rearrange formulas to highlight a quantity of interest, using the same reasoning used in solving equations. For example, rearrange Ohm's law V = IR to highlight resistance R.*

Solve equations and inequalities in one variable.

- A-REI.B.3 Solve linear equations and inequalities in one variable, including equations with coefficients represented by letters.
- A-REI.B.4 Solve quadratic equations in one variable.
 - Use the method of completing the square to transform any quadratic equation in x into a. an equation of the form $(x - p)^2 = q$ that has the same solutions. Derive the guadratic formula from this form.

Solve systems of equations.

A-REI.C.5 Prove that, given a system of two equations in two variables, replacing one equation by the sum of that equation and a multiple of the other produces a system with the same solutions.

Represent and solve equations and inequalities graphically.

- A-REI.D.10 Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line).
- **A-REI.D.11** Explain why the x-coordinates of the points where the graphs of the equations y = f(x)and y = g(x) intersect are the solutions of the equation f(x) = g(x); find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where f(x) and/or g(x) are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.*

Translate between the geometric description and the equation for a conic section.

G-GPE.A.1 Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.



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Focus Standards for Mathematical Practice

- **MP.1** Make sense of problems and persevere in solving them. Students discover the value of equating factored terms of a polynomial to zero as a means of solving equations involving polynomials. Students solve rational equations and simple radical equations, while considering the possibility of extraneous solutions and verifying each solution before drawing conclusions about the problem. Students solve systems of linear equations and linear and quadratic pairs in two variables. Further, students come to understand that the complex number system provides solutions to the equation $x^2 + 1 = 0$ and higher-degree equations.
- **MP.2 Reason abstractly and quantitatively.** Students apply polynomial identities to detect prime numbers and discover Pythagorean triples. Students also learn to make sense of remainders in polynomial long division problems.
- **MP.4 Model with mathematics.** Students use primes to model encryption. Students transition between verbal, numerical, algebraic, and graphical thinking in analyzing applied polynomial problems. Students model a cross-section of a riverbed with a polynomial, estimate fluid flow with their algebraic model, and fit polynomials to data. Students model the locus of points at equal distance between a point (focus) and a line (directrix) discovering the parabola.
- **MP.7 Look for and make use of structure.** Students connect long division of polynomials with the long-division algorithm of arithmetic and perform polynomial division in an abstract setting to derive the standard polynomial identities. Students recognize structure in the graphs of polynomials in factored form and develop refined techniques for graphing. Students discern the structure of rational expressions by comparing to analogous arithmetic problems. Students perform geometric operations on parabolas to discover congruence and similarity.
- MP.8 Look for and express regularity in repeated reasoning. Students understand that polynomials form a system analogous to the integers. Students apply polynomial identities to detect prime numbers and discover Pythagorean triples. Students recognize factors of expressions and develop factoring techniques. Further, students understand that all quadratics can be written as a product of linear factors in the complex realm.

Terminology

New or Recently Introduced Terms

- Axis of Symmetry (The *axis of symmetry of a parabola* given by a focus point and a directrix is the perpendicular line to the directrix that passes through the focus.)
- **Dilation at the Origin** (A dilation at the origin D_k is a horizontal scaling by k > 0 followed by a vertical scaling by the same factor k. In other words, this dilation of the graph of y = f(x) is the graph of the equation $y = kf\left(\frac{1}{k}x\right)$. A dilation at the origin is a special type of a dilation.)



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- End Behavior (Let *f* be a function whose domain and range are subsets of the real numbers. The end behavior of a function *f* is a description of what happens to the values of the function
 - as x approaches positive infinity and
 - as x approaches negative infinity.)
- Even Function (Let f be a function whose domain and range is a subset of the real numbers. The function f is called *even* if the equation f(x) = f(-x) is true for every number x in the domain. Even-degree polynomial functions are sometimes even functions, such as $f(x) = x^{10}$, and sometimes not, such as $g(x) = x^2 x$.)
- Odd Function (Let f be a function whose domain and range is a subset of the real numbers. The function f is called *odd* if the equation f(-x) = -f(x) is true for every number x in the domain. Odd-degree polynomial functions are sometimes odd functions, such as $f(x) = x^{11}$, and sometimes not, such as $h(x) = x^3 x^2$.)
- **Parabola** (A *parabola* with *directrix line L* and *focus point F* is the set of all points in the plane that are equidistant from the point F and line L.)
- **Pythagorean Triple** (A *Pythagorean triple* is a triplet of positive integers (a, b, c) such that $a^2 + b^2 = c^2$. The triple (3, 4, 5) is a Pythagorean triple, but $(1, 1, \sqrt{2})$ is not, even though the numbers are side lengths of an isosceles right triangle.)
- Rational Expression (A rational expression is either a numerical expression or a variable symbol or the result of placing two previously generated rational expressions into the blanks of the addition operator (__+__), the subtraction operator (__-__), the multiplication operator (__×__), or the division operator (__÷__).)
- A Square Root of a Number (A square root of a number x is a number whose square is x. In symbols, a square root of x is a number a such that a² = x. Negative numbers do not have any real square roots, zero has exactly one real square root, and positive numbers have two real square roots.)
- The Square Root of a Number (Every positive real number x has a unique positive square root called *the square root* or *principal square root* of x; it is denoted \sqrt{x} . The square root of zero is zero.)
- Vertex of a Parabola (The vertex of a parabola is the point where the axis of symmetry intersects the parabola.)

Familiar Terms and Symbols¹⁰

- Algebraic Expression
- Arithmetic Sequence
- Binomial
- Coefficient of a Monomial
- Constant Function
- Cubic Function
- Degree of a Monomial

¹⁰ These are terms and symbols students have seen previously.



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- Degree of a Polynomial Function
- Degree of a Polynomial in One Variable
- Discriminant of a Quadratic Function
- Equivalent Polynomial Expressions
- Function
- Graph of *f*
- Graph of y = f(x)
- Increasing/Decreasing
- Like Terms of a Polynomial
- Linear Function
- Monomial
- Numerical Expression
- Numerical Symbol
- Polynomial Expression
- Polynomial Function
- Polynomial Identity
- Quadratic Function
- Relative Maximum
- Relative Minimum
- Sequence
- Standard Form of a Polynomial in One Variable
- Terms of a Polynomial
- Trinomial
- Variable Symbol
- Zeros or Roots of a Function

Suggested Tools and Representations

- Graphing Calculator
- Wolfram Alpha Software
- GeoGebra Software



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Preparing to Teach a Module

Preparation of lessons will be more effective and efficient if there has been an adequate analysis of the module first. Each module in A Story of Functions can be compared to a chapter in a book. How is the module moving the plot, the mathematics, forward? What new learning is taking place? How are the topics and objectives building on one another? The following is a suggested process for preparing to teach a module.

Step 1: Get a preview of the plot.

- A: Read the Table of Contents. At a high level, what is the plot of the module? How does the story develop across the topics?
- B: Preview the module's Exit Tickets to see the trajectory of the module's mathematics and the nature of the work students are expected to be able to do.

Note: When studying a PDF file, enter "Exit Ticket" into the search feature to navigate from one Exit Ticket to the next.

1	Algebra I-M1_Teacher_Edition.pdf - Adobe Reader	
File	Edit View Window Help	
2) Open 📑 🔁 17 / 327 71.2% 💌 💽 🔻 Tools Fill & Sign	Comme
	Find ×	
	Exit Ticket 💌	
R	Previous Next	
ľ	NYS COMMON CORE MATHEMATICS CURRICULUM	U
	ALGEBRA I	
	Name Date	
	Lesson 1: Graphs of Piecewise Linear Functions	
	Exit Ticket	
	The graph in the Exploratory Challenge is made by combining pieces of nine linear functions (it is a piecewise linear function). Each linear function is defined over an interval of time, represented on the horizontal axis. List those nine time intervals.	

Step 2: Dig into the details.

- A: Dig into a careful reading of the Module Overview. While reading the narrative, liberally reference the lessons and Topic Overviews to clarify the meaning of the text—the lessons demonstrate the strategies, show how to use the models, clarify vocabulary, and build understanding of concepts.
- B: Having thoroughly investigated the Module Overview, read through the Student Outcomes of each lesson (in order) to further discern the plot of the module. How do the topics flow and tell a coherent story? How do the outcomes move students to new understandings?

Step 3: Summarize the story.

Complete the Mid- and End-of-Module Assessments. Use the strategies and models presented in the module to explain the thinking involved. Again, liberally reference the lessons to anticipate how students who are learning with the curriculum might respond.



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Preparing to Teach a Lesson

A three-step process is suggested to prepare a lesson. It is understood that at times teachers may need to make adjustments (customizations) to lessons to fit the time constraints and unique needs of their students. The recommended planning process is outlined below. Note: The ladder of Step 2 is a metaphor for the teaching sequence. The sequence can be seen not only at the macro level in the role that this lesson plays in the overall story, but also at the lesson level, where each rung in the ladder represents the next step in understanding or the next skill needed to reach the objective. To reach the objective, or the top of the ladder, all students must be able to access the first rung and each successive rung.

Step 1: Discern the plot.

- A: Briefly review the module's Table of Contents, recalling the overall story of the module and analyzing the role of this lesson in the module.
- B: Read the Topic Overview related to the lesson, and then review the Student Outcome(s) and Exit Ticket of each lesson in the topic.
- C: Review the assessment following the topic, keeping in mind that assessments can be found midway through the module and at the end of the module.

Step 2: Find the ladder.

- A: Work through the lesson, answering and completing each question, example, exercise, and challenge.
- B: Analyze and write notes on the new complexities or new concepts introduced with each question or problem posed; these notes on the sequence of new complexities and concepts are the rungs of the ladder.
- C: Anticipate where students might struggle, and write a note about the potential cause of the struggle.
- D: Answer the Closing questions, always anticipating how students will respond.



Step 3: Hone the lesson.

Lessons may need to be customized if the class period is not long enough to do all of what is presented and/or if students lack prerequisite skills and understanding to move through the entire lesson in the time allotted. A suggestion for customizing the lesson is to first decide upon and designate each question, example, exercise, or challenge as either "Must Do" or "Could Do."

A: Select "Must Do" dialogue, questions, and problems that meet the Student Outcome(s) while still providing a coherent experience for students; reference the ladder. The expectation should be that the majority of the class will be able to complete the "Must Do" portions of the lesson within the allocated time. While choosing the "Must Do" portions of the lesson, keep in mind the need for a balance of dialogue and conceptual questioning, application problems, and abstract problems, and a balance between students using pictorial/graphical representations and abstract representations. Highlight dialogue to be included in the delivery of instruction so that students have a chance to articulate and consolidate understanding as they move through the lesson.



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B: "Must Do" portions might also include remedial work as necessary for the whole class, a small group, or individual students. Depending on the anticipated difficulties, the remedial work might take on different forms as suggested in the chart below.

Anticipated Difficulty	"Must Do" Remedial Problem Suggestion
The first problem of the lesson is too challenging.	Write a short sequence of problems on the board that provides a ladder to Problem 1. Direct students to complete those first problems to empower them to begin the lesson.
There is too big of a jump in complexity between two problems.	Provide a problem or set of problems that bridge student understanding from one problem to the next.
Students lack fluency or foundational skills necessary for the lesson.	Before beginning the lesson, do a quick, engaging fluency exercise ¹¹ . Before beginning any fluency activity for the first time, assess that students have conceptual understanding of the problems in the set and that they are poised for success with the easiest problem in the set.
More work is needed at the concrete or pictorial level.	Provide manipulatives or the opportunity to draw solution strategies.
More work is needed at the abstract level.	Add a set of abstract problems to be completed toward the end of the lesson.

- C: "Could Do" problems are for students who work with greater fluency and understanding and can, therefore, complete more work within a given time frame.
- D: At times, a particularly complex problem might be designated as a "Challenge!" problem to provide to advanced students. Consider creating the opportunity for students to share their "Challenge!" solutions with the class at a weekly session or on video.
- E: If the lesson is customized, be sure to carefully select Closing questions that reflect such decisions, and adjust the Exit Ticket if necessary.

¹¹Look for fluency suggestions at <u>www.eureka-math.org</u>.







Assessment Summary

Assessment Type	Administered	Format	Standards Addressed
Mid-Module Assessment Task	After Topic B	Constructed response with rubric	N-Q.A.2, A-SSE.A.2, A-APR.B.2, A-APR.B.3, A-APR.C.4, A-REI.A.1, A-REI.B.4b, F-IF.C.7c
End-of-Module Assessment Task	After Topic D	Constructed response with rubric	N-Q.A.2, A.SSE.A.2, A.APR.B.2, A-APR.B.3, A-APR.C.4, A-APR.D.6, A-REI.A.1, A-REI.A.2, A-REI.B.4b, A-REI.C.6, A-REI.C.7, F-IF.C.7c, G-GPE.A.2



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Lesson 1: Successive Differences in Polynomials

Student Outcomes

 Students write explicit polynomial expressions for sequences by investigating successive differences of those sequences.

Lesson Notes

This first lesson of the year tells students that this course is about thinking and reasoning with mathematics. It reintroduces the study of polynomials in a surprising new way involving sequences. This offers a chance to evaluate how much students recall from Algebra I. The lesson starts with discussions of expressions, polynomials, sequences, and equations. In this lesson, students continue the theme that began in Grade 6 of evaluating and building expressions. Explore ways to test students' recall of the vocabulary terms listed at the end of this lesson.

Throughout this lesson, listen carefully to students' discussions. Their reactions will indicate how to best approach the rest of the module. The homework set to this lesson should also offer insight into how much they remember from previous grades and how well they can read instructions. In particular, if they have trouble with evaluating or simplifying expressions or solving equations, then consider revisiting Lessons 6–9 in Algebra I, Module 1, and Lesson 2 in Algebra I, Module 4. If they are having trouble solving equations, use Lessons 10–12, 15–16, and 19 in Algebra I, Module 1 to give them extra practice.

Finally, the use of the term *constant* may need a bit of extra discussion. It is used throughout this PK-12 curriculum in two ways: either as a constant number (e.g., the a in $ax^2 + bx + c$ is a number chosen once-and-for-all at the beginning of a problem) or as a constant rate (e.g., a copier that reproduces at a constant rate of 40 copies/minute). Both uses are offered in this lesson.

Classwork

MP.1

MP.8

Opening Exercise (7 minutes)

This exercise provides an opportunity to think about and generalize the main concept of today's lesson: that the second differences of a quadratic polynomial are constant. This generalizes to the n^{th} differences of a degree n polynomial. The goal is to help students investigate, discuss, and generalize the second and higher differences in this exercise.

Present the exercise to students and ask them (in groups of two) to study the table and explain to their partner how to calculate each line in the table. If they get stuck, help them find entry points into this question, possibly by drawing segments connecting the successive differences on their papers (e.g., connect 5.76 and 11.56 to 5.8 and ask, "How are these three numbers related?"). This initial problem of the school year is designed to encourage students to persevere and look for and express regularity in repeated reasoning.

Teachers may also use the Opening Exercise to informally assess students' pattern-finding abilities and fluency with rational numbers.

Scaffolding:

Before presenting the problem below, consider starting by displaying the first two rows of the table on the board and asking students to investigate the relationship between them, including making a conjecture about the nature of the relationship.



Lesson 1:

Successive Differences in Polynomials





John noticed patterns i	in the arrangeme	nt of numbers in	the table below.					
<u>Number</u>	2.4	3.4	4.4		5.4		6.4	
<u>Square</u>	5.76	11.56	19.36		29.16		40.96	
First Differences	5.	8	7.8	9.8		11.8		
Second Differences		2	2		2			
7. 4^2 , and then use the To find 7. 4^2 , John assu	e pattern to find 8 med the next ter	ue, he used it to 8.4 ² . <i>m in the first diff</i>	find the value of 7.4	² . Explain	how he u since 13.	sed the p 8 is 2 ma	battern to find	
7. 4 ² , and then use the To find 7. 4 ² , John assu Therefore, the next ter calculator, we also find	tern would contine the pattern to find 8 sumed the next tern im in the square n of $7.4^2 = 54.76$.	ue, he used it to a. 4 ² . m in the first diff umbers would he	find the value of 7.4 erences would have ave to be 40.96 + 1	 4². Explain to be 13.8 : 3.8, which is 	how he u since 13. is 54.76.	sed the p 8 is 2 mc Checkin	pattern to find ore than 11.8. ng with a	
7. 4^2 , and then use the To find 7. 4^2 , John assu Therefore, the next ter calculator, we also find To find 8. 4^2 , we follow	tern would contine a pattern to find 8 sumed the next tern or in the square n d $7.4^2 = 54.76$.	ue, he used it to 2.4 ² . m in the first diff umbers would he ss: The next term	find the value of 7.4 erences would have ave to be 40.96 + 1	 Explain to be 13.8: 3.8, which tes would hold 	how he u since 13. is 54.76. ive to be	sed the p 8 is 2 mc Checkin	oattern to find ore than 11.8. ng with a affolding:	
7. 4^2 , and then use the To find 7. 4^2 , John assu Therefore, the next ter calculator, we also find To find 8. 4^2 , we follow 15. 8, so the next term Check: 8. $4^2 = 70.56$.	tern would contin e pattern to find 8 umed the next tern m in the square n d $7.4^2 = 54.76$. v the same proces n in the square num.	ue, he used it to 2.4 ² . m in the first diff umbers would he ss: The next term mbers would be !	find the value of 7.4 erences would have ave to be $40.96 + 1$ a in the first difference 54.76 + 15.8, which	k ² . Explain to be 13.8 3.8, which tes would ho h is 70.56.	how he u since 13. is 54.76. uve to be	sed the p 8 is 2 ma Checkin Sca Fo gra	attern to find ore than 11. 8. In gwith a affolding: In students wor ade level, cons	king belo ider using
To find 7. 4^2 , and then use the To find 7. 4^2 , John assu Therefore, the next ter calculator, we also find To find 8. 4^2 , we follow 15. 8, so the next term Check: 8. $4^2 = 70.56$.	tern would contine a pattern to find 8 umed the next tern im in the square in d 7. $4^2 = 54.76$. v the same process in the square num- each row of numb	ue, he used it to 2.4 ² . <i>m in the first diff</i> <i>umbers would he</i> <i>ss: The next term</i> <i>mbers would be</i> ers in the table?	find the value of 7.4 erences would have ave to be 40.96 + 1 n in the first difference 54.76 + 15.8, whice	k ² . Explain to be 13.8 : 3.8, which the swould ho h is 70.56.	how he u since 13. is 54. 76. ive to be	sed the p 8 is 2 ma Checkin Sca Fo gra po an	affolding: affolding: affolding: affolding cons ade level, cons ositive integers d correspondir	king belo ider using {1, 2, 3, ng square

Discuss with students the relationship between each row and the row above it and how to label the rows based upon that relationship. Feel free to have this discussion before or after they find 7.4^2 and 8.4^2 . They are likely to come up with labels such as *subtract* or *difference* for the third and fourth row. However, guide them to call the third and fourth rows *First Differences* and *Second Differences*, respectively.

Discussion (3 minutes)

The pattern illustrated in the Opening Exercise is a particular case of a general phenomenon about polynomials. In Algebra I, Module 3, students saw how to recognize linear functions and exponential functions by recognizing similar growth patterns; that is, linear functions grow by a *constant difference over successive intervals of equal length*, and exponential functions grow by a *constant factor over successive intervals of equal length*. This lesson sees the generalization of the linear growth pattern to polynomials of second degree (quadratic expressions) and third degree (cubic expressions).

Discussion

Let the sequence $\{a_0, a_1, a_2, a_3, ...\}$ be generated by evaluating a polynomial expression at the values 0, 1, 2, 3, ... The numbers found by evaluating $a_1 - a_0, a_2 - a_1, a_3 - a_2, ...$ form a new sequence, which we will call the *first differences* of the polynomial. The differences between successive terms of the first differences sequence are called the *second differences* and so on.

It is a good idea to use an actual sequence of numbers such as the square numbers $\{1, 4, 9, 16, ...\}$ to help explain the meaning of the terms *first differences* and *second differences*.



Successive Differences in Polynomials







Example 1 (4 minutes)

Although it may be tempting to work through Example 1 using numbers instead of *a* and *b*, using symbols *a* and *b* actually makes the *structure* of the first differences sequence obvious, whereas numbers could hide that structure. Also, working with constant coefficients gives the generalization all at once.

Note: Consider using Example 1 to informally assess students' fluency with algebraic manipulations.

Example 1	
What is the sequence of first differences for the linear polynomial given by $ax + b$, where a and b a coefficients?	ire constant
The terms of the first differences sequence are found by subtracting consecutive terms in the sequence generated by the polynomial expression $ax + b$, namely, $\{b, a + b, 2a + b, 3a + b, 4a + b,\}$. 1 st term: $(a + b) - b = a$, 2 nd term: $(2a + b) - (a + b) = a$, 3 rd term: $(3a + b) - (2a + b) = a$, 4 th term: $(4a + b) - (3a + b) = a$. The first differences sequence is $\{a, a, a, a,\}$. For first-degree polynomial expressions, the first differences are constant and equal to a.	Scaffolding: Try starting the example by first asking students to generate sequences of first differences for $2x + 3$, $3x - 1$, and $4x + 2$. For example, the sequence generated by $2x + 3$ is $\{3, 5, 7, 9,\}$, and its sequence of first differences is $\{2, 2, 2, 2, 2\}$.
What is the sequence of second differences for $ax + b$? Since $a - a = 0$, the second differences are all 0. Thus, the sequence of second differences is $\{0, 0, 0, 0,\}$.	These three sequences can then be used as a source of examples from which to make and verify conjectures.

- How is this calculation similar to the arithmetic sequences you studied in Algebra I, Module 3?
 - The constant derived from the first differences of a linear polynomial is the same constant addend used to define the arithmetic sequence generated by the polynomial. That is, the a in A(n) = an + b for $n \ge 0$. Written recursively this is A(0) = b and A(n + 1) = A(n) + a for $n \ge 0$.

For Examples 2 and 3, let students work in groups of two to fill in the blanks of the tables (3 minute maximum for each table). Walk around the room, checking student work for understanding. Afterward, discuss the paragraphs below each table as a whole class.



Successive Differences in Polynomials







Example 2 (5 minutes)

Example 2							
Find the f	irst, second, and third dif	ferences of the polynomial	$ax^2 + bx + c$ by filling in the b	lanks in the following table.			
x	$ax^2 + bx + c$	First Differences	Second Differences	Third Differences			
0	С						
		a + b					
1	a + b + c		2 <i>a</i>				
		3a + b		0			
2	4a + 2b + c		2 <i>a</i>				
		5a + b		0			
3	9a + 3b + c		2 <i>a</i>				
		7a + b		0			
4	16a + 4b + c		2 <i>a</i>				
		9a + b					
5	25a + 5b + c						

The table shows that the second differences of the polynomial $ax^2 + bx + c$ all have the constant value 2a. The second differences hold for any sequence of values of x where the values in the sequence differ by 1, as the Opening Exercise shows. For example, if we studied the second differences for x-values π , π + 1, π + 2, π + 3, ..., we would find that the second differences would also be 2a. In your homework, you will show that this fact is indeed true by finding the second differences for the values n + 0, n + 1, n + 2, n + 3, n + 4.

Ask students to describe what they notice in the sequences of first, second, and third differences. Have them make a conjecture about the third and fourth differences of a sequence generated by a third degree polynomial.

Students are likely to say that the third differences have the constant value 3a (which is incorrect). Have them work through the next example to help them discover what the third differences really are. This is a good example of why it is necessary to follow up conjecture based on observation with proof.

Example 3 (7 minutes)

Example 3							
Find the second, third, and fourth differences of the polynomial $ax^3 + bx^2 + cx + d$ by filling in the blanks in the following table.							
x	$ax^3 + bx^2 + cx + d$	First Differences	Second Differences	Third Differences	Fourth Differences		
0	d						
1	a+b+c+d	a+b+c	6a + 2b				
		7a + 3b + c		<u>6a</u>			
2	8a + 4b + 2c + d		12a + 2b		0		
		19a + 5b + c		<u>6a</u>			
3	27a + 9b + 3c + d		18a + 2b		0		
		37a + 7b + c		<u>6a</u>			
4	64a + 16b + 4c + d		24a + 2b				
		61a + 9b + c					
5	125a + 25b + 5c + d						



Successive Differences in Polynomials





The third differences of $ax^3 + bx^2 + cx + d$ all have the constant value 6*a*. Also, if a different sequence of values for *x* that differed by 1 was used instead, the third differences would still have the value 6*a*.

- Ask students to make a conjecture about the fourth differences of a sequence generated by a degree 4 polynomial. Students who were paying attention to their (likely wrong) conjecture of the third differences before doing this example may guess that the fourth differences are constant and equal to $(1 \cdot 2 \cdot 3 \cdot 4)a$, which is 24*a*. This pattern continues: the n^{th} differences of any sequence generated by an n^{th} degree polynomial with leading coefficient *a* will be constant and have the value $a \cdot (n!)$.
- Ask students to make a conjecture about the $(n + 1)^{st}$ differences of a degree *n* polynomial, for example, the 5th differences of a fourth-degree polynomial.

Students are now ready to tackle the main goal of this lesson—using differences to recognize polynomial relationships and build polynomial expressions.

Example 4 (7 minutes)

When collecting bivariate data on an event or experiment, the data does not announce, "I satisfy a quadratic relationship," or "I satisfy an exponential relationship." There need to be ways to recognize these relationships in order to model them with functions. In Algebra I, Module 3, students studied the conditions upon which they could conclude that the data satisfied a linear or exponential relationship. Either the first differences were constant, or *first factors* were constant. By checking that the second or third differences of the data are constant, students now have a way to recognize a quadratic or cubic relationship and can write an equation to describe that relationship (A-CED.A.3, F-BF.A.1a).

Give students an opportunity to attempt this problem in groups of two. Walk around the room helping them find the leading coefficient.

v to help you decide	e. (The first difference	es have already been com	puted for you.)	
x	у	First Differences	Second Differences	Third Differences
0	2			
-		-1		
1	1		6	
		5		6
2	6		12	
		17		6
3	23		18	
		35		6
4	58		24	
		59		
5	117			



Successive Differences in Polynomials



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Find the equation of the form $y = ax^3 + bx^2 + cx + d$ that all ordered pairs (x, y) above satisfy. Give evidence that your equation is correct. Since third differences of a cubic polynomial are equal to 6a, using the table above, we get 6a = 6, so that a = 1. Also,

Since third alignments of a cubic polynomial are equal to 6a, using the table above, we get 6a = 6, so that a = 1. Also, since (0, 2) satisfies the equation, we see that d = 2. Thus, we need only find b and c. Substituting (1, 1) and (2, 6) into the equation, we get

$$1 = 1 + b + c + 2$$

6 = 8 + 4b + 2c + 2.

Subtracting two times the first equation from the second, we get 4 = 6 + 2b - 2, so that b = 0. Substituting 0 in for b in the first equation gives c = -2. Thus, the equation is $y = x^3 - 2x + 2$.

After finding the equation, have students check that the pairs (3, 23) and (4, 58) satisfy the equation.

Help students to persevere in finding the coefficients. They will most likely try to plug three ordered pairs into the equation, which gives a 3×3 system of linear equations in a, b, and c after they find that d = 2. Using the fact that the third differences of a cubic polynomial are 6a will greatly simplify the problem. (It implies a = 1 immediately, which reduces the system to the easy 2×2 system above.) Walk around the room as they work, and ask questions that lead them to realize that they can use the third differences fact if they get too stuck. Alternatively, find a student who used the fact, and then have the class discuss and understand his or her approach.

Closing (7 minutes)

- What are some of the key ideas that we learned today?
 - Sequences whose second differences are constant satisfy a quadratic relationship.
 - Sequences whose third differences are constant satisfy a cubic relationship.

The following terms were introduced and taught in Module 1 of Algebra I. The terms are listed here for completeness and reference.

Relevant Vocabulary

```
NUMERICAL SYMBOL: A numerical symbol is a symbol that represents a specific number. Examples: 1, 2, 3, 4, \pi, -3.2.
VARIABLE SYMBOL: A variable symbol is a symbol that is a placeholder for a number from a specified set of numbers. The
set of numbers is called the domain of the variable. Examples: x, y, z.
ALGEBRAIC EXPRESSION: An alaebraic expression is either
     a numerical symbol or a variable symbol or
1.
     the result of placing previously generated algebraic expressions into the two blanks of one of the four operators
2.
     ((\_) + (\_), (\_) - (\_), (\_) \times (\_), (\_) \div (\_)) or into the base blank of an exponentiation with an exponent that
     is a rational number.
Following the definition above, ((x) \times (x)) \times (x)) + ((3) \times (x)) is an algebraic expression, but it is generally written
more simply as x^3 + 3x.
NUMERICAL EXPRESSION: A numerical expression is an algebraic expression that contains only numerical symbols (no variable
symbols) that evaluates to a single number. Example: The numerical expression \frac{(3\cdot2)^2}{12} evaluates to 3.
MONOMIAL: A monomial is an algebraic expression generated using only the multiplication operator (\_ \times \_). The
expressions x^3 and 3x are both monomials.
```

BINOMIAL: A *binomial* is the sum of two monomials. The expression $x^3 + 3x$ is a binomial.



Successive Differences in Polynomials







POLYNOMIAL EXPRESSION: A polynomial expression is a monomial or sum of two or more monomials.

SEQUENCE: A sequence can be thought of as an ordered list of elements. The elements of the list are called the *terms of the sequence*.

ARITHMETIC SEQUENCE: A sequence is called *arithmetic* if there is a real number d such that each term in the sequence is the sum of the previous term and d.

Exit Ticket (5 minutes)











Name

Date _____

Lesson 1: Successive Differences in Polynomials

Exit Ticket

1. What type of relationship is indicated by the following set of ordered pairs? Explain how you know.

x	у
0	0
1	2
2	10
3	24
4	44

2. Find an equation that all ordered pairs above satisfy.









Exit Ticket Sample Solutions

1.	What type of I	relationship i	s indicated by th	ne following set of ordered pa	iirs? Explain how you know.	
		x	y	First Differences	Second Differences	
		0	0			
				2		
		1	2		6	
				8		
		2	10		6	
				14		
		3	24		6	
				20		
		4	44			
2.	Find an equati Since $(0,0)$ sa (2,10), we ha	ion that all or tisfies an equive	rdered pairs abo uation of the for	we satisfy. $m y = ax^2 + bx + c$, we hav 2 = a + b	e that $c = 0$. Using the points (1,	2) and
				$10 = \mathbf{4a} + \mathbf{2b}$		
	Subtracting tw equation gives	vice the first $b = -1$. The formula $b = -1$ is the first of the fir	equation from the form $y = 3x^2 - x^2$	he second gives $6 = 2a$, which x is the equation.	h means $a=3$. Substituting 3 int	o the first
	OR					
	Since the pairs so $a = 3$. Since Thus, $y = 3x^2$	s satisfy a que ce $(0, 0)$ satis -x is the eq	adratic relations sfies the equatio quation that is so	hip, the second differences m n, $c = 0$. Using the point (1, atisfied by these points.	ust be equal to $2a$. Therefore, $6 = 2$), we have that $2 = 3 + b + 0$, s	= 2a, so $b = -1.$

Problem Set Sample Solutions

t	$36 - 16t^2$	First Differences	Second Differences
0	36		
		-16	
1	20		-32
		-48	
2	-28		-32
		-80	
3	-108		-32
		-112	
4	-220		-32
		-144	
5	-364		

EUREKA MATH

Successive Differences in Polynomials







2.	Create a table to find the third differences for the polynomial $s^3 - s^2 + s$ for integer values of s from -3 to 3.						
	S	$s^3 - s^2 + s$	First Differences	Second Differences	Third Differences		
	-3	-39					
			25				
	-2	-14		-14			
			11		6		
	-1	-3		-8			
			3		6		
	0	0		-2			
			1		6		
	1	1		4			
			5		6		
	2	6		10			
			15				
	3	21					
3.	Create a table of second difference	values for the polynomial x es are all equal to 2.	² , using $n, n + 1, n +$	2, $n + 3$, $n + 4$ as value	les of <i>x</i> . Show that the		
	<i>x</i>	x- 	First Differences	Second Diffe	rences		
	n	<i>n</i> -	2n + 1				
	n + 1	1 $n^2 + 2n + 1$	2n + 3	2			
	n + 2	2 $n^2 + 4n + 4$	2	2			
	n + 1	3 $n^2 + 6n + 9$	2n + 5	2			
			2				

			2n + 7		
	<i>n</i> + 4	$n^2 + 8n + 16$			
Show that difference	t the set o es.) Find t	f ordered pairs (x, y) in the equation of the form	the table below satisfies a q $y = ax^2 + bx + c$ that all o	uadratic relationship. (Hint: F f the ordered pairs satisfy.	ind second

x	0	1	2	3	4	5
у	5	4	-1	-10	-23	-40

Students show that second differences are constant and equal to -4. The equation is $y = -2x^2 + x + 5$.

5. Show that the set of ordered pairs (x, y) in the table below satisfies a cubic relationship. (Hint: Find third differences.) Find the equation of the form $y = ax^3 + bx^2 + cx + d$ that all of the ordered pairs satisfy.

x	0	1	2	3	4	5
у	20	4	0	20	76	180
Students show that third differences are constant and equal to 12. The equation is $y = 2x^3 - 18x + 20$.						



4.

Successive Differences in Polynomials







v	0	1	2	3	4	5		
d	0	5	19.5	43.5	77	120		
	•	·	•	•				
a.	What type of	relationship is indi	cated by the set	of ordered pairs	\$?			
	Students sho quadratic.	w that second diffe	rences are const	ant and equal to	9.5. Therefore,	, the relationship is		
b.	Assuming that reaches 60 n	at the relationship compared to the relation of $v=6$.	continues to hold	d, find the distar	ice required to st	op the car when the spee		
	172.5 ft							
c.	Extension: Findistance d . $d = 4.75v^2$	ind an equation tha + 0.25v <i>(Note: St</i>	t describes the r udents do not no	relationship betw eed to find the e	veen the speed o	of the car v and its stoppir er part (b).)		
7. Us	e the polynomial	expressions $5x^2 +$	x + 1 and $2x + 1$	3 to answer the	questions belov	v.		
a.	Create a tabl	e of second differer	nces for the poly	momial $5x^2 + x$	+ 1 for the integ	ger values of <i>x</i> from 0 to 5		
	x	$5x^2 + x + 1$	First	Differences	Second Dij	fferences		
	0	1		6				
	1	7		16	10)		
	2	23		16	10			
	3	49		26	10)		
	4	85		36	1()		
	5	131		46				
b.	Justin claims that for $n \ge 2$, the n^{th} differences of the sum of a degree n polynomial and a linear polynomial are the same as the n^{th} differences of just the degree n polynomial. Find the second differences for the sum $(5x^2 + x + 1) + (2x + 2)$ of a degree 2 and a degree 1 achieve the advector statements are the same the second differences for the sum							
	$(5x^2 + x + 1) + (2x + 3)$ of a degree 2 and a degree 1 polynomial, and use the calculation to explain why Justin might be correct in general.							
	Students con correct becau higher) differ degree n pol	pute that the secor ise the differences of ences of the degree ynomial contribute	nd differences an of the sum are th e 1 polynomial a to the n th differe	re constant and o he sum of the di <u>f</u> ire constant and ence of the sum.	equal to 10, just ferences. Since t equal to zero, or	as in part (a). Justin is he second (and all other ly the $\mathbf{n}^{ heta}$ differences of the second second the second sec		
c.	Jason thinks he can generalize Justin's claim to the product of two polynomials. He claims that for $n \ge 2$, the $(n + 1)^{\text{st}}$ differences of the product of a degree n polynomial and a linear polynomial are the same as the n^{th} differences of the degree n polynomial. Use what you know about second and third differences (from Examples 2 and 3) and the polynomial $(5x^2 + x + 1)(2x + 3)$ to show that Jason's generalization is incorrect.							
	The second d 10. Since (5 that the third	ifferences of a quad $x^2 + x + 1)(2x + 3)$ I differences of $(5x^2)$	$dratic polynomia (3) = 10x^3 + 17)^2 + (2x + 1)(2x + 1)^2$	and are $2a$, so the $(x^2 + 5x + 3)$ are always (second difference d third difference 60, which is not 1	es of $5x^2 + x + 1$ are alw es are equal to 6a, we ha 10.		







Lesson 2: The Multiplication of Polynomials

Student Outcomes

• Students develop the distributive property for application to polynomial multiplication. Students connect multiplication of polynomials with multiplication of multi-digit integers.

Lesson Notes

This lesson begins to address standards **A-SSE.A.2** and **A-APR.C.4** directly and provides opportunities for students to practice MP.7 and MP.8. The work is scaffolded to allow students to discern patterns in repeated calculations, leading to some general polynomial identities that are explored further in the remaining lessons of this module.

As in the last lesson, if students struggle with this lesson, they may need to review concepts covered in previous grades, such as:

- The connection between area properties and the distributive property: Grade 7, Module 6, Lesson 21.
- Introduction to the table method of multiplying polynomials: Algebra I, Module 1, Lesson 9.
- Multiplying polynomials (in the context of quadratics): Algebra I, Module 4, Lessons 1 and 2.

Since division is the inverse operation of multiplication, it is important to make sure that your students understand how to multiply polynomials before moving on to division of polynomials in Lesson 3 of this module. In Lesson 3, division is explored using the *reverse tabular method*, so it is important for students to work through the table diagrams in this lesson to prepare them for the upcoming work.

There continues to be a sharp distinction in this curriculum between justification and proof, such as justifying the identity $(a + b)^2 = a^2 + 2ab + b$ using area properties and proving the identity using the distributive property. The key point is that the area of a figure is *always* a nonnegative quantity and so cannot be used to prove an algebraic identity where the letters can stand for negative numbers (there is no such thing as a geometric figure with negative area). This is one of many reasons that manipulatives such as Algebra Tiles need to be handled with extreme care: depictions of *negative area* actually teach incorrect mathematics. (A correct way to model expressions involving the subtraction of two positive quantities using an area model is depicted in the last problem of the Problem Set.)

The tabular diagram described in this lesson is purposely designed to look like an area model without actually being an area model. It is a convenient way to keep track of the use of the distributive property, which is a basic property of the number system and is assumed to be true for all real numbers—regardless of whether they are positive or negative, fractional or irrational.

Classwork

Opening Exercise (5 minutes)

The Opening Exercise is a simple use of an area model to justify why the distributive property works when multiplying 28×27 . When drawing the area model, remember that it really matters that the length of the side of the big square is about $2\frac{1}{2}$ times the length of the top side of the upper right rectangle (20 units versus 8 units) in the picture below and similarly for the lengths going down the side of the large rectangle. It should be an accurate representation of the area of a rectangular region that measures 28 units by 27 units.













Example 1 (9 minutes)

MP.7

Explain that the goal today is to generalize the Opening Exercise to multiplying polynomials. Start by asking students how the expression (x + 8)(x + 7) is similar to the expression 28×27 . Then suggest that students replace 20 with x in the area model. Since x in (x + 8)(x + 7) can stand for a negative number, but lengths and areas are always positive, an area model cannot be used to represent the polynomial expression (x + 8)(x + 7) without also saying that x > 0. So it is not correct to say that the area model above (with 20 replaced by x) represents the polynomial expression (x + 8)(x + 7) for all values of x. The tabular method below is meant to remind students of the area model as a visual representation, but it is not an area model.



- Explain how the result $x^2 + 15x + 56$ is related to 756 determined in the Opening Exercise.
 - If x is replaced with 20 in $x^2 + 15x + 56$, then the calculation becomes the same as the one shown in the Opening Exercise: $(20)^2 + 15(20) + 56 = 400 + 300 + 56 = 756$.



The Multiplication of Polynomials

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engage



- How can we multiply these binomials without using a table?
 - Think of x + 8 as a single number and distribute over x + 7:



Next, distribute the x over x + 8 and the 7 over x + 8. Combining like terms shows that $(x + 8)(x + 7) = x^2 + 15x + 56$.

$$(x+8)(x+7) = (x+8) \cdot x + (x+8) \cdot 7 = x^2 + 8x + 7x + 56$$

- What property did we repeatedly use to multiply the binomials?
 - The distributive property
- The table in the calculation above looks like the area model in the Opening Exercise. What are the similarities? What are the differences?
 - The expressions placed in each table entry correspond to the expressions placed in each rectangle of the area model. The sum of the table entries represents the product, just as the sum of the areas of the sub-rectangles is the total area of the large rectangle.
 - One difference is that we might have x < 0 so that 7x and 8x are negative, which does not make sense in an area model.
- How would you have to change the table so that it represents an area model?
 - First, all numbers and variables would have to represent positive lengths. So, in the example above, we would have to assume that x > 0. Second, the lengths should be commensurate with each other; that is, the side length for the rectangle represented by 7 should be slightly shorter than the side length represented by 8.
- How is the tabular method similar to the distributive property?
 - The sum of the table entries is equal to the result of repeatedly applying the distributive property to (x + 8)(x + 7). The tabular method graphically organizes the results of using the distributive property.
- Does the table work even when the binomials do not represent lengths? Why?
 - Yes it does because the table is an easy way to summarize calculations done with the distributive property—a property that works for all polynomial expressions.

Scaffolding:

If students need to work another problem, ask students to use an area model to find 16×19 and then use the tabular method to find (x + 6)(x + 9).

Exercises 1–2 (6 minutes)

MP.7

Allow students to work in groups or pairs on these exercises. While Exercise 1 is analogous to the previous example, in Exercise 2, students may need time to think about how to handle the zero coefficient of x in $x^2 - 2$. Allow them to struggle and discuss possible solutions.



The Multiplication of Polynomials









Example 2 (6 minutes)

Prior to Example 2, consider asking students to find the products of each of these expressions.

$$(x-1)(x+1) (x-1)(x2 + x + 1) (x-1)(x3 + x2 + x + 1)$$

Students may work on this in mixed-ability groups and come to generalize the pattern.

Scaffolding:

For further scaffolding, consider asking students to see the pattern using numerical expressions, such as:

$$(2-1)(2^{1}+1)$$
$$(2-1)(2^{2}+2+1)$$
$$(2-1)(2^{3}+2^{2}+2+1).$$

Can they describe in words or symbols the meaning of these quantities?

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The Multiplication of Polynomials





Example 2

Multiply the polynomials $(x - 1)(x^4 + x^3 + x^2 + x + 1)$ using a table. Generalize the pattern that emerges by writing down an identity for $(x - 1)(x^n + x^{n-1} + \dots + x^2 + x + 1)$ for n a positive integer.

MP.8



- What quadratic identity from Algebra I does the identity above generalize?
 - This generalizes $(x 1)(x + 1) = x^2 1$, or more generally, the difference of squares formula $(x y)(x + y) = x^2 y^2$ with y = 1. We will explore this last identity in more detail in Exercises 4 and 5.

Exercises 3-4 (10 minutes)

Before moving on to Exercise 3, it may be helpful to scaffold the problem by asking students to multiply (x - y)(x + y)and $(x - y)(x^2 + xy + y^2)$. Ask students to make conjectures about the form of the answer to Exercise 3.

Exercises 3–4

3. Multiply (x - y)(x³ + x²y + xy² + y³) using the distributive property and combine like terms. How is this calculation similar to Example 2?
Distribute the expression x³ + x²y + xy² + y³ through x - y to get
(x - y)(x³ + x²y + xy² + y³) = x⁴ + x³y + x²y² + xy³ - x³y - x²y² - xy³ - y⁴ = x⁴ - y⁴
Substitute 1 in for y to get the identity for n = 3 in Example 2.
This calculation is similar to Example 2 because it has the same structure. Substituting 1 for y results in the same expression as Example 2.

Exercise 3 shows why the mnemonic FOIL is not very helpful—and in this case does not make sense. By now, students should have had enough practice multiplying to no longer require such mnemonics to help them. They understand that the multiplications they are doing are really *repeated use of the distributive property*, an idea that started when they learned the multiplication algorithm in Grade 4. However, it may still be necessary to summarize the process with a mnemonic. If this is the case, try *Each-With-Each*, or EWE, which is short for the process of multiplying each term of one polynomial with each term of a second polynomial and combining like terms.



The Multiplication of Polynomials





To introduce Exercise 4, consider starting with a group activity to help illuminate the generalization. For example, students could work in groups again to investigate the pattern found in expanding these expressions.

$(x^2 + y^2)(x^2 - y^2)$	
$(x^3 + y^3)(x^3 - y^3)$	
$(x^4 + y^4)(x^4 - y^4)$	
$(x^5 + y^5)(x^5 - y^5)$	

Scaffolding: For further scaffolding, consider asking students to see the pattern using numerical expressions, such as: $(3^2 + 2^2)(3^2 - 2^2)$ $(3^3 + 2^3)(3^3 - 2^3)$ $(3^4 + 2^4)(3^4 - 2^4).$ How do $13 \cdot 5$ and $3^4 - 2^4$ relate to the first line? How do

 $35 \cdot 19$ and $3^6 - 2^6$ relate to the second line? Etc.



Sample student work:

 x^{n} y^{n} y^{n} y^{2n}

 $(x^n - y^n)(x^n + y^n) = x^{2n} - y^{2n}$

The generalized identity $x^{2n} - y^{2n} = (x^n - y^n)(x^n + y^n)$ is used several times in this module. For example, it helps to recognize that $2^{130} - 1$ is not a prime number because it can be written as $(2^{65} - 1)(2^{65} + 1)$. Some of the problems in the Problem Set rely on this type of thinking.

Closing (4 minutes)

Ask students to share two important ideas from the day's lesson with their neighbor. You can also use this opportunity to informally assess their understanding.

- Multiplication of two polynomials is performed by repeatedly applying the distributive property and combining like terms.
- There are several useful identities:
 - (a + b)(c + d) = ac + ad + bc + bd (an example of *each-with-each*)
 - $(a+b)^2 = a^2 + 2ab + b^2$
 - $(x^n y^n)(x^n + y^n) = x^{2n} y^{2n}$, including $(x y)(x + y) = x^2 y^2$ and $(x^2 y^2)(x^2 + y^2) = x^4 y^4$
 - $(x-1)(x^n + x^{n-1} + \dots + x^2 + x + 1) = x^{n+1} 1$





• (Optional) Consider a quick white board activity in which students build fluency with applying these identities.

The vocabulary used in this lesson was introduced and taught in Algebra I. The definitions included in this lesson are for reference. To support students, consider creating a poster with these vocabulary words for the classroom wall.

Relevant Vocabulary

EQUIVALENT POLYNOMIAL EXPRESSIONS: Two polynomial expressions in one variable are *equivalent* if, whenever a number is substituted into all instances of the variable symbol in both expressions, the numerical expressions created are equal.

POLYNOMIAL IDENTITY: A *polynomial identity* is a statement that two polynomial expressions are equivalent. For example, $(x + 3)^2 = x^2 + 6x + 9$ for any real number x is a polynomial identity.

COEFFICIENT OF A MONOMIAL: The *coefficient of a monomial* is the value of the numerical expression found by substituting the number 1 into all the variable symbols in the monomial. The coefficient of $3x^2$ is 3, and the coefficient of the monomial $(3xyz) \cdot 4$ is 12.

TERMS OF A POLYNOMIAL: When a polynomial is expressed as a monomial or a sum of monomials, each monomial in the sum is called a *term* of the polynomial.

LIKE TERMS OF A POLYNOMIAL: Two terms of a polynomial that have the same variable symbols each raised to the same power are called *like terms*.

STANDARD FORM OF A POLYNOMIAL IN ONE VARIABLE: A polynomial expression with one variable symbol, *x*, is in *standard form* if it is expressed as

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

where *n* is a non-negative integer, and $a_0, a_1, a_2, ..., a_n$ are constant coefficients with $a_n \neq 0$.

A polynomial expression in x that is in standard form is often just called a polynomial in x or a polynomial.

The degree of the polynomial in standard form is the highest degree of the terms in the polynomial, namely n. The term $a_n x^n$ is called the *leading term* and a_n (thought of as a specific number) is called the *leading coefficient*. The constant term is the value of the numerical expression found by substituting 0 into all the variable symbols of the polynomial, namely a_0 .

Exit Ticket (5 minutes)









Name

Date _____

Lesson 2: The Multiplication of Polynomials

Exit Ticket

Multiply $(x - 1)(x^3 + 4x^2 + 4x - 1)$ and combine like terms. Explain how you reached your answer.









Exit Ticket Sample Solutions



Problem Set Sample Solutions

1.	Complete the following statements by filling in the blanks.					
	a.	$(a+b)(c+d+e) = ac+ad+ae+__+__+__+__$	bc, bd, be			
	b.	$(r-s)^2 = \left(\underline{\qquad}\right)^2 - \left(\underline{\qquad}\right)rs + s^2$	r, 2			
	c.	$(2x+3y)^2 = (2x)^2 + 2(2x)(3y) + (___)^2$	3у			
	d.	$(w-1)(1+w+w^2) = -1$	<i>w</i> ³			
	е.	$a^2 - 16 = (a + __)(a - \)$	4, 4			
	f.	$(2x+5y)(2x-5y) = \ \$	$4x^2$, $25y^2$			
	g.	$(2^{21} - 1)(2^{21} + 1) = \ 1$	2 ⁴²			
	h.	$[(x-y)-3][(x-y)+3] = (\)^2 - 9$	<i>x</i> – <i>y</i>			

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Lesson 2: The Multiplication of Polynomials





3. Multiply and combine like terms to write as the sum or difference of monomials. a. 2a(5+4a)b. $x^3(x+6)+9$ $8a^2 + 10a$ $x^4 + 6x^3 + 9$ d. $2^{23}(2^{84}-2^{81})$ $\frac{1}{8}(96z+24z^2)$ c. $2^{107} - 2^{104}$ $12z + 3z^2$ e. (x-4)(x+5)f. (10w - 1)(10w + 1) $x^2 + x - 20$ $100w^2 - 1$ g. $(3z^2 - 8)(3z^2 + 8)$ h. $(-5w-3)w^2$ $-5w^3 - 3w^2$ $9z^4 - 64$ $8\gamma^{1000}(\gamma^{12200}+0.125\gamma)$ j. $(2r+1)(2r^2+1)$ i. $8y^{13200} + y^{1001}$ $4r^3 + 2r^2 + 2r + 1$ I. $(w-1)(w^5 + w^4 + w^3 + w^2 + w + 1)$ k. $(t-1)(t+1)(t^2+1)$ $t^4 - 1$ $w^6 - 1$ m. (x+2)(x+2)(x+2)n. n(n+1)(n+2) $x^3 + 6x^2 + 12x + 8$ $n^3 + 3n^2 + 2n$ o. n(n+1)(n+2)(n+3)p. n(n+1)(n+2)(n+3)(n+4) $n^4 + 6n^3 + 11n^2 + 6n$ $n^{5} + 10n^{4} + 35n^{3} + 50n^{2} + 24n$ r. $(x+1)(x^5 - x^4 + x^3 - x^2 + x - 1)$ q. $(x+1)(x^3-x^2+x-1)$ $x^4 - 1$ $x^{6} - 1$ s. $(x+1)(x^7 - x^6 + x^5 - x^4 + x^3 - x^2 + x - 1)$ t. $(m^3 - 2m + 1)(m^2 - m + 2)$ $x^{8} - 1$ $m^5 - m^4 + 3m^2 - 5m + 2$ Polynomial expressions can be thought of as a generalization of place value. 4. Multiply 214×112 using the standard paper-and-pencil algorithm. а. 214 x112 428 214 +214 Multiply $(2x^2 + x + 4)(x^2 + x + 2)$ using the tabular method and combine like terms. b. 2x2 4 X x² 2×4 X3 4x2 X2 $(2x^{2} + x + 4)(x^{2} + x + 2) = 2x^{4} + 3x^{3} + 9x^{2} + 6x + 8$ $2x^3$ 4× X 2×4 8 4x2 2x 2 9x 8 6x



Lesson 2: The Multiplication of Polynomials

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EUREKA MATH

The Multiplication of Polynomials



Lesson 2:



Q Lesson 3: The Division of Polynomials

Student Outcomes

 Students develop a division algorithm for polynomials by recognizing that division is the inverse operation of multiplication.

Lesson Notes

This lesson begins to address standards **A-SSE.A.2** and **A-APR.C.4** and provides many opportunities for students to practice MP.7 and 8. Students explore reversing the tabular method they learned in Lesson 2 as a means to divide two polynomials. They develop a procedure for polynomial division using a table. The problems are scaffolded to lead students to discover this method in small groups, allowing them to discern patterns in repeated calculations, reinforcing the work from Lessons 1 and 2. Discussion questions and the lesson closure are key to guiding struggling students. In this lesson it is critical to emphasize the relationships between the table entries and the position of the dividend and divisor and how the diagonals in the table sum to the terms of the dividend. In the next lesson, students will connect this method to the traditional long division algorithm of polynomial division while reinforcing the relationship between polynomial division and integer arithmetic. All of the problems in this lesson divide without a remainder; division with remainders is addressed in later lessons in this module. General polynomial identities that are used heavily and explored further in the remaining lessons of this module are also touched on in this lesson.

Since division is the inverse operation of multiplication, it is important to make sure that students understand how to multiply polynomials before moving on to division of polynomials found in this lesson. If students are struggling with the content of this lesson, they may need to review problems from Lesson 2 in this module and the following lessons from previous grades:

- Studying the connection between area properties and the distributive property: Grade 7, Module 6, Lesson 21.
- Introduction to the tabular method of multiplying polynomials: Algebra I, Module 1, Lesson 9.

In the early lessons on division (Lessons 3–7), the issue of potential zeros in the denominator is not emphasized but becomes prominent in Lesson 21 when rational expressions are introduced. When introducing polynomial division in this lesson, choose whether or not to emphasize that it is necessary to exclude certain values of x that cause division by zero.

Classwork

Opening Exercise (3 minutes)

This exercise provides students with an opportunity to practice the tabular method of multiplication of polynomials. This problem is continued in the Discussion and Exploratory Challenge that follow.

Scaffolding:

For students who are unfamiliar with the vocabulary in this lesson (divisor, quotient, dividend, factor, product), take time to pre-teach these terms. Use a vocabulary journal or notebook to help them learn and use proper terminology throughout the lesson.

A visual representation like the one below may be of assistance.

(Factor)(Factor) = Product

 $\frac{\text{Dividend}}{\text{Divisor}} = \text{Quotient}$









Opening Exercise a. Multiply these polynomials using the tabular method. $(2x + 5)(x^2 + 5x + 1)$ The product is $2x^3 + 15x^2 + 27x + 5$. b. How can you use the expression in part (a) to quickly multiply $25 \cdot 151$? If you let x = 10, then the product is $2(10)^3 + 15(10)^2 + 27(10) + 5 = 2000 + 1500 + 270 + 5$ = 3775.

Discussion (5 minutes)

Lead a discussion that connects multiplication and division. Display the following problem on the board, and ask how it could be transformed into a division problem.

$$25 \cdot 151 = 3,775$$

- How can a multiplication problem be rewritten as a division problem?
 - One of the factors is the divisor and the other is the quotient. The product is the dividend.

$$\frac{3775}{25} = 151$$
 OR $\frac{3775}{151} = 25$

How can we rewrite the Opening Exercise as a division problem?

$$(2x+5)(x^2+5x+1) = 2x^3+15x^2+27x+5$$

It would be rewritten the same way. One of the factors is the divisor and the other is the quotient. The product is the dividend.

$$\frac{2x^3 + 15x^2 + 27x + 5}{2x + 5} = x^2 + 5x + 1 \quad \text{OR} \quad \frac{2x^3 + 15x^2 + 27x + 5}{x^2 + 5x + 1} = 2x + 5$$

- Let x = 10. Substitute that value into each polynomial, and compare the results of multiplying and dividing the polynomials with the arithmetic problem. How do polynomial multiplication and division compare to multiplication and division of integers?
 - When x = 10, the polynomial problems result in the same values as the number problems. This reinforces the fact that arithmetic operations with polynomials are similar to arithmetic operations with integers.

Exploratory Challenge (27 minutes): Reverse the Tabular Method to Divide Polynomials

Give students time to discuss how they would fill in the rows and columns in the table below. Let them struggle to make sense of the problem and look for patterns. If they are proficient with the tabular method of multiplication, they should quickly discover a method for populating the table and then verifying that the quotient is in the top row. Read through this entire section before attempting this with students in order to clearly understand how the process works. Start this challenge by asking students to consider how they might reverse the tabular method for multiplying to solve a polynomial division problem.









Encourage them to discuss where to position the polynomials now that they have the "answer" (the product) and one of the factors. In the last lessons on multiplication, the factors positioned along the top and right side of the table and the product positioned along the left and bottom came from *summing* the like terms in each diagonal. The student pages pose the problem and provide an empty two-row by three-column table. If needed, use the scaffolded tables provided in the teacher notes that follow to provide more support for students.



The partially completed tables shown below provide some suggestions for scaffolding this lesson exploration (if needed). Encourage students to place the dividend where the product would result from multiplying and the quotient where one of the factors would be located. Then, have them work backward to show that the top row of this table is the quotient, $x^2 + 5x + 1$. Since the table is two by three, it would make sense that 2x + 5 is positioned along the vertical side on the right. Notice that the terms of the quotient are located around the left and bottom side of the table. This is where they would appear if these terms represented the result of multiplying two polynomials whose product was $2x^3 + 15x^2 + 27x + 5$. Notice also that the divisor is positioned along the right side of the table (as was found with one of the factors in the earlier tabular method multiplication problems). The arrows in the diagram indicate that the diagonal entries must sum to the term outside the table (e.g., $15x^2 = 5x^2 + 10x^2$).





Lesson 3: The Division of Polynomials



After students have had a few minutes to discuss their ideas in groups, lead a short discussion (as needed) if they appear to be stuck. Use these questions to prompt groups while also circulating around the room.

- The quotient will be the polynomial that would go along the top of the table. Remember we know it should be $x^2 + 5x + 1$. How can you get started to confirm this using the dividend and the divisor?
 - We need to think about how the divisor terms fill back into the table, and then we need to use the 2x and the 5 terms to determine the other factor for each column at the top of the table.
- Are there any cells in the table that we can fill in based on the information we have? What must be in the top left cell? Why?
 - The $2x^3$ term will be the top left entry, and the 5 will be the bottom right entry. Because the diagonals add to produce the terms of the product, we know that the top left and bottom right entries must be the first and last terms of the dividend since those terms do not involve combining any like terms when we compute the product.
- What must the first term of the missing polynomial in the top row be?
 - It would have to be x^2 since that term and 2x must multiply to be $2x^3$.
- What goes in the rest of the cells in the first column? How can you continue this pattern?
 - Multiply x^2 and 5 to get $5x^2$. Then the 2^{nd} column in the first row would need to be $10x^2$ because $15x^2 = 10x^2 + 5x^2$. Since the 2^{nd} column of the first row is now known, we can figure out the remaining term of the quotient since that term times 2x would have to equal $10x^2$.
- Compare your work on this problem with the Opening Exercise. How could you verify that $x^2 + 5x + 1$ really is the quotient? Explain?
 - In the Opening Exercise, we multiplied the quotient and divisor and got the dividend. So $(2x + 5)(x^2 + 5x + 1)$ must equal $2x^3 + 15x^2 + 27x + 5$.

Describe the process you used to determine your answer to Exercise 1.
 Student descriptions will vary but should be similar to the detailed directions provided on the following page.

Check each group's work before moving on to the next exercise. Have a couple of groups present their solution methods to the Exercises 1 and 2 on the board, especially if other groups seem to be struggling. Then, if needed, go over the steps outlined below in detail to clarify this process for all students. Provide additional scaffolding if needed.

Step 1—Draw a table with factors along the right sides and bottom corner, and fill in known entries along the top left and bottom right.





Lesson 3: The Division of Polynomials







Step 2—Use the top left entry to find the second-degree term along the top. It has to be x^2 because $2x \cdot x^2 = 2x^3$.



Step 3—Use x^2 and 5 to find the bottom left entry, $5x^2$, and then use that entry to find the top middle term. It has to be $10x^2$ because $5x^2 + 10x^2 = 15x^2$.



Step 4—Repeat these steps to fill in the bottom entry in the 2nd and last columns.





Lesson 3: The Division of Polynomials



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Students do not already know the quotient in Exercise 3, making this more challenging than the previous exercises.



After these two exercises, lead a short discussion to help students learn how to reverse their thinking about the tabular method of multiplication to divide polynomials.

- . We will call this approach to dividing polynomials the reverse tabular method. Let's make a conjecture about the number of rows and columns you will need to perform the division. How can you predict how many rows you will need in your table? How can you predict how many columns you will need in your table?
 - The number of rows will be one more than the degree of the divisor. The number of columns will be one more than the difference between the degree of the dividend and the degree of the divisor.





MP.3



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ALGEBRA II

If necessary, remind groups to include a 0 coefficient term place holder when missing a needed term.

5. Test your conjectures. Use the *reverse tabular method* to find the quotient. $\frac{3x^5 - 2x^4 + 6x^3 - 4x^2 - 24x + 16}{x^2 + 4}$ The quotient is $3x^3 - 2x^2 - 6x + 4$.

6. What is the quotient of $\frac{x^5-1}{x-1}$? What is the quotient of $\frac{x^6-1}{x-1}$? The quotients are $x^4 + x^3 + x^2 + x + 1$ and $x^5 + x^4 + x^3 + x^2 + x + 1$, respectively.

After students complete Exercise 6, see if they can extend the patterns to make predictions about similar problems.

- MP.7 & MP.8
- What is the result of dividing $\frac{x^3-1}{x-1}$? • $x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$
- What is the result of dividing $\frac{x^n 1}{x 1}$? $x^{n-1} + x^{n-2} + \dots + x + 1$

• What is the result of dividing
$$\frac{x^5+1}{x+1}$$
?

$$x^4 - x^3 + x^2 - x + 1$$

Scaffolding:

If students are struggling to extend this pattern in Exercise 6, begin with simpler problems such as:

$$\frac{(x^2 - 1)}{(x - 1)} = x + 1$$
$$\frac{x^3 - 1}{x - 1} = x^2 + x + 1$$
$$\frac{x^4 - 1}{x - 1} = x^3 + x^2 + x + 1.$$

They should solve these simpler problems using the reverse tabular method. Perhaps have different groups each work a different problem, and then record the results on the board. Conduct a class discussion so they can generalize the pattern.

Wrap up this lesson by having groups present their solutions to Exercises 3–5 and discuss the strategies they used to solve the problems. It is fine if two groups present the same problem since they may have had slightly different approaches to completing the table. Students may notice that it is easier to fill in the table by columns. Some may have started working from the lower right corner where the constant is located. Students may realize that they do not need to fill in every cell to finish the problem but may wish to do so to check their work. If time permits, have them verify their conjectures to the follow-up questions for Exercise 6.

Closing (5 minutes)

These questions help to reinforce the relationship between multiplication and division and some of the patterns that emerge in using the reverse tabular method to divide polynomials with no remainder. Have students respond to the questions below (either in writing or with a partner) to provide a summary and formative assessment information. The questions that ask about the degree and leading coefficient help to reinforce the **A.APR** standards addressed in this lesson.





- What strategies were helpful when you set up and solved these problems? What patterns did you notice as you solved these problems?
 - The number of rows will be one more than the degree of the divisor. The number of columns will be one more than the difference between the degree of the dividend and the degree of the divisor. For example, when dividing $2x^4 x^3 + 4x^2 1$ by $x^3 + 2x + 1$, we will have four rows since the divisor has degree 3. Then we will need 4 3 + 1 = 2 columns for the quotient.

Lesson 3

ALGEBRA II

- You need to use a place holder for missing terms. For example, $x^2 + 1 = x^2 + 0x + 1$.
- The diagonals must add to the correct terms, and the cells in the table must be the product of the factors along the sides of the table.
- What happens to the degree of the product when you multiply two polynomials?
 - When you multiply polynomials, the degree of the product will be the sum of the degrees of each factor.
- What happens to the degree of the quotient when you divide two polynomials?
 - When you divide polynomials, the degree of the quotient will be the difference of the degrees of the dividend and divisor.
- What happens to leading coefficients when you multiply or divide polynomials?
 - The leading coefficient of the result is the product or quotient of the leading coefficients of the original polynomials.

Exit Ticket (5 minutes)







Name

Date _____

Lesson 3: The Division of Polynomials

Exit Ticket

Find the quotient. Justify your answer.

 $\frac{x^5 + 2x^4 - 7x^2 - 19x + 15}{x^2 + 2x + 5}$









Exit Ticket Sample Solutions

Find the quotient. Justify your an	nswer.			
	x^5 -	$2x^4 - 7$	$x^2 - 19x$	+ 15
		$x^{2} + $	2x + 5	
The quotient is $x^3 - 5x + 3$.				
	X ³	Dx2	-5x	3
	7×5	Dx4	-5X ³	3x2
	x5 2x4	0x3	-10x2	6x
	2x4 5x3	Ox2	-25x	15
	0x3 -7x2	-19x	15	

Problem Set Sample Solutions

Use the reverse tabular method to solve these division problems.		
1.	$\frac{2x^3 + x^2 - 16x + 15}{2x - 3}$	$x^2 + 2x - 5$
2.	$\frac{\left(3x^5+12x^4+11x^3+2x^2-4x-2\right)}{(3x^2-1)}$	$x^3 + 4x^2 + 4x + 2$
3.	$\frac{x^3 - 4x^2 + 7x - 28}{x^2 + 7}$	<i>x</i> – 4
4.	$\frac{x^4 - 2x^3 - 29x - 12}{x^3 + 2x^2 + 8x + 3}$	<i>x</i> – 4
5.	$\frac{6x^5 + 4x^4 - 6x^3 + 14x^2 - 8}{6x + 4}$	$x^4 - x^2 + 3x - 2$
6.	$\frac{(x^3-8)}{(x-2)}$	$x^2 + 2x + 4$
7.	$\frac{x^3+2x^2+2x+1}{x+1}$	$x^2 + x + 1$
8.	$\frac{x^4 + 2x^3 + 2x^2 + 2x + 1}{x + 1}$	$x^3 + x^2 + x + 1$





9. Use the results of Problems 7 and 8 to predict the quotient of $\frac{x^5+2x^4+2x^3+2x^2+2x+1}{x+1}$ Explain your prediction. Then check your prediction using the reverse tabular method. The quotient is $x^4 + x^3 + x^2 + x + 1$. In Problems 7 and 8, the result is a polynomial of degree one less than the dividend where all the coefficients were 1. The dividend in this problem has the same structure except it was degree 5, and it is also divided by x + 1. 10. Use the results of Problems 7–9 above to predict the quotient of $\frac{x^4-2x^3+2x^2-2x+1}{x-1}$. Explain your prediction. Then check your prediction using the reverse tabular method. The quotient is $x^3 - x^2 + x - 1$. 11. Make and test a conjecture about the quotient of $\frac{x^6+x^5+2x^4+2x^3+2x^2+x+1}{x^2+1}$. Explain your reasoning. The quotient is $x^4 + x^3 + x^2 + x + 1$. Since we are missing the x term, there will not be two x^5 or two x terms. Otherwise it will follow the same pattern as Problems 7–9. 12. Consider the following quotients: $\frac{4x^2+8x+3}{2x+1}$ and $\frac{483}{21}$ How are these expressions related? a. If we let x = 10, then $4x^2 + 8x + 3 = 4(10^2) + 8(10) + 3 = 483$ and 2x + 1 = 2(10) + 1 = 21, so $\frac{4x^2+8x+3}{2x+1}=\frac{483}{21}.$ Find each quotient. b. $\frac{4x^2 + 8x + 3}{2x + 1} = 2x + 3 \text{ and } \frac{483}{21} = 23$ Explain the connection between the quotients. c. If we let x = 10, then. 2x + 3 = 2(10) + 3 = 23.





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Lesson 4: Comparing Methods—Long Division, Again?

Student Outcomes

Students connect long division of polynomials with the long division algorithm of arithmetic and use this
algorithm to rewrite rational expressions that divide without a remainder.

Lesson Notes

This lesson reinforces the analogous relationship between arithmetic of numbers and the arithmetic of polynomials (**A-APR.6**, **A-APR.7**). These standards address working with rational expressions and focus on using a long division algorithm to rewrite simple rational expressions. In addition, it provides another method for students to fluently calculate the quotient of two polynomials after the Opening Exercises.

Classwork

Opening

Have students work individually on the Opening Exercises to confirm their understanding of the previous lesson's outcomes. Circulate around the room to observe their progress, or have students check their work with a partner after a few minutes. Today's lesson will transition to another method for dividing polynomials.

Opening Exercises (5 minutes)



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Division and multiplication of polynomials are very similar to those operations with real numbers. In these problems, if x = 10, the result would match an arithmetic problem. Yesterday, students divided two polynomials using the reverse tabular method. Today, the goal is to see how polynomial division is related to the long division algorithm learned in elementary school.

Discussion (5 minutes)

We have seen how division of polynomials relates to multiplication and that both of these operations are similar to the arithmetic operations you learned in elementary school.

- Can we relate division of polynomials to the long division algorithm?
 - We would need to use the fact that the terms of a polynomial expression represent place value when x = 10.

Prompt students to consider the long division algorithm they learned in elementary school, and ask them to apply it to evaluate $1573 \div 13$. Have a student model the algorithm on the board as well. The solution to this problem is included in the example below.

Example 1 (5 minutes): The Long Division Algorithm for Polynomial Division

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MP.7
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When solving the problem in Example 1, be sure to record the polynomial division problem next to the arithmetic problem already on the board. Guide students through this example to demonstrate the parallels between the long division algorithm for numbers and this method. Emphasize that the long division algorithm they learned in elementary school is a special case of polynomial long division. They should record the steps on their handouts or in their notebooks. Have students check their work by solving this problem using the reverse tabular method. Use the questions below while working the example.

See the sample problem written out after Example 1, and use the questions that follow as discussion points while modeling this algorithm.



Comparing Methods—Long Division, Again?





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ALGEBRA II

- What expression multiplied by x will result in x³?
 - $\square x^2$

 When you do long division, you multiply the first digit of the quotient by the divisor and then subtract the result. It works the same with polynomial division. How do we represent multiplication and subtraction of polynomials?

- You apply the distributive property to multiply, and to subtract you add the opposite.
- Then, we repeat the process to determine the next term in the quotient. What do we need to bring down to complete the process?
 - You should bring down the next term.



The completed board work for this example should look something like this:



Example 2 (5 minutes): The Long Division Algorithm for Polynomial Division

Any two numbers can be divided as long as the divisor is not equal to 0. Similarly, any two

polynomials can be divided as long as the divisor is not equal to 0. Note: The number 0 is also a polynomial. Because the class is now dealing with a general case of polynomials and not simply numbers, it is possible to solve problems where the coefficients of the terms are any real numbers. It would be difficult, but not impossible, if the coefficients of the terms of the polynomials were irrational. In the next example, model again how this process works. Be sure to point out that students must use a 0 coefficient place holder for the missing x term.

Example 2
Use the long division algorithm for polynomials to evaluate
$\frac{2x^3 - 4x^2 + 2}{2x - 2}$
The quotient is $x^2 - x - 1$.



Comparing Methods—Long Division, Again?



- For further scaffolding, consider starting with a simpler problem, such as $126 \div 18$. Have students compare this problem to the polynomial division problem $(x^2 + 2x + 6) \div$ (x + 8) by explaining the structural similarities. Have students consider this as the teacher places them side by side on the board. This shows students that if x = 10, the polynomial division problem is analogous to the integer division problem.
- For advanced learners, challenge them to create two examples, a numerical one and a polynomial one, that illustrate the structural similarities. Note, however, that not every problem will work nicely. For example, $800 \div 32 = 25$, but $8x^2 \div (3x + 2) \neq 2x + 5$ because there are many polynomials in *x* that evaluate to 25 when x = 10.



Before beginning the next exercises, take the time to reinforce the idea that polynomial division is analogous to whole number division by posing a reflection question. Students can discuss this with a partner or respond in writing.

- Why are we able to do long division with polynomials?
 - Polynomials form a system analogous to the integers. The same operations that hold for integers hold for polynomials.

Exercises 1-8 (15 minutes)

These problems start simple and become more complicated. Monitor student progress as they work. Have students work these problems independently or in pairs, and use this as an opportunity to informally assess their understanding. After students have completed the exercises, post the solutions on the board but not the work. Have students with errors team up with a partner and trade papers. Ask students to find the mistakes in their partner's work. Choose an incorrect solution to display on the board, and then lead a class discussion to point out where students are likely to make errors and how to prevent them. Students typically make careless errors in multiplying or subtracting terms. Other errors can occur if they forget to include the zero coefficient place holder terms when needed. If students appear to be running short on time, have them check every other result using the reverse tabular method. Alternately, students could check their work using multiplication.

Exe	rcises 1–8
Use tabi	the long division algorithm to determine the quotient. For each problem, check your work by using the reverse Jlar method.
1.	$\frac{x^2+6x+9}{x+3}$
	<i>x</i> + 3
2.	$\frac{7x^3 - 8x^2 - 13x + 2}{7x - 1}$
	$x^2 - x - 2$
3.	$\frac{x^3-27}{x-3}$
	$x^2 + 3x + 9$
4.	$\frac{2x^4 + 14x^3 + x^2 - 21x - 6}{2x^2 - 3}$
	$x^2 + 7x + 2$
5.	$\frac{5x^4 - 6x^2 + 1}{x^2 - 1}$
	$5x^2 - 1$

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6.	$\frac{x^6 + 4x^4 - 4x - 1}{x^3 - 1}$ $x^3 + 4x + 1$
7.	$\frac{2x^7 + x^5 - 4x^3 + 14x^2 - 2x + 7}{2x^2 + 1}$ $x^5 - 2x + 7$
8.	$\frac{x^6 - 64}{x + 2}$ $x^5 - 2x^4 + 4x^3 - 8x^2 + 16x - 32$

Closing (5 minutes)

Ask students to summarize the important parts of this lesson either in writing, to a partner, or as a class. Use this opportunity to informally assess their understanding prior to starting the Exit Ticket. Important elements are included in the Lesson Summary box below. The questions that follow are recommended to guide the discussions with sample student responses included in italics. Depending on the structure of the closure activity, the sample responses would be similar to student-written, partner, or whole-class summaries.

- Which method do you prefer, long division or the reverse tabular method?
 - Student responses will vary. The reverse tabular method may appeal to visual learners. The long division algorithm works well as long as you avoid careless mistakes.
- Is one method easier than another?
 - This will depend on student preferences, but some will like the connection to prior methods for dividing and multiplying. Perhaps when many terms are missing (as in Exercise 8), the reverse tabular method can go more quickly than long division.
- What advice would you give to a friend that is just learning how to do these problems quickly and accurately?
 - ^D Be careful when multiplying terms and working with negative terms.

Lesson Summary

The long division algorithm to divide polynomials is analogous to the long division algorithm for integers. The long division algorithm to divide polynomials produces the same results as the reverse tabular method.

Exit Ticket (5 minutes)



Comparing Methods—Long Division, Again?



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Lesson 4: Comparing Methods—Long Division, Again?

Exit Ticket

Write a note to a friend explaining how to use long division to find the quotient.

 $\frac{2x^2 - 3x - 5}{x + 1}$



Comparing Methods—Long Division, Again?







Exit Ticket Sample Solutions

Write a note to a friend explaining how to use long division to find the quotient.

 $2x^2 - 3x - 5$ *x* + 1

Set up the divisor outside the division symbol and the dividend underneath it. Then ask yourself what number multiplied by x is $2x^2$. Then multiply that number by x + 1, and record the results underneath $2x^2 - 3x$. Subtract these terms and bring down the -5. Then repeat the process.

Problem Set Sample Solutions

Use the long division algorithm to determine the quotient in problems 1–5. $2x^3 - 13x^2 - x + 3$ 1. 2x+1 $x^2 - 7x + 3$ $3x^3+4x^2+7x+22$ 2. x+2 $3x^2 - 2x + 11$ $\frac{x^4 + 6x^3 - 7x^2 - 24x + 12}{x^2 - 4}$ 3. $x^2 + 6x - 3$ 4. $(12x^4 + 2x^3 + x - 3) \div (2x^2 + 1)$ $6x^2 + x - 3$ 5. $(2x^3 + 2x^2 + 2x) \div (x^2 + x + 1)$ 2xUse long division to find the polynomial, p, that satisfies the equation below. 6. $2x^4 - 3x^2 - 2 = (2x^2 + 1)(p(x))$ $p(x) = x^2 - 2$ 7. Given $q(x) = 3x^3 - 4x^2 + 5x + k$. Determine the value of k so that 3x - 7 is a factor of the polynomial q. а. k = -28What is the quotient when you divide the polynomial q by 3x - 7? b. $x^2 + x + 4$



Comparing Methods-Long Division, Again?









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Comparing Methods—Long Division, Again?









Student Outcomes

- Students perform arithmetic operations on polynomials and write them in standard form.
- Students understand the structure of polynomial expressions by quickly determining the first and last terms if the polynomial were to be written in standard form.

Lesson Notes

In this lesson, students work with all four polynomial operations. The first part of the lesson is a relay exercise designed to build fluency, and the second part of the lesson includes combining two or more operations to write a polynomial in standard form. Prepare a set of notecards as described below for Exercises 1–15.

The Algebra Progressions name three different forms for a quadratic expression: standard, factored, and vertex. Examples of these forms are shown below.

- Standard Form: $x^2 + 4x 5$
- Factored Form: (x + 5)(x 1)
- Vertex Form: $(x + 2)^2 9$

It is possible to define a standard and factored form of a degree *n* polynomial expression as well. The final lessons in this module introduce the fundamental theorem of algebra, which states that a degree *n* polynomial with real coefficients can be written as the product of *n* linear factors. Precise definitions of the following terms are provided for teacher reference at the end of this lesson: monomial, polynomial expression, coefficient of a monomial, degree of a monomial, terms of a polynomial, standard form of a polynomial in one variable, and degree of a polynomial in one variable. This lesson concludes by challenging students to quickly determine the first and last term of a polynomial expression if it were to be written in standard form. Students who have this capacity can quickly analyze the end behavior of the graph of a polynomial function or compute key features such as a *y*-intercept without having to fully rewrite the expression in standard form. This allows for quick analysis when applying properties of polynomials in future mathematics courses such as Precalculus and Advanced Topics or Calculus.

Scaffolding:

Before beginning these exercises, it may be necessary to review the definition and/or post an example of a polynomial in standard form. Also consider reminding students of the vertex and factored forms of quadratic expressions using the examples in the Lesson Notes if needed. For example,

- Standard Form:
 - $x^2 + 4x 5$ Factored Form:
- (x+5)(x-1)• Vertex Form:
 - $(x+2)^2 9$

Classwork

Exercises 1–15 (20 minutes): Polynomial Pass

Prior to the lesson, write Exercises 1–15 on index cards, one exercise per card. On the back of each card, write the solution to the previous exercise. For example, on the back of the card for Exercise 2, write the answer to Exercise 1. On the back of the index card for Exercise 1, write the answer to Exercise 15. Students should be seated in a circle if possible. If more than 15 different cards are needed, create additional exercises that require addition, subtraction, multiplication, or division without remainder of linear, quadratic, or cubic polynomial expressions.









To complete the exercise, have students work in pairs. Pass out cards to pairs in numerical order so that when the cards are passed, each pair gets a new exercise that contains the answer to the one they just worked on the back of the card. (Alternatively, have students work individually and make two sets of cards so that cards 16–30 are duplicates of 1–15.) Allow one minute for students to work each calculation on a separate sheet of paper, and then have them pass the card to the next pair and receive a card from the previous pair. They should check their answers and begin the next exercise. After all the rounds are completed, have students complete the graphic organizer on their student handouts to model each operation.

Exercises 1–15: Polynomial Pass				
Per	Perform the indicated operation to write each polynomial in standard form.			
1.	$(x^2 - 3)(x^2 + 3x - 1)$	$x^4 + 3x^3 - 4x^2 - 9x + 3$		
2.	$(5x^2 - 3x - 7) - (x^2 + 2x - 5)$	$4x^2-5x-2$		
3.	$\frac{x^3-8}{x-2}$	$x^2 + 2x + 4$		
4.	(x+1)(x-2)(x+3)	$x^3 + 2x^2 - 5x - 6$		
5.	(x+1) - (x-2) - (x+3)	- <i>x</i>		
6.	$(x+2)(2x^2-5x+7)$	$2x^3 - x^2 - 3x + 14$		
7.	$\frac{x^3 - 2x^2 - 65x + 18}{x - 9}$	$x^2 + 7x - 2$		
8.	$(x^2 - 3x + 2) - (2 - x + 2x^2)$	$-x^2-2x$		
9.	$(x^2 - 3x + 2)(2 - x + 2x^2)$	$2x^4 - 7x^3 + 9x^2 - 8x + 4$		
10.	$\frac{x^3 - x^2 - 5x - 3}{x - 3}$	$x^2 + 2x + 1$		
11.	$(x^2 + 7x - 12)(x^2 - 9x + 1)$	$x^4 - 2x^3 - 74x^2 + 115x - 12$		
12.	$(2x^3 - 6x^2 - 7x - 2) + (x^3 + x^2 + 6x - 12)$	$3x^3 - 5x^2 - x - 14$		
13.	$(x^3-8)(x^2-4x+4)$	$x^5 - 4x^4 + 4x^3 - 8x^2 + 32x - 32$		
14.	$\frac{x^3 - 2x^2 - 5x + 6}{x + 2}$	$x^2 - 4x + 3$		
15.	$(x^3 + 2x^2 - 3x - 1) + (4 - x - x^3)$	$2x^2 - 4x + 3$		









Exercise 16 (5 minutes)

Use this exercise to help students see the structure of polynomial expressions and to create a graphic organizer of how to perform the four polynomial operations for future reference. They should complete this exercise either with a partner or in groups of four.

Exercises 16				
16. I	Review Exercises 1–15 and then select one exercise for ea an example. Be sure to show all your work.	ew Exercises 1–15 and then select one exercise for each category and record the steps in the operation below as cample. Be sure to show all your work.		
	Addition Exercise	Multiplication Exercise		
	Subtraction Exercise	Division Exercise		

Exercises 17–20 (5 minutes)

Before starting the next exercises, lead a short discussion transitioning into the problems below that combine polynomial operations. Say (or paraphrase) the following:

- In the previous exercises, you applied one operation to two or three polynomials, but many times we work with expressions that contain more than one operation. In the next exercises, more than one operation is indicated. How do you determine which operation to perform first?
 - The parentheses and the use of the fraction bar for division tell us which operations to perform first.
 The order of operations also can be applied to polynomials. For example, you have to multiply before you can add or subtract.

After this discussion, have students work with the same partner (or group) that they worked with on Exercise 16. Have different groups present their work to the class. Students could use white boards or chart paper to present their solutions to the class.

Exercises 17–20
For Exercises 17–20, rewrite each polynomial in standard form by applying the operations in the appropriate order.
17.
$$\frac{(x^2+5x+20)+(x^2+6x-6)}{x+2}$$
2x + 7
18.
$$(x^2-4)(x+3) - (x^2+2x-5)$$

$$x^3 + 2x^2 - 6x - 7$$



Lesson 5: Putting It All Together





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19. \frac{(x-3)^3}{x^2-6x+9}
x - 3
20. (x+7)(2x-3) - (x^3 - 2x^2 + x - 2) \div (x-2)
x^2 + 11x - 22
```

Exercise 21 (4 minutes)

This exercise, along with Exercise 22, helps students understand that they can learn quite a bit about the nature of a polynomial expression without performing all the operations required to write it in standard form.

- Sometimes we do not need to perform the entire operation to understand the structure of an expression.
 Can you think of a situation where we might only need to know the first term or the last term of a polynomial expression?
 - If we wanted to understand the shape of the graph of a polynomial, like $p(x) = x^2 + 2x + 3$, we would only need to know the coefficient and degree of the first term to get a general idea of its behavior.
 - The constant term of a polynomial expression indicates the *y*-intercept of the corresponding graph.

```
Exercise 21

21. What would be the first and last terms of the polynomial if it was rewritten in standard form? Answer these quickly without performing all of the indicated operations.

a. (2x^3 - x^2 - 9x + 7) + (11x^2 - 6x^3 + 2x - 9)

First term: -4x^3, Last term: -2

b. (x - 3)(2x + 3)(x - 1)

First term: 2x^3, Last term: 9

c. (2x - 3)(3x + 5) - (x + 1)(2x^2 - 6x + 3)

First term: -2x^3, Last term: -18

d. (x + 5)(3x - 1) - (x - 4)^2

First term: 2x^2, Last term: -21
```

After students complete this exercise, lead a short discussion.

- How did you determine your answer quickly?
 - I knew that the first term would be the one with the highest degree, so I focused on that operation.
 I knew that the last term would be the constant term unless it turned out to equal 0.
- Did any of the solutions surprise you?
 - In parts (a) and (c), I had to pay attention to the order of the terms and which operation would produce the largest degree term.



MP.7



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Exercise 22 (4 minutes)



Closing (2 minutes)

Consider having students record their answers to these questions in writing or by sharing their thoughts with a partner.

- How is polynomial arithmetic similar to integer arithmetic?
 - The four operations produce a new polynomial. The four operations can be combined by following the order of operations conventions.
- How can you quickly determine the first and last terms of a polynomial without performing all of the operations needed to rewrite it in standard form?
 - Analyze the problem to identify the highest degree terms and perform the indicated operation on only those terms. Do the same with the lowest degree terms to determine the last term.

problem (or one similar to it): Generate three different polynomial expressions NOT already expressed in standard form that would have a first term of $-3x^3$ and a last term of $\frac{1}{2}$ if the polynomial expression was written in standard form.

Precise definitions of terms related to polynomials are presented here. These definitions were first introduced in Algebra I, Modules 1 and 4. These definitions are for teacher reference and can be shared with students at the teacher's discretion. Following these definitions are discussion questions for closing this lesson.

Relevant Vocabulary

POLYNOMIAL EXPRESSION: A *polynomial expression* is either a numerical expression or a variable symbol or the result of placing two previously generated polynomial expressions into the blanks of the addition operator $(_ + _)$ or the multiplication operator $(_ \times _)$.

The definition of polynomial expression includes subtraction $(a - b = a + (-1 \cdot b))$, exponentiation by a nonnegative integer $(x^3 = (x \cdot x) \cdot x)$, and dividing by a nonzero number (multiplying by the multiplicative inverse). Because subtraction, exponentiation, and division still apply, the regular notation for these operations is still used. In other $(x^3 - 3x)$

words, polynomials are still written simply as $\frac{(x^3-3x)}{1}$ instead of

$$\frac{1}{2} \cdot \left(\left((x \cdot x) \cdot x \right) + \left(-1 \cdot (3 \cdot x) \right) \right)$$





Lesson 5:





Note, however, that the definition excludes dividing by x or dividing by any polynomial in x.

All polynomial expressions are algebraic expressions.

MONOMIAL: A monomial is a polynomial expression generated using only the multiplication operator $(- \times -)$.

Monomials are products whose factors are numerical expressions or variable symbols.

COEFFICIENT OF A MONOMIAL: The coefficient of a monomial is the value of the numerical expression found by substituting the number 1 into all the variable symbols in the monomial.

Sometimes the coefficient is considered as a constant (like the constant a in ax^2) instead of an actual number. In those cases, when a monomial is expressed as a product of a constant and variables, then the constant is called a constant coefficient.

DEGREE OF A MONOMIAL: The *degree of a* nonzero *monomial* is the sum of the exponents of the variable symbols that appear in the monomial.

For example, the degree of $7x^2y^4$ is 6, the degree of 8x is 1, the degree of 8 is 0, and the degree of $9x^2y^3x^{10}$ is 15.

TERMS OF A POLYNOMIAL: When a polynomial is expressed as a monomial or a sum of monomials, each monomial in the sum is called a *term of the polynomial*.

A monomial can now be described as a polynomial with only one term. But please realize that the word term means many, seemingly disparate things in mathematics (e.g., sin(x) is often called a term of the expression sin(x) + cos(x)), whereas monomial is a specific object. This is why monomial is defined first, and then used to define a term of a polynomial.

STANDARD FORM OF A POLYNOMIAL IN ONE VARIABLE: A polynomial expression with one variable symbol x is in *standard form* if it is expressed as,

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$
,

where n is a nonnegative integer, and $a_0, a_1, a_2, ..., a_n$ are constant coefficients with $a_n \neq 0$.

A polynomial expression in x that is in standard form is often called a *polynomial in* x.

The degree of the polynomial in standard form is the highest degree of the terms in the polynomial, namely n. The term $a_n x^n$ is called the *leading term* and a_n is called the *leading coefficient*. The *constant term* is the value of the numerical expression found by substituting 0 into all the variable symbols of the polynomial, namely a_0 .

In general, one has to be careful about determining the degree of polynomials that are not in standard form: For example, the degree of the polynomial,

$$(x+1)^2 - (x-1)^2$$
,

is not 2 since its standard form is 4x.

DEGREE OF A POLYNOMIAL IN ONE VARIABLE: The *degree of a polynomial expression in one variable* is the degree of the polynomial in standard form that is equivalent to it.

Exit Ticket (5 minutes)







Name

Date _____

Lesson 5: Putting It All Together

Exit Ticket

Jenny thinks that the expression below is equal to $x^2 - 4$. If you agree, show that she is correct. If you disagree, show that she is wrong by rewriting this expression as a polynomial in standard form.

$$\frac{(x-2)^3}{x-2}$$





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Exit Ticket Sample Solutions

MP.3

Jenny thinks that the expression below is equal to $x^2 - 4$. If you agree, show that she is correct. If you disagree, show that she is wrong by rewriting this expression as a polynomial in standard form.

$$\frac{(x-2)^3}{x-2}$$

Multiple approaches are possible to justify why Jenny is incorrect. One possible solution is shown below.

Jenny is incorrect. To perform this operation, you can first divide by x - 2 and then expand the quotient.

$$\frac{(x-2)^3}{x-2} = (x-2)^2$$
$$= x^2 - 4x + 4$$

Problem Set Sample Solutions

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For	Problems 1–7, rewrite each expression as a polynomial in standard form.
1.	$(3x-4)^3$
	$27x^3 - 108x^2 + 144x - 64$
2.	$(2x^2 - x^3 - 9x + 1) - (x^3 + 7x - 3x^2 + 1)$
	$-2x^3+5x^2-16x$
3.	$(x^2 - 5x + 2)(x - 3)$
	$x^3 - 8x^2 + 17x - 6$
	4 3 < 7 0
4.	$\frac{x^2 - x^2 - 6x^2 - 9x + 27}{x - 3}$
	$x^3 + 2x^2 - 9$
5.	(x+3)(x-3) - (x+4)(x-4)
	7
6.	$(x+3)^2 - (x+4)^2$
	-2x - 7
7.	$\frac{x^2-5x+6}{x-3}+\frac{x^3-1}{x-1}$
	$x^2 + 2x - 1$







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ALGEBRA II

For Problems 8–9: Quick, what would be the first and last terms of the polynomial if it was written in standard form? 8. $2(x^2 - 5x + 4) - (x + 3)(x + 2)$ The first and last term are x^2 and 2. $(x-2)^5$ 9. The first and last terms are x^4 and 16. 10. The profit a business earns by selling x items is given by the polynomial function p(x) = x(160 - x) - (100x + 500).What is the last term in the standard form of this polynomial? What does it mean in this situation? The last term is -500, so that p(0) = -500. This means that if no items are sold, the company would lose \$500. 11. Explain why these two quotients are different. Compute each one. What do they have in common? Why? $\frac{(x-2)^4}{x-2}$ and $\frac{x^4-16}{x-2}$ The quotients are $x^3 - 6x^2 + 12x - 8$ and $x^3 + 2x^2 + 4x + 8$. They are different because the dividends are not equivalent expressions. The quotients have the first and last terms in common because division is going to reduce the degree by the difference of the degrees of the numerator and denominator, and their leading coefficients were both one. When multiplying, the last term of a polynomial in standard form is the product of the lowest degree terms in each factor. Therefore, when dividing, the last term of the quotient will be the quotient of the last term of the dividend and divisor. 12. What are the area and perimeter of the figure? Assume there is a right angle at each vertex. 2x + 1515x + 106x + 810x + 30The missing horizontal side length is 8x + 15. The missing vertical side length is 9x + 2. I determined these lengths by subtracting the vertical lengths and by subtracting the horizontal lengths. The perimeter is 50x + 80. I got this by adding the lengths of all of the sides together. The area can be found by splitting the shape either horizontally or vertically into two rectangles. If split vertically, the areas of the rectangles are (15x + 10)(2x + 15) and (6x+8)(8x+15). The total area of the figure is the sum of these two products, $78x^2 + 399x + 270$.

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Lesson 5: Putting It All Together





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Lesson 6: Dividing by x - a and by x + a

Student Outcomes

Students work with polynomials with constant coefficients to derive and use polynomial identities.

Lesson Notes

Students extend their understanding of polynomial division to abstract situations that involve division by x - a and by x + a. Through this work they derive the fundamental identities for the difference of two squares and the sum and difference of two cubes. Further, they connect this work back to divisibility of integers. This lesson bridges the first five lessons of Topic A of this module with the upcoming lessons in Topic B in which students extend work with division and polynomial identities to factoring. In those lessons, they learn the usefulness of the factored form of a polynomial expression to solve polynomial equations and analyze the zeros of the graph of a polynomial function. This lesson addresses aspects of several standards (most notably **A.SSE.A.2** and **A.APR.C.4**) in a way that also emphasizes MP.7 and MP.8. Students recognize and then generalize patterns and use them to fluently rewrite polynomial expressions and perform polynomial operations.

Classwork

Opening (1 minute)

Students may choose to solve the problems using either the reverse tabular method or long division. Encourage and model both approaches throughout this lesson. Have students work with a partner on these problems and then randomly select pairs to present each of the problems on the board. Begin by paraphrasing the following statement as students start the Opening Exercise:

 Today we want to observe patterns when we divide certain types of polynomials and make some generalizations to help us quickly compute quotients without having to do the work involved with the reverse tabular method or the long division algorithm.

Opening Exercise (4 minutes)

Opening Exercise

Find the following quotients, and write the quotient in standard form.

a. $\frac{x^2 - 9}{x - 3}$ x + 3



Provide support for struggling learners by asking them to recall the two methods for dividing polynomials: the reverse tabular method and the long division algorithm. Provide a table with the dividend and divisor in the proper location or a long division problem with the missing zero coefficient terms already in place.

$$x-3)\overline{x^{2}+0x-9}$$
$$x-3)\overline{x^{3}+0x^{2}+0x-27}$$

Or using the tabular method as shown.

$$\begin{array}{c|cccc} x & +3 \\ \hline x^2 & 3x \\ \hline -3x & -9 \\ 0x & -9 \end{array} \begin{array}{c} x \\ -3 \\ -3 \end{array}$$

Ask advanced learners to generate a similar sequence of problems that have a quotient equal to x - 4 and to then explain how they determined their expressions.







b. $\frac{x^3 - 27}{x - 3}$ $x^2 + 3x + 9$ c. $\frac{x^4 - 81}{x - 3}$ $x^3 + 3x^2 + 9x + 27$

Discussion (5 minutes)

Have students come to the board to present their solutions. After students check and correct their work, discuss the patterns that they notice in these problems.

- What patterns do you notice in the Opening Exercise?
 - The expression x 3 divides without a remainder into all three dividends, which means it is a factor of each dividend.
 - The dividends are differences of powers of x and powers of 3. For example, $x^3 27 = x^3 3^3$.
 - The degree of the quotient is 1 less than the degree of the dividend. The terms of the quotient are products of powers of *x* and powers of 3. The exponents on *x* decrease by one, and the exponents on 3 increase by one. Each term is positive.
- Use the patterns you observed in the Opening Exercise to determine the quotient of $\frac{x^5-243}{x-3}$. Explain your reasoning.
 - Since $243 = 3^5$, we should be able to apply the same pattern, and the quotient should be $x^4 + 3x^3 + 9x^2 + 27x + 81$.
- Test your conjecture by using long division or the reverse tabular method to compute the quotient.
 - The result is the same.

Exercise 1 (5 minutes)

Have students work in groups of two or three to complete these problems. Have them make and test conjectures about the quotient that results in each problem. Have the groups divide up the work so at least two students are working on each problem. Then have them share their results in their small groups.

Exercise 1

1. Use patterns to predict each quotient. Explain how you arrived at your prediction, and then test it by applying the reverse tabular method or long division.

a. $\frac{x^2 - 144}{x - 12}$

The quotient is x + 12. I arrived at this conclusion by noting that $144 = 12^2$, so I could apply the patterns in the previous problems to obtain the result.





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Once this exercise concludes and students have presented their work to the class, they should be ready to generalize a pattern for the quotient of $\frac{x^n - a^n}{x - a}$. The next example establishes this identity for the case where n = 2.

Example 1 (4 minutes)

The reverse tabular method can be used to compute quotients like the ones in the Opening Exercise and Exercise 1 for any constant a. In this way, it is possible to verify that the patterns noticed work for any value of a. In this example and Exercises 1 and 2, students work with specific values for the exponents. An interesting extension for advanced students would be to show that

$$\frac{x^n - a^n}{x - a} = x^{n-1} + ax^{n-2} + a^2 x^{n-3} + \dots + a^{n-2} x + a^{n-1} x^0$$

using the reverse tabular method.

Scaffolding:

Provide additional support here by considering specific values of *a* for each part. For example, ask students to work with

 $\frac{x^{3}-8}{x-2}$ $\frac{x^{3}-27}{x-3}$ $\frac{x^{3}-64}{x-4}$

and then ask them to solve the problem using the tabular method for the variables x and a.



Dividing by x - a and by x + a









Exercise 2 (7 minutes)



Before moving on, discuss these results as a whole class. It may be necessary to model a solution to the third question MP.7 below if students are still struggling with connecting division back to multiplication.

- What patterns do you notice in the quotient?
 - The terms are always added, and each term is a product of a power of x and a power of a. As the powers of x decrease by 1 for each consecutive term, the powers of a increase by 1.
- How do these patterns compare to the ones you observed in the opening exercises?
 - They support the patterns we discovered earlier. This work shows that we can quickly compute the quotient for any problem that fits the pattern.
- How can you rewrite these division problems as multiplication problems?
 - The dividend is equal to the product of the quotient and the divisor. For example, $x^3 - a^3 = (x - a)(x^2 + ax + a^2)$. The other problems would be $x^2 - a^2 = (x - a)(x + a)$ and $x^4 - a^4 = (x - a)(x^3 + ax^2 + a^2x + a^3)$.



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Exercise 3 (6 minutes)

The focus shifts to division by x + a. The expression x + a divides into the difference of two squares $x^2 - a^2$ without a remainder. Some students may be surprised that it does not divide without a remainder into the difference of two cubes. However, x + a does divide without a remainder into the sum of two cubes but not into the sum of two squares. Prior to this point, students have not worked with polynomial division problems that result in a remainder. While discussing these results as a class, draw a parallel to division of integers. Polynomial division with remainders is addressed in later lessons in this same module. At this point, lead students to conclude that some of these quotients produce identities that may be helpful for quickly dividing polynomials and some do not.

Ask students to think about how these problems would be different if dividing by x + a. Have students discuss their ideas with a partner before starting this exercise. Students will most likely assume that there are similar patterns for dividing the sums of squares and cubes, but they may be surprised by their results to parts (b) and (c). For groups that finish early, have them guess and check the results of dividing $x^4 + a^4$ and $x^4 - a^4$ by x + a. Provide additional concrete examples with numerical values of a such as a = 2, 3, 4, ... if needed to reinforce this concept.

The focus of this part of the lesson is to derive the three identities provided in the Lesson Summary and for students to realize that x + a and x - a do not divide into the sum of squares $x^2 + a^2$ without a remainder.



Students consider the special case when a = 1 for different values of n. They should be able to quickly generalize a

Exercise 4 (5 minutes)

pattern. In part (b), introduce the use of the ellipsis (...) to indicate the missing powers of x when displaying the general result since all of the terms cannot be written. In this exercise, students are asked to look for patterns in the quotient r^{n} -1

Lesson 6:

 $\frac{x^{n}-1}{x-1}$ for integer exponents *n* greater than 1.











Closing (5 minutes)

The summary details the identities derived in this lesson. Ask students to summarize the important results of this lesson either in writing, to a partner, or as a class. Take the opportunity to informally assess their understanding of this lesson before moving on to the Exit Ticket. Note that the last identity has not been formally derived; inductive reasoning was used to generalize the pattern based on the work done in Exercise 4.

Lesson Summary Based on the work in this lesson, it can be concluded that the following statements are true for all real values of x and a: $\begin{array}{c} x^2 - a^2 = (x-a)(x+a)\\ x^3 - a^3 = (x-a)(x^2+ax+a^2)\\ x^3 + a^3 = (x+a)(x^2-ax+a^2),\\ \end{array}$ and it seems that the following statement is also an identity for all real values of x and a: $\begin{array}{c} x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + x^{n-3} + \cdots + x^1 + 1), \text{ for integers } n > 1. \end{array}$

Exit Ticket (3 minutes)

In this Exit Ticket, students actually apply the identities they worked with to determine quotients. This Exit Ticket allows the teacher to test their fluency in working with these new relationships. The Lesson Summary is reinforced by having them rewrite each quotient as a product.







Name ____

Date _____

Lesson 6: Dividing by x - a and by x + a

Exit Ticket

Compute each quotient using the identities you discovered in this lesson.



 $\frac{x^3+1000}{x+10}$ 2.

3. $\frac{x^5-1}{x-1}$





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Exit Ticket Sample Solutions

Compute each quotient using the identities you discovered in this lesson.
1.
$$\frac{x^4 - 16}{x - 2}$$

 $\frac{x^4 - 16}{x - 2} = x^3 + 2x^2 + 4x + 8$
2. $\frac{x^3 + 1000}{x + 10}$
 $\frac{x^3 + 1000}{x + 10} = x^2 - 10x + 100$
3. $\frac{x^5 - 1}{x - 1}$
 $\frac{x^5 - 1}{x - 1} = x^4 + x^3 + x^2 + x + 1$

Problem Set Sample Solutions



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2. In the next exercises, you can use the same identities you applied in the previous problem. Fill in the blanks in the problems below to help you get started. Check your work by using the reverse tabular method or long division to make sure you are applying the identities correctly. $\frac{16x^2 - 121}{4x - 11} = \frac{()^2 - ()^2}{4x - 11} = () + 11$ a. 4x. 11. 4xb. $\frac{25x^2-49}{5x+7} = \frac{(-)^2-(-)^2}{5x+7} = (-) - (-) = -$ 5x, 7, 5x, 7, 5x - 7c. $\frac{8x^3-27}{2x-3} = \frac{(\)^3-(\)^3}{2x-3} = (\)^2 + (\)(\) + (\)^2 =$ _____ $2x, 3, 2x, 2x, 3, 3, 4x^2 + 6x + 9$ Show how the patterns and relationships learned in this lesson could be applied to solve the following arithmetic 3. problems by filling in the blanks. a. $\frac{625-81}{16} = \frac{()^2-(9)^2}{25-()} = () + () = 34$ $\frac{25^2 - 9^2}{25 - 9} = 25 + 9 = 34$ b. $\frac{1000-27}{7} = \frac{()^3-()^3}{()-3} = ()^2 + (10)() + ()^2 =$ $\frac{10^3 - 3^3}{10 - 3} = 10^2 + 10(3) + 3^2 = 139$ c. $\frac{100-9}{7} = \frac{(_)^2 - (_)^2}{(_)-3} = _$ $\frac{10^2 - 3^2}{10 - 3} = 10 + 3 = 13$ d. $\frac{1000+64}{14} = \frac{()^3+()^3}{()+()} = ()^2 - ()() + ()^2 =$ $\frac{10^3+4^3}{10+4}=10^2-10(4)+4^2=76$ 4. Apply the identities from this lesson to compute each quotient. Check your work using the reverse tabular method or long division. $16x^2 - 9$ а. 4x + 34x - 3 $\frac{81x^2-25}{18x-10}$ b. $\frac{9}{2}x + \frac{5}{2}$

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Lesson 6:





 $27x^3 - 8$ c. 3x-2 $9x^2 + 6x + 4$ Extend the patterns and relationships you learned in this lesson to compute the following quotients. Explain your 5. reasoning, and then check your answer by using long division or the tabular method. $8+x^{3}$ a. 2+xThe quotient is $4 - 2x + x^2$. This problem has the variable and constant terms reversed using the commutative property, so it is the same as computing $(x^3 + 8) \div (x + 2)$. $\frac{x^4 - y^4}{x - y}$ b. The quotient is $x^3 + x^2y + xy^2 + y^3$. This problem is similar to Opening Exercise part (c), except that instead of 81 and 3 in the dividend and quotient, we have a power of y. You can also extend the patterns for $\frac{x^3-a^3}{x-a} = x^2 + ax^2 + a^2$ using the variable y instead of the variable a. $\frac{27x^3+8y^3}{3x+2y}$ c. The quotient is $9x^2 - 6xy + 4y^2$. In this example, 3x is in the x position, and 2y is in the a position. Then, the divisor fits the pattern of $x^3 + a^3$. $\frac{x^7-y^7}{x-y}$ d. The quotient is $x^6 + x^5y + x^4y^2 + x^3y^3 + x^2y^4 + xy^5 + y^6$. In this problem, replace 1 with y and extend the powers of y pattern using the identities in the Lesson Summary.



Dividing by x - a and by x + a









Student Outcomes

Students perform arithmetic by using polynomial identities to describe numerical relationships.

Lesson Notes

Students continue exploring the usefulness of polynomial identities to perform arithmetic calculations. This work reinforces the essential understanding of standards **A-APR** and **A-SEE**. The lesson concludes by discussing prime and composite numbers and using polynomial identities to check whether a number is prime or composite. This lesson ties into the work in the next lesson, which further investigates prime numbers.

The tree diagram analysis, touched upon later in the lesson, offers a connection to some of the probability work from later in this course.

Classwork

Opening (1 minute)

Students perform arithmetic that they might not have thought possible without the assistance of a calculator or computer. To motivate this lesson, mention a multiplication problem of the form (a - b)(a + b) that is difficult to calculate without pencil and paper, such as $87 \cdot 93$. Perhaps even time students to see how long it takes them to do this calculation without a calculator.

 Today we use the polynomial identities derived in Lesson 6 to perform a variety of calculations quickly using mental arithmetic.

Opening Exercise (3 minutes)

Have students complete the following exercises. Ask students to discuss their ideas with a partner, and then have them summarize their thoughts on the lesson handout. These two exercises build upon the concept of division of polynomials developed in previous lessons by addressing both multiplication *and* division.

Opening Exercise a. How are these two equations related? \$\frac{x^2-1}{x+1} = x - 1\$ and \$x^2 - 1 = (x+1)(x-1)\$ They represent the same relationship between the expressions \$x^2 - 1\$, \$x - 1\$, and \$x + 1\$ as long as \$x ≠ -1\$. One shows the relationship as division and the other as multiplication. b. Explain the relationship between the polynomial identities \$x^2 - 1 = (x + 1)(x - 1\$)\$ and \$x^2 - a^2 = (x - a)(x + a)\$. The expression \$x^2 - 1\$ is of the form \$x^2 - a^2\$, with \$a = 1\$. Note that this works with \$a = -1\$ as well.







Discussion (8 minutes)

Call on a student to share his or her solutions to the Opening Exercise. Then invite other students to add their thoughts to the discussion. This discussion should show students how to apply the difference of two squares identity to quickly find the product of two numbers. Use the questions below to prompt a discussion.

- Consider $(x 1)(x + 1) = x^2 1$. If x = 100, what number sentence is represented by this identity? Which side of the equation is easier to compute?
 - This is $99 \cdot 101 = 100^2 1$.
 - Computing $100^2 1$ is far easier than the original multiplication.
- Now let's consider the more general $x^2 a^2 = (x a)(x + a)$. Keep x = 100, and test some small positive integer values for a. What multiplication problem does each one represent?
- How does the identity $(x a)(x + a) = x^2 a^2$ make these multiplication problems easier?
 - Let x = 100 and a = 5.

$$(x-a)(x+a) = x^2 - a^2$$

95 \cdot 105 = (100 - 5)(100 + 5) = 100^2 - 5^2

Therefore, $95 \cdot 105 = 100^2 - 5^2 = 10000 - 25 = 9975$.

Let x = 100 and a = 7.

$$(x-a)(x+a) = x^2 - a^2$$

93 · 107 = (100 - 7)(100 + 7) = 100² - 7²

Therefore, $93 \cdot 107 = 100^2 - 7^2 = 10000 - 49 = 9951$.

- Do you notice any patterns?
 - The products in these examples are differences of squares.
 - The factors in the product are exactly 'a' above and 'a' below 100.
- How could you use the difference of two squares identity to multiply $92 \cdot 108$? How did you determine the values of x and a?
 - You could let x = 100 and a = 8. We must figure out each number's distance from 100 on the number line.
- How would you use the difference of two squares identity to multiply $87 \cdot 93$? What values should you select for x and a? How did you determine them?
 - We cannot use 100, but these two numbers are 3 above and 3 below 90. So we can use

$$(90-3)(90+3) = 90^2 - 3^2 = 8100 - 9 = 8091.$$

• In general, *x* is the mean of the factors, and *a* is half of the absolute value of the difference between the factors.

Depending on the level of students, it may be appropriate to wait until after Exercise 1 to make a generalized statement about how to determine the x and a values used to solve these problems. They may need to experiment with some additional problems before they are ready to generalize a pattern.









Exercise 1 (4 minutes)

Have students work individually and then check their answers with a partner. Make sure they write out their steps as in the sample solutions. After a few minutes, invite students to share one or two solutions on the board.

Exercise 1			
1.	Compute the following products using the identity $x^2 - a^2 = (x - a)(x + a)$. Show your steps.		ity $x^2 - a^2 = (x - a)(x + a)$. Show your steps.
	a.	6 · 8	$6 \cdot 8 = (7 - 1)(7 + 1)$
			$=7^2-1^2$
			= 49 - 1
			= 48
	h	11.10	$11 \cdot 19 = (15 - 4)(15 + 4)$
	υ.	11 • 17	$=15^2-4^2$
			= 225 - 16
			= 209
		00.47	$23 \cdot 17 = (20 + 3)(20 - 3)$
	с.	23 · 17	$= 20^2 - 3^2$
			= 400 - 9
			= 391
			$34 \cdot 26 - (30 \pm 4)(30 - 4)$
	d.	34 · 26	$34^{\circ}20 = (30 \pm 4)(30 \pm 4)$
			$= 30^{-} - 4^{-}$
			= 900 - 16
			= 884

Discussion (5 minutes)

At this point, make sure students have a clear way to determine how to write a product as the difference of two squares. Then put these problems on the board.

 $56 \cdot 63$ $24 \cdot 76$ $998 \cdot 1002$

Give them a few minutes to struggle with these problems. While it is possible to use the identity to rewrite each expression, the first two problems do not make for an easy calculation when written as the difference of two squares. The third problem is easy even though the numbers are large.

- Which product is easier to compute using mental math? Explain your reasoning.
 - ^a The last one is the easiest. In the first one, the numbers have a mean of 59.5, which is not easy to square mentally. The second example would be $50^2 26^2$, which is not so easy to calculate mentally.
- Can the product of any two positive integers be written as the difference of two squares?
 - Yes, but not all of them will be rewritten in a form that makes computation easy.
- If you wanted to impress your friends with your mental math abilities, and they gave you these three problems to choose from, which one would you pick and why?

509 · 493 511 · 489 545 · 455

^D This middle one is the easiest since the numbers are 11 above and below the number 500.



MP.3







Discussion (10 minutes)

At this point, it is possible to introduce the power of algebra over the calculator.

• The identity $x^2 - a^2$ is just the n = 2 case of the identity

$$x^{n} - a^{n} = (x - a)(x^{n-1} + ax^{n-2} + a^{2}x^{n-2} + \dots + a^{n-1}).$$

How might we use this general identity to quickly count mentally?
 To see how, let's doodle. A *tree* is a figure made of points called *vertices* and segments called *branches*. My tree splits into two branches at each vertex.



How many vertices does my tree have?

Allow students to count the vertices for a short while, but don't dwell on the answer.

• It is difficult to count the vertices of this tree, so let's draw it in a more organized way.

Present the following drawing of the tree, with vertices aligned in rows corresponding to their levels.





Lesson 7: Mental Math



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For the following question, give students time to write or share their thinking with a neighbor.

- How many vertices are in each level? Find a formula to describe the number of vertices in level n.
 - The number of vertices in each level follows this sequence: $\{1, 2, 4, 8, 16, ...\}$, so in level *n* there are 2^{n-1} vertices.
- How many vertices are there in all 5 levels? Explain how you know.
 - The number of vertices in our tree, which has five levels, is $2^4 + 2^3 + 2^2 + 2 + 1$. First, we recognize that 2 - 1 = 1, so we can rewrite our expression as $(2 - 1)(2^4 + 2^3 + 2^2 + 2 + 1)$. If we let x = 2, this numerical expression becomes a polynomial expression.

$$2^{4} + 2^{3} + 2^{2} + 2 + 1 = (2 - 1)(2^{4} + 2^{3} + 2^{2} + 2 + 1)$$

= $(x - 1)(x^{4} + x^{3} + x^{2} + x + 1)$
= $x^{5} - 1$
= $2^{5} - 1$
= 31

- How could you find the total number of vertices in a tree like this one with n levels? Explain.
 - Repeating what we did with n = 5 in the previous step, we have

$$2^{n-1} + 2^{n-2} + \dots + 2 + 1 = (2 - 1)(2^{n-1} + 2^{n-2} + \dots + 2 + 1)$$
$$= (x - 1)(x^{n-1} + x^{n-2} + \dots + x + 1)$$
$$= x^n - 1$$
$$= 2^n - 1.$$

Thus, a tree like this one with *n* levels has $2^n - 1$ vertices.

Now, suppose I drew a tree with 30 levels:



- How many vertices would a tree with 30 levels have?
 - According to the formula we developed in the last step, the number of vertices is
 - $2^n 1 = 2^{30} 1$.

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MP.3

Lesson 7: Mental Math



- Would you prefer to count all 1,073,741,823 vertices?
 - □ No

Discussion (5 minutes)

This discussion is designed to setup the general identity for $x^n - a^n$ to identify some composite numbers in the next lesson.

- Recall that a *prime number* is a positive integer greater than 1 whose only
 positive integer factors are 1 and itself. A *composite number* can be written as
 the product of positive integers with at least one factor that is not 1 or itself.
- Suppose that *a*, *b*, and *n* are positive integers with b > a. What does the identity $x^n a^n = (x a)(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \dots + a^{n-1})$ suggest about whether or not the number $b^n a^n$ is prime?
 - We see that $b^n a^n$ is divisible by b a and that $b^n - a^n = (b - a)(b^{n-1} + ab^{n-2} + \dots + a^{n-1}).$
 - If b a = 1, then we do not know if $b^n a^n$ is prime because we do not know if $b^{n-1} + ab^{n-2} + \dots + a^{n-1}$ is prime. For example, $15^2 14^2 = 225 196 = 29$ is prime, but $17^2 16^2 = 289 256 = 33$ is composite.
 - But, if b a > 1, then we know that $b^n a^n$ is not prime.
- Use the identity $b^n a^n = (b a)(b^{n-1} + ab^{n-2} + a^2b^{n-3} + \dots + a^{n-1})$ to determine whether or not 143 is prime. Check your work using a calculator.
 - Let b = 12, a = 1, and n = 2. Since b a is a factor of $b^n a^n$, and b a = 12 1 = 11, we know that 11 is a factor of 143, which means that 143 is not prime. The calculator shows $143 = 11 \cdot 13$.
- We could have used a calculator to determine that $11 \cdot 13 = 143$, so that 143 is not prime. Will a calculator help us determine whether $2^{100} 1$ is prime? Try it.
 - The calculator will have difficulty calculating a number this large.
- Can we determine whether or not $2^{100} 1$ is prime using identities from this lesson?
 - We can try to apply the following identity.

$$x^{n} - 1 = (x - 1)(x^{n-1} + x^{n-2} + \dots + 1)$$

- If we let n = 100, then this identity does not help us because 1 divides both composites and primes. $2^{100} - 1 = (2 - 1)(2^{99} + 2^{98} + \dots 2 + 1)$
- But, what if we look at this problem a bit differently?

 $2^{100} - 1 = (2^4)^{25} - 1 = 16^{25} - 1 = (16 - 1)(16^{24} + 16^{23} + \dots + 16 + 1)$

- We can see now that $2^{100} 1$ is divisible by 15, so $2^{100} 1$ is not prime.
- What can we conclude from this discussion?
 - If we can write a positive integer as the difference of squares of nonconsecutive integers, then that integer is composite.







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If students are struggling with the words *prime* and *composite*, try doing a quick T-chart activity in which students classify numbers as prime or composite on either side of the T.

Scaffolding:

Lesson 7

ALGEBRA II



Exercises 2–3 (4 minutes)



Closing (2 minutes)

Ask students to write a mental math problem that they can now do easily and to explain why the calculation can be done simply.

Ask students to summarize the important parts of the lesson, either in writing, to a partner, or as a class. Use this opportunity to informally assess their understanding of the lesson. The following are some important summary elements:

Lesson Summary Based on the work in this lesson, students can convert differences of squares into products (and vice versa) using $x^2 - a^2 = (x - a)(x + a).$ If x, a, and n are integers with $(x - a) \neq \pm 1$ and n > 1, then numbers of the form $x^n - a^n$ are not prime because $x^{n} - a^{n} = (x - a)(x^{n-1} + ax^{n-2} + a^{2}x^{n-3} + \dots + a^{n-2}x + a^{n-1}).$

Exit Ticket (3 minutes)







Name_____

Date _____

Lesson 7: Mental Math

Exit Ticket

1. Explain how you could use the patterns in this lesson to quickly compute (57)(43).

2. Jessica believes that $10^3 - 1$ is divisible by 9. Support or refute her claim using your work in this lesson.





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Exit Ticket Sample Solutions



Problem Set Sample Solutions

1.	Usin state	sing an appropriate polynomial identity, quickly compute the following products. Show each step. Be sure to tate your values for x and a .	
	a.	41 · 19	$a = 11, x = 30$ $(x + a)(x - a) = x^{2} - a^{2}$ $(30 + 11)(30 - 11) = 30^{2} - 11^{2}$ $= 900 - 121$ $= 779$
	b.	993 · 1,007	$a = 7, x = 1000$ $(x - a)(x + a) = x^{2} - a^{2}$ $(1000 - 7)(1000 + 7) = 1000^{2} - 7^{2}$ $= 10000000 - 49$ $= 999951$
	c.	213 · 187	$a = 13, x = 200$ $(x - a)(x + a) = x^{2} - a^{2}$ $(200 - 13)(200 + 13) = 200^{2} - 13^{2}$ $= 40000 - 169$ $= 39831$
	d.	29 · 51	$a = 11, x = 40$ $(x - a)(x + a) = x^{2} - a^{2}$ $(40 - 11)(40 + 11) = 40^{2} - 11^{2}$ $= 1600 - 121$ $= 1479$
	e.	125 · 75	a = 25, x = 100 (x - a)(x + a) = x ² - a ² (100 - 25)(100 + 25) = 100 ² - 25 ² = 10000 - 625 = 9375





NYS COMMON CORE MATHEMATICS CURRICULUM



ALGEBRA II

2. Give the general steps you take to determine x and a when asked to compute a product such as those in Problem 1. The number x is the mean (average is also acceptable) of the two factors, and a is the positive difference between xand either factor. 3. Why is $17 \cdot 23$ easier to compute than $17 \cdot 22$? The mean of 17 and 22 is 19.5, whereas the mean of 17 and 23 is the integer 20. I know that the square of 20 is 400 and the square of 3 is 9. However, I cannot quickly compute the squares of 19.5 and 2.5.4. Rewrite the following differences of squares as a product of two integers. $81 - 1 = 9^2 - 1^2$ 81 - 1a. = (9-1)(9+1) $= 8 \cdot 10$ $\mathbf{400} - \mathbf{121} = \mathbf{20}^2 - \mathbf{11}^2$ 400 - 121b. =(20-11)(20+11) $= 9 \cdot 31$ Quickly compute the following differences of squares. 5. $64^2 - 14^2 = (64 - 14)(64 + 14)$ $64^2 - 14^2$ а. $= 50 \cdot 78$ = 3900 $112^2 - 88^2 = (112 - 88)(112 + 88)$ $112^2 - 88^2$ b. $= 24 \cdot 200$ = 4800 $785^2 - 215^2 = (785 - 215)(785 + 215)$ $785^2 - 215^2$ c. $= 570 \cdot 1000$ = 570 000 6. Is 323 prime? Use the fact that $18^2 = 324$ and an identity to support your answer. *No*, 323 is not prime because it is equal to $18^2 - 1$. Therefore, 323 = (18 - 1)(18 + 1). Note: This problem can also be solved through factoring. 7. The number $2^3 - 1$ is prime and so are $2^5 - 1$ and $2^7 - 1$. Does that mean $2^9 - 1$ is prime? Explain why or why not. $2^9 - 1 = (2^3)^3 - 1$ $=(2^{3}-1)((2^{3})^{2}+2^{3}(1)+1^{2})$ The factors are 7 and 73. As such, $2^9 - 1$ is not prime. 8. Show that 9, 999, 999, 991 is not prime without using a calculator or computer. Note that $9\,999\,999\,991 = 10\,000\,000\,000 - 9$. Since 10^{10} is the square of 10^5 , 10,000,000,000 is the square of 100 000. Since 9 is the square of 3, 9 999 999 $991 = 100 000^2 - 3^2$, which is divisible by 100 000 - 3and by $100\,000 + 3$.



Lesson 7: Mental Math



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Lesson 8: The Power of Algebra—Finding Primes

Student Outcomes

• Students apply polynomial identities to the detection of prime numbers.

Lesson Notes

This lesson applies the identities students have been working with in previous lessons to finding prime numbers, a rich topic with a strong historical background. Many famous mathematicians have puzzled over prime numbers, and their work is the foundation for mathematics used today in the RSA encryption algorithm that provides for secure Internet transmissions. This is an engaging topic for students and is readily accessible to them because of its current use in providing safe and secure electronic communications and transactions. Students will be actively engaging several mathematical practice standards during this lesson, including making sense of problems (MP.1), looking for patterns, and seeing the structure in expressions (MP.7 and MP.8), as they investigate patterns with prime numbers. The lesson includes many opportunities to prove conjectures (MP.3) as students gain experience using algebraic properties to prove statements about integers. Several excellent resources are available for students wishing to learn more about prime numbers, their history, and their application to encryption and decryption. A good starting place for additional exploration about prime numbers is the website The Prime Pages, http://primes.utm.edu/.

Classwork

Opening (10 minutes)

To motivate students, show the YouTube video on RSA encryption (<u>http://www.youtube.com/watch?v=M7kEpw1tn50</u>) to the class. This video introduces students to encryption and huge numbers. Encryption algorithms are the basis of all secure Internet transactions. Today, many encryption algorithms rely on very large prime numbers or very large composite numbers that are the product of two primes to create an encryption key. Often these numbers are Mersenne Primes—primes of the form $2^p - 1$, where p is itself prime. Interestingly, not all numbers in this form are prime. As of December 2013, only 43 Mersenne Primes have been discovered. Encourage students to research the following terms: Mersenne Primes, Data Encryption, and RSA. The Opening Exercise along with the first examples engage students in the exploration of large primes.

Mathematicians have tried for centuries to find a formula that always yields a prime number but have been unsuccessful in their quest. The search for large prime numbers and a formula that will generate all the prime numbers has provided fertile ground for work in number theory. The mathematician Pierre de Fermat (1601–1665, France) applied the difference of two squares identity to factor very large integers as the product of two prime numbers. Up to this point, students have worked with numbers that can be expressed as the difference of two perfect squares. If a prime number could be written as a difference of perfect squares $a^2 - b^2$, then it would have to be of the form (a + b)(a - b), where a and b are consecutive whole numbers and a + b is prime. The challenge is that not every pair of consecutive whole numbers yields a prime number 3 + 4 = 7 is prime, but 4 + 5 = 9 is not. This idea is further addressed in the last exercise and in the Problem Set.



The Power of Algebra—Finding Primes







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Opening Exercise (10 minutes): When is $2^n - 1$ prime and when is it composite?

Before beginning this exercise, have students predict when an expression in this form will be prime and when it will be composite. Questions like this are ideal places to engage students in constructing viable arguments.

- When will this expression be prime and when will it be composite?
 - ^a Student responses will vary. Some may say always prime or always composite. A response that goes back to the identity $x^n 1 = (x 1)(x^{n-1} + x^{n-2} + \dots + 1)$ from the previous lesson is showing some good initial thinking.

Students should work the Opening Exercise in small groups. After about seven minutes of group work, have one student from each group come up and fill in the table values and the supporting work. Then, lead a whole group discussion to debrief this problem.

Scaffolding:

For more advanced students, consider posing the question: Can you construct an expression that always yields a prime number? Do this before starting the Opening Exercise, and then ask them to test their expressions.

If students are having a hard time constructing an expression, consider asking the following questions: Can you construct an expression that will always yield an even number? Can you construct an expression that will always yield an odd number?

If n is an integer, then 2n is always an even number, and 2n + 1 is always an odd number.

Opening Exercise: When is $2^n - 1$ prime and when is it composite?

xponent	Expression	Value	Prime or Composite?
n	$2^{n} - 1$		Justify your answer if composite.
1	$2^{1} - 1$	1	Prime
2	$2^2 - 1$	3	Prime
3	$2^3 - 1$	7	Prime
4	$2^4 - 1$	15	Composite $(3 \cdot 5)$
5	$2^{5}-1$	31	Prime
6	2 ⁶ – 1	63	Composite (7 · 9)
7	$2^{7} - 1$	127	Prime
8	$2^8 - 1$	255	Composite $(5 \cdot 51)$
9	$2^9 - 1$	511	Composite (7 · 73)
10	$2^{10} - 1$	1023	Composite $(32 - 1)(32 + 1)$
11	$2^{11} - 1$	2047	Composite (23 · 89)

Answers will vary. Suggested responses are in the discussion questions.

Encourage students to use tools strategically as they work with these problems. They should have a calculator available to determine if the larger numbers are composite. When debriefing, point out the fact that students can use the difference of two squares identity to factor these expressions when the exponent is an even number.

Use these questions to lead a short discussion on the results of this Opening Exercise.



The Power of Algebra—Finding Primes



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MP.2 MP.7 & MP.8

MP.3



- What patterns do you notice about which expressions are composite and which are prime?
 - When the exponent is an even number greater than 2, the result is composite and can be factored using this identity: $2^{2n} 1 = (2^n + 1)(2^n 1)$.
 - ^D When the exponent is a prime number, the result is sometimes prime and sometimes not prime. $2^{11} - 1$ was the first number with a prime exponent that was composite.
 - When the exponent is a composite odd number, the expression appears to be composite, but we have yet to prove that.

The statements above are examples of the types of patterns students should notice as they complete the Opening Exercise. If the class was not able to prove that the case for even exponents resulted in a composite number, encourage them to consider the identities learned in the last lesson involving the difference of two squares. See if they can solve the problem with that hint. Of course, that technique does not work when the exponent is odd. Make sure students have articulated the answer to the last problem. To transition to the next section, ask students how they might prove that $2^{ab} - 1$ is composite when the exponent *ab* is an odd composite number.

Example 1 (5 minutes): Proving a Conjecture

This example and the next exercise prove patterns students noticed in the table in the Opening Exercise. Some scaffolding is provided, but feel free to adjust as needed for students. Give students who need less support the conjecture on the board for Example 1 (without the additional scaffolding on the student pages); others may need more assistance to get started.

Example 1: Proving a Conjecture If *m* is a positive odd composite number, then $2^m - 1$ is a composite number. Conjecture: $x^{n} - 1 = (x - 1)(x^{n-1} + x^{n-2} + \dots + x^{n-1} + 1)$ Start with an identity: In this case, x = 2, so the identity above becomes: $2^{m} - 1 = (2 - 1)(2^{m-1} + 2^{m-2} + \dots + 2^{1} + 1)$ $= (2^{m-1} + 2^{m-2} + \dots + 2^1 + 1),$ and it is not clear whether or not $2^m - 1$ is composite. Rewrite the expression: Let m = ab be a positive odd composite number. Then a and b must also be odd, or else the product ab would be even. The smallest such number m is 9, so we have $a \ge 3$ and b > 3.Then we have $2^m - 1 = (2^a)^b - 1$ $=(2^a-1)\underbrace{((2^a)^{b-1}+(2^a)^{b-2}+\dots+(2^a)^1+1)}_{\text{Some number larger than }1}.$ Since $a \ge 3$, we have $2^a \ge 8$; thus, $2^a - 1 \ge 7$. Since the other factor is also larger than 1, $2^m - 1$ is composite, and we have proven our conjecture.



The Power of Algebra—Finding Primes







Exercises 1–3 (4 minutes)

In these exercises, students confirm the conjecture proven in Example 1. Emphasize that it does not really matter what the 2nd factor is once the first one is known. There is more than one way to solve each of these problems depending on how students decide to factor the exponent on the 2. Students should work in small groups on these exercises. Encourage them to use a calculator to determine the prime factors of 537 in Exercise 3. Have different groups present their results.

Discussion (4 minutes)

Cryptography is the science of making codes, and *cryptanalysis* is the science of breaking codes. The rise of Internet commerce has created a demand for encoding methods that are hard for unintended observers to decipher. One encryption method, known as RSA encryption, uses very large numbers with hundreds of digits that are the product of two primes; the product of the prime factors is called the *key*. The key itself is made public so anyone can encode using this system, but in order to break the code, you would have to know how to factor the key, and that is what is so difficult.

- You had a hint in Exercise 3 that made it easier for you to factor a very large number, but what if you do not have any hints?
 - It would be almost impossible to factor the number because you would have to check all the prime numbers up to the square root of the exponent to find the factors.

If you know the key, then decoding is not particularly difficult. Programmers select a number that is almost impossible to factor without significant time and computing power and use this as the key to encode data and communications. The last exercise illuminates the logic behind modern encryption algorithms.



The Power of Algebra—Finding Primes







Exercise 4 (6 minutes): How quickly can a computer factor a very large number?

To determine if a number *m* is prime, it is possible to just try dividing by every prime number *p* that is less than \sqrt{m} . This takes the longest time if *m* happens to be a perfect square. In this exercise, students consider the speed it takes a certain computer algorithm to factor a square of a large prime number; this is the case where it should take the algorithm the longest to find the factorization. Actual algorithms used to factor large numbers are quicker than this, but it still takes a really long time and works to the advantage of people who want to encrypt information electronically using these large numbers. Introduce the problem and review the table as a whole class; then, have students answer the question in small groups. Have students working in groups apply the given function to estimate the time it would take to factor a 32-digit number. Make sure they convert their answer to years. Because we are using an exponential function, the factorization time grows very rapidly. Take the time after Exercise 4 to remind students that exponential functions increase very rapidly over intervals of equal length, ideas that were introduced in Algebra I and will be revisited in Module 3 of Algebra II.

	The time in seconds required to factor an <i>n</i> -digit number of the form p^2 , where <i>p</i> is a large prime, can roughly be approximated by $f(n) = 3.4 \times 10^{(n-13)/2}$. Some values of this function are listed in the table below.					
	p	<i>p</i> ²	Number of Digits	Time needed to factor the number (sec)		
	10,007	100, 140, 049	9	0.034		
	100,003	10,000,600,009	11	0.34		
	1,000,003	1,000,006,000,009	13	3.4		
	10,000,019	100, 000, 380, 000, 361	15	34		
	100,000,007	10,000,001,400,000,049	17	340		
	1000.000.007	1,000,000,014,000,000,049	19	3.400		

After allowing groups to take a few minutes to evaluate the function and convert their answer to years, connect this exercise to the context of this situation by summarizing the following points.

- Using a very fast personal computer with a straightforward algorithm, it would take about 342 years to factor a 32-digit number, making any secret message encoded with that number obsolete before it could be cracked with that computer.
- However, we have extremely fast computers (much faster than one personal computer) and very efficient algorithms designed for those computers for factoring numbers. These computers can factor a number thousands of times faster than the computer used above, but they are still not fast enough to factor huge composite numbers in a reasonable amount of time.



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 In 2009, computer scientists were able to factor a 232-digit number in two years by distributing the work over hundreds of fast computers running at the same time. That means any message encoded using that 232-digit number would take two years to decipher, by which time the message would no longer be relevant. Numbers used to encode secret messages typically contain over 300 digits, and extremely important secret messages use numbers that have over 600 digits—a far bigger number than any bank of computers can currently factor in a reasonable amount of time.

Closing (2 minutes)

There are better ways of factoring numbers than just checking all of the factors, but even advanced methods take a long time to execute. Products of primes of the magnitude of 2^{2048} are almost impossible to factor in a reasonable amount of time, which is how mathematics is used to guarantee the security of electronic transactions. Give students a few minutes to summarize what they have learned in writing or by discussing it with a partner before starting the Exit Ticket.

- Polynomial identities can help us prove conjectures about numbers and make calculations easier.
- The field of number theory has contributed greatly to the fields of cryptography and cryptanalysis (code-making and code-breaking).

Exit Ticket (4 minutes)





Lesson 8

ALGEBRA II





Name

Date _____

Lesson 8: The Power of Algebra—Finding Primes

Exit Ticket

Express the prime number 31 in the form $2^p - 1$ where p is a prime number and as a difference of two perfect squares using the identity $(a + b)(a - b) = a^2 - b^2$.



The Power of Algebra—Finding Primes







Exit Ticket Sample Solutions

Express the prime number 31 in the form $2^p - 1$ where p is a prime number and as a difference of two perfect squares using the identity $(a + b)(a - b) = a^2 - b^2$. $31 = (16 - 15)(16 + 15) \qquad \qquad 31 = 2^5 - 1$ $= 16^2 - 15^2$

Problem Set Sample Solutions

Factor $4^{12} - 1$ in two different ways using the identity $x^n - a^n = (x - a)(x^n + ax^{n-1} + a^2x^{n-2} + \dots + a^n)$ and 1. the difference of squares identity. $(4^6 - 1)(4^6 + 1)$ $(4-1)(4^{11}+4^{10}+\ldots+4+1)$ 2. Factor $2^{12} + 1$ using the identity $x^n + a^n = (x + a)(x^n - ax^{n-1} + a^2x^{n-2} - \dots + a^n)$ for odd numbers n. $(2^4)^3 + 1 = (2^4 + 1)((2^4)^2 - 2^4 + 1)$ 3. Is 10,000,000,001 prime? Explain your reasoning. No, because it is of the form $10^{10} + 1$, which could be written as $(10^2)^5 + 1 = (10^2 + 1)((10^2)^4 - \dots + 1)$. 4. Explain why $2^n - 1$ is never prime if *n* is a composite number. If n is composite, then it can be written in the form n = ab, where a and b are integers larger than 1. Then $2^n - 1 = 2^{ab} - 1 = (2^a)^b - 1 = (2^a - 1)((2^a)^{b-1} + \dots + 2^a + 1)$. For a > 1, this number will be composite because $2^a - 1$ will be larger than 1. Fermat numbers are of the form $2^n + 1$ where n is a positive integer. 5. Create a table of Fermat numbers for odd values of *n* up to 9. а. $2^{n} + 1$ n $2^1 + 1 = 3$ 1 $2^3 + 1 = 9$ 3 $2^5 + 1 = 33$ 5 7 $2^7 + 1 = 129$ 9 $2^9 + 1 = 513$ Explain why if n is odd, the Fermat number $2^n + 1$ will always be divisible by 3. b. The Fermat number $2^n + 1$ will factor as $(2 + 1)(2^{n-1} - 2^{n-2} + \dots + 1)$ using the identity in Exercise 2.



The Power of Algebra—Finding Primes





ALGEBRA II

				ALC
c.	Complete th	e table of values for even values of <i>n</i>	up to 12.	
		<i>n</i>	3 <i>n</i>	+ 1
		2	2^{2} +	$\frac{+1}{1-5}$
		4	$2^4 + 2^4$	1 = 3 1 = 17
		6	2 ⁶ + 1	1 = 65
		8	2 ⁸ + 1	= 257
		10	$2^{10} + 1$	= 1,025
		12	$2^{12} + 1$	= 4,097
d.	Show that if Let $n = 2k$,	<i>n</i> can be written in the form $2k$ whe where <i>k</i> is odd. Then $2^n + 1 = 2^{2k}$	ere k is odd, then $+1=(2^2)^k+1$	n $2^n + 1$ is divisible by 5. L = $(2^2 + 1) ((2^2)^{k-1} + \dots + 2^2 + 1).$
				number larger than 1
	Since $2^2 + 1$	= 5, we know that 5 is a factor of 2	$n^n + 1$. This only	holds when k is an odd number because
	that is the o	nly case when we can factor expression	ons of the form :	$x^{\kappa}+1.$
e.	Which even that might b The powers when the ex	numbers are not divisible by an odd in e prime. of 2 are the only positive integers the ponent n in $2^n + 1$ is a power of 2, the ponent n in 2^n	number? Make at are not divisib he Fermat numb	a conjecture about the only Fermat numbers the by any odd numbers. This implies that her $2^n + 1$ might be prime.
Com	plete this table	e to explore which numbers can be explore	xpressed as the o	difference of two perfect squares.
	Number	Difference of Two Squares	Number	Difference of Two Squares
	1	$1^2 - 0^2 = 1 - 0 = 1$	11	$6^2 - 5^2 = 36 - 25 = 11$
	2	Not possible	12	$4^2 - 2^2 = 16 - 4 = 12$
	3	$2^2 - 1^2 = 4 - 1 = 3$	13	$7^2 - 6^2 = 49 - 36 = 13$
	4	$2^2 - 0^2 = 4 - 0 = 4$	14	Not possible
	5	$3^2 - 2^2 = 9 - 4 = 5$	15	$8^2 - 7^2 = 64 - 49 = 15$
	6	Not possible	16	$5^2 - 3^2 = 25 - 9 = 16$
	7	$4^2 - 3^2 = 16 - 9 = 7$	17	$9^2 - 8^2 = 81 - 64 = 17$
	8	$3^2 - 1^2 = 9 - 1 = 8$	18	Not nossible
	0	$F^2 = A^2 - 2F = 16 - 9$	10	$10^2 9^2 - 100 91 - 10$
	10	3 - 4 = 23 - 10 = 9	20	$10^{-9} - 100^{-81} - 10^{-81}$
	10	Not possible	20	0 4 - 30 10 - 20
a.	For which odd numbers does it appear to be possible to write the number as the difference of two squares? It appears that we can write any positive odd number as the difference of two squares.			
b.	For which ev	en numbers does it appear to be pos	sible to write th	e number as the difference of two squares?
	It appears th	nat we can write any multiple of 4 as	the difference o	f two squares.
c.	Suppose tha do you notic	t n is an odd number that can be exp e about a and b ?	pressed as $n=a$	$a^2 - b^2$ for positive integers a and b . What
	When n is o	dd, a and b are consecutive whole nu	imbers and $a + b$	b = n.
d.	Suppose tha do you notic	t n is an even number that can be ex se about a and b ?	pressed as $n=$	$a^2 - b^2$ for positive integers a and b . What
	When n is an even number that can be written as a difference of squares, then n is a multiple of 4, and a and b are either consecutive even integers or consecutive odd integers. We also have $a + b = \frac{n}{2}$.			



6.

The Power of Algebra—Finding Primes





Lesson 8 M1

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7. Express the numbers from 21 to 30 as the difference of two squares, if possible. This is not possible for 22, 26, and 30. Otherwise we have the following. $27 = 14^2 - 13^2$ $21 = 11^2 - 10^2$ $23 = 12^2 - 11^2$ $28 = 8^2 - 6^2$ $24 = 7^2 - 5^2$ $29 = 15^2 - 14^2$ $25 = 13^2 - 12^2$ Prove this conjecture: Every positive odd number m can be expressed as the difference of the squares of two 8. consecutive numbers that sum to the original number *m*. Let m be a positive odd number. Then for some integer n, m = 2n + 1. We will look at the consecutive a. integers *n* and n + 1. Show that n + (n + 1) = m. n + (n + 1) = n + n + 1= 2n + 1= mWhat is the difference of squares of n + 1 and n? b. $(n+1)^2 - n^2 = n^2 + 2n + 1 - n^2$ = 2n + 1= mWhat can you conclude from parts (a) and (b)? c. We can write any positive odd number m as the difference of squares of two consecutive numbers that sum to m. 9. Prove this conjecture: Every positive multiple of 4 can be expressed as the difference of squares of two numbers that differ by 2. Use the table below to organize your work for parts (a)-(c). Write each multiple of 4 in the table as a difference of squares. a. Difference of squares 4nb n а $a^2 - b^2$ $2^2 - 0^2$ 1 4 2 0 $3^2 - 1^2$ 2 8 3 1 3 12 **4**² -- 2² 4 2 4 16 **5**² -- 3² 5 3 5 20 $6^2 - 4^2$ 4 6 n **4***n* $)^{2} - ($ n + 1n-1What do you notice about the numbers *a* and *b* that are squared? How do they relate to the number *n*? b. The values of a and b in the differences of two squares differ by 2 every time. They are one larger and one smaller than n; that is, a = n + 1 and b = n - 1.



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Given a positive integer of the form 4n, prove that there are integers a and b so that $4n = a^2 - b^2$ and that c. a - b = 2. (Hint: Refer to parts (a) and (b) for the relationship between n and a and b.) Define a = n + 1 and b = n - 1. Then we can calculate $a^2 - b^2$ as follows. $a^2 - b^2 = (n+1)^2 - (n-1)^2$ $= \big((n+1) + (n-1)\big)\big((n+1) - (n-1)\big)$ = (2n)(2)= 4nWe can also see that a - b = (n + 1) - (n - 1)= n + 1 - n + 1= 2. Thus every positive multiple of 4 can be written as a difference of squares of two integers that differ by 2. 10. The steps below prove that the only positive even numbers that can be written as a difference of square integers are the multiples of 4. That is, completing this exercise will prove that it is impossible to write a number of the form 4n-2 as a difference of square integers. Let *m* be a positive even integer that we can write as the difference of square integers $m = a^2 - b^2$. Then m = (a + b)(a - b) for integers a and b. How do we know that either a and b are both even or a and b are both odd? If one of a and b is even and the other one is odd, then one of a^2 and b^2 is even and the other one is odd. Since the difference of an odd and an even number is odd, this means that $m = a^2 - b^2$ would be odd. Since we know that m is even, it must be that either a and b are both even or a and b are both odd. Is a + b even or odd? What about a - b? How do you know? b. Since a and b are either both odd or both even, we know that both a + b and a - b are even. Is 2 a factor of a + b? Is 2 a factor of a - b? Is 4 a factor of (a + b)(a - b)? Explain how you know. c. Because a + b and a - b are both even, 2 is a factor of both a + b and a - b. Thus, $2^2 = 4$ is a factor of (a+b)(a-b). d. Is 4 a factor of any integer of the form 4n - 2? No. If 4 were a factor of 4n - 2, we could factor it out: $4n - 2 = 4\left(n - \frac{1}{2}\right)$. But this means that $n - \frac{1}{2}$ is an integer, which it clearly is not. This means that 4 is not a factor of any number of the form 4n - 2. What can you conclude from your work in parts (a)-(d)? e. If m is a positive even integer and m can be written as the difference of two square integers, then m cannot be of the form 4n-2 for any integer n. Another way to say this is that the positive integers of the form



The Power of Algebra—Finding Primes

4n-2 for some integer n cannot be written as the difference of two square integers.







11. Explain why the prime number 17 can only be expressed as the difference of two squares in only one way, but the composite number 24 can be expressed as the difference of two squares in more than one way.

Since every odd number can be expressed as the difference of two squares, $a^2 - b^2 = (a + b)(a - b)$, the number 17 must fit this pattern. Because 17 is prime, there is only one way to factor 17, which is $17 = 1 \cdot 17$.

Let a + b = 17 and a - b = 1. The two numbers that satisfy this system of equations are 8 and 9. Thus,

$$17 = 1 \cdot 17$$

= (9 - 8)(9 + 8)
= 9² - 8².

A composite number has more than one factorization, not all of which will lead to writing the number as the difference of squares of two integers. For the number 24, you could use

$$24 = 2 \cdot 12$$

= (7 - 5)(7 + 5)
= 7² - 5².

Or, you could use

MP.3

$$24 = 4 \cdot 6$$

= (5 - 1)(5 + 1)
= 5² - 1².

12. Explain why you cannot use the factors of 3 and 8 to rewrite 24 as the difference of two square integers.

If $24 = 3 \cdot 8$, then a - b = 3 and a + b = 8. The solution to this system of equations is (5.5, 2.5). If we are restricting this problem to the set of whole numbers, then you cannot apply the identity to rewrite 24 as the difference of two perfect squares where a and b are whole numbers. It certainly is true that $24 = (5.5 - 2.5)(5.5 + 2.5) = 5.5^2 - 2.5^2$, but this is not necessarily an easy way to calculate 24.



The Power of Algebra—Finding Primes









Student Outcomes

- Students understand that the sum of two square roots (or two cube roots) is not equal to the square root (or cube root) of their sum.
- Students convert expressions to simplest radical form.
- Students understand that the product of conjugate radicals can be viewed as the difference of two squares.

Lesson Notes

Because this lesson deals with radicals, it might seem out of place amid other lessons on polynomials. A major theme, however, is the parallelism between the product of conjugate radicals and the difference of two squares. There is also parallelism between taking the square root or cube root of a sum and taking the square of a sum; both give rise to an error sometimes called the freshman's dream or the illusion of linearity. If students are not careful, they may easily conclude that $(x + y)^n$ and $x^n + y^n$ are equivalent expressions for all $n \ge 0$, when this is only true for n = 1. Additionally, this work with radicals prepares students for later work with radical expressions in Module 3. Throughout this lesson, students employ MP.7, as they see complicated expressions as being composed of simpler ones. Additionally, the Opening Exercise offers further practice in making a conjecture (MP.3).

Scaffolding:

If necessary, remind students that $\sqrt{2}$ and $-\sqrt[3]{4}$ are irrational numbers. They cannot be written in the form $\frac{p}{q}$ for integers p and q. They are numbers, however, and, just like $\sqrt{4}$ and $-\sqrt[3]{64}$, can be found on a number line. On the number line, $\sqrt{2} = 1.414...$ is just to the right of 1.4, and $-\sqrt[3]{4} = -1.5874...$ is just to the left of -1.587.

Classwork

Opening Exercise (3 minutes)

The Opening Exercise is designed to show students that they need to be cautious when working with radicals. The multiplication and division operations combine with radicals in a predictable way, but the addition and subtraction operations do not.

The square root of a product is the product of the two square roots. For example,

$$\sqrt{4 \cdot 9} = \sqrt{2 \cdot 2 \cdot 3 \cdot 3}$$
$$= \sqrt{6 \cdot 6}$$
$$= 6$$
$$= 2 \cdot 3$$
$$= \sqrt{4} \cdot \sqrt{9}.$$

Similarly, the square root of a quotient is the quotient of the two square roots: $\sqrt{\frac{25}{16}} = \frac{5}{4} = \frac{\sqrt{25}}{\sqrt{16}}$. And the same holds

true for multiplication and division with cube roots, but not for addition or subtraction with square or cube roots.







Begin by posing the following question for students to work on in groups.

	Scaffolding:
Opening Exercise Which of these statements are true for all $a, b > 0$? Explain your conjecture. i. $2(a + b) = 2a + 2b$ ii. $\frac{a + b}{2} = \frac{a}{2} + \frac{b}{2}$ iii. $\sqrt{a + b} = \sqrt{a} + \sqrt{b}$	If necessary, circulate to help students get started by suggesting that they substitute numerical values for <i>a</i> and <i>b</i> . Perfect squares like 1 and 4 are good values to start with when square roots are in the
	expression.

Discussion (3 minutes)

Students should be able to show that the first two equations are true for all a and b, and they should be able to find values of a and b for which the third equation is not true. (In fact, it is always false, as students will show in the Problem Set.)

- Can you provide cases for which the third equation is not true? (Remind them that a single counterexample is sufficient to make an equation untrue in general.)
 - Students should give examples such as $\sqrt{9} + \sqrt{16} = 3 + 4 = 7$, but $\sqrt{9 + 16} = \sqrt{25} = 5$.

Point out that just as they have learned (in Lesson 2, if not before) that the square of (x + y) is not equal to the sum of x^2 and y^2 (for $x, y \neq 0$), so it is also true that the square root of (x + y) is not equal to the sum of the square roots of x and y (for x, y > 0). Similarly, the cube root of (x + y) is not equal to the sum of the cube roots of x and y (for $x, y \neq 0$).

Example 1 (2 minutes)

Explain to students that an expression is in *simplest radical form* when the radicand (the expression under the radical sign) has no factor that can be raised to a power greater than or equal to the index (either 2 or 3), and there is no radical in the denominator. Present the following example.

	easier to express in simplest
Example 1	
Express $\sqrt{50} - \sqrt{18} + \sqrt{8}$ in simplest radical form and combine like terms.	$\sqrt{8}$
$\sqrt{50} = \sqrt{25 \cdot 2} = \sqrt{25} \cdot \sqrt{2} = 5\sqrt{2}$	$\sqrt{6}$
$\sqrt{18} = \sqrt{9 \cdot 2} = \sqrt{9} \cdot \sqrt{2} = 3\sqrt{2}$	$\sqrt{18}$
$\sqrt{8} = \sqrt{4\cdot 2} = \sqrt{4}\cdot\sqrt{2} = 2\sqrt{2}$	$\sqrt{20}$
Therefore, $\sqrt{50} - \sqrt{18} + \sqrt{8} = 5\sqrt{2} - 3\sqrt{2} + 2\sqrt{2} = 4\sqrt{2}$.	$\sqrt{20} + \sqrt{8}$
	$\sqrt{20} - \sqrt{18}$.





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Scaffolding:

If necessary, ask students

about expressions that are



Scaffolding:

Exercises 1–5 (8 minutes)

The following exercises make use of the rules for multiplying and dividing radicals. Express each expression in simplest radical form and combine like terms.



In the example and exercises above, we repeatedly used the following properties of radicals (write the following statements on the board).

$\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$	$\sqrt[3]{a} \cdot \sqrt[3]{b} = \sqrt[3]{ab}$
$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$	$\frac{\sqrt[3]{a}}{\sqrt[3]{b}} = \sqrt[3]{\frac{a}{b}}$

- When do these identities make sense?
 - Students should answer that the identities make sense for the square roots whenever $a \ge 0$ and $b \ge 0$, with $b \ne 0$ when b is a denominator. They make sense for the cube roots for all a and b, with $b \ne 0$ when b is a denominator.

Example 2 (8 minutes)

This example is designed to introduce conjugates and their properties.

Example 2 Multiply and combine like terms. Then explain what you notice about the two different results. $(\sqrt{3} + \sqrt{2})(\sqrt{3} + \sqrt{2})$ $(\sqrt{3} + \sqrt{2})(\sqrt{3} - \sqrt{2})$





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Solution (with teacher comments and a question):

The first product is $\sqrt{3} \cdot \sqrt{3} + 2(\sqrt{3} \cdot \sqrt{2}) + \sqrt{2} \cdot \sqrt{2} = 5 + 2\sqrt{6}$.

The second product is $\sqrt{3} \cdot \sqrt{3} - (\sqrt{3} \cdot \sqrt{2}) + (\sqrt{3} \cdot \sqrt{2}) - \sqrt{2} \cdot \sqrt{2} = 3 - 2 = 1.$

The first product is an irrational number; the second is an integer.

The second product has the nice feature that the radicals have been eliminated. In that case, the two factors are given a special name: two binomials of the form $\sqrt{a} + \sqrt{b}$ and $\sqrt{a} - \sqrt{b}$ are called *conjugate radicals*:

 $\sqrt{a} + \sqrt{b}$ is the conjugate of $\sqrt{a} - \sqrt{b}$, and

 $\sqrt{a} - \sqrt{b}$ is the conjugate of $\sqrt{a} + \sqrt{b}$.

More generally, for any expression in two terms, at least one of which contains a radical, its conjugate is an expression consisting of the same two terms but with the opposite sign separating the terms. For example, the conjugate of $2 - \sqrt{3}$ is $2 + \sqrt{3}$, and the conjugate of $\sqrt[3]{5} + \sqrt{3}$ is $\sqrt[3]{5} - \sqrt{3}$.

- What polynomial identity is suggested by the product of two conjugates?
 - Students should answer that it looks like the difference of two squares. The product of two conjugates has the form of the difference of squares: $(x + y)(x - y) = x^2 - y^2$.

The following exercise focuses on the use of conjugates.





Exercise 6 (5 minutes)

Exercise 6 6. Find the product of the conjugate radicals. $(\sqrt{5} + \sqrt{3})(\sqrt{5} - \sqrt{3})$ 5 - 3 = 2 $(7 + \sqrt{2})(7 - \sqrt{2})$ 49 - 2 = 47 $(\sqrt{5} + 2)(\sqrt{5} - 2)$ 5 - 4 = 1

- In each case in Exercise 6, is the result the difference of two squares?
 - Yes. For example, if we think of 5 as $(\sqrt{5})^2$ and 3 as $(\sqrt{3})^2$, then $5 3 = (\sqrt{5})^2 (\sqrt{3})^2$.



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Example 3 (6 minutes)

This example is designed to show how division by a radical can be reduced to division by an integer by multiplication by the conjugate radical in the numerator and denominator.

Example 3 Write $\frac{\sqrt{3}}{5-2\sqrt{3}}$ in simplest radical form. $\frac{\sqrt{3}}{5-2\sqrt{3}} = \frac{\sqrt{3}}{5-2\sqrt{3}} \cdot \frac{5+2\sqrt{3}}{5+2\sqrt{3}} = \frac{\sqrt{3}(5+2\sqrt{3})}{25-12} = \frac{5\sqrt{3}+6}{13}$

The process for simplifying an expression with a radical in the denominator has two steps:

- 1. Multiply the numerator and denominator of the fraction by the *conjugate* of the denominator.
- 2. Simplify the resulting expression.

Closing (5 minutes)

Radical expressions with the same index and same radicand combine in the same way as like terms in a
polynomial when performing addition and subtraction.

For example, $\sqrt[3]{3} + \sqrt{2} + 5\sqrt[3]{3} - \sqrt{7} + \sqrt{3} + \sqrt{7} + 3\sqrt{2} = 6\sqrt[3]{3} + 4\sqrt{2} + \sqrt{3}$.

- Simplifying an expression with a radical in the denominator relies on an application of the difference of squares formula.
 - For example, to simplify $\frac{3}{\sqrt{2}+\sqrt{3}}$, we treat the denominator like a binomial.

Substitute $\sqrt{2} = x$ and $\sqrt{3} = y$, and then

$$\frac{3}{\sqrt{2}+\sqrt{3}} = \frac{3}{x+y} = \frac{3}{x+y} \cdot \frac{x-y}{x-y} = \frac{3(x-y)}{x^2-y^2}$$

Since $x = \sqrt{2}$ and $y = \sqrt{3}$, $x^2 - y^2$ is an integer. In this case, $x^2 - y^2 = -1$.

$$\frac{3}{\sqrt{2}+\sqrt{3}} \cdot \frac{\sqrt{2}-\sqrt{3}}{\sqrt{2}-\sqrt{3}} = \frac{3(\sqrt{2}-\sqrt{3})}{2-3} = -3(\sqrt{2}-\sqrt{3})$$





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Ask students to summarize the important parts of the lesson, either in writing, to a partner, or as a class. Use this as an opportunity to informally assess understanding of the lesson. The following are some important summary elements.



Exit Ticket (5 minutes)









Name

Date

Lesson 9: Radicals and Conjugates

Exit Ticket

- 1. Rewrite each of the following radicals as a rational number or in simplest radical form.
 - a. $\sqrt{49}$
 - b. $\sqrt[3]{40}$
 - c. √242
- 2. Find the conjugate of each of the following radical expressions.
 - a. $\sqrt{5} + \sqrt{11}$
 - b. $9 \sqrt{11}$
 - c. $\sqrt[3]{3} + 1.5$
- 3. Rewrite each of the following expressions as a rational number or in simplest radical form.
 - a. $\sqrt{3}(\sqrt{3}-1)$
 - b. $(5 + \sqrt{3})^2$
 - c. $(10 + \sqrt{11})(10 \sqrt{11})$





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Exit Ticket Sample Solution

1.	Rewrite each of the following radicals as a rational number or in simplest radical form.		
	a.	$\sqrt{49}$	7
	b.	³ √40	2 ³ √5
	c.	$\sqrt{242}$	11√2
2.	Find	the conjugate of each of the following radical expression	15.
	a.	$\sqrt{5} + \sqrt{11}$	$\sqrt{5} - \sqrt{11}$
	b.	$9-\sqrt{11}$	$9 + \sqrt{11}$
	c.	$\sqrt[3]{3} + 1.5$	$\sqrt[3]{3} - 1.5$
3.	Rewi	ite each of the following expressions as a rational numb	er or in simplest radical form.
	a.	$\sqrt{3}(\sqrt{3}-1)$	$3-\sqrt{3}$
	b.	$\left(5+\sqrt{3}\right)^2$	$28 + 10\sqrt{3}$
	c.	$(10+\sqrt{11})(10-\sqrt{11})$	89

Problem Set Sample Solutions

Problem 10 is different from the others and may require some discussion and explanation before students work on it. Consider explaining that the converse of an *if-then* theorem is obtained by interchanging the clauses introduced by *if* and then, and that the converse of such a theorem is not necessarily a valid theorem. The converse of the Pythagorean theorem will be important for the development of a formula leading to Pythagorean triples in Lesson 10.

1.	Express each of the following as a rational number or in simplest radical form. Assume that the symbols a , b , and x represent positive numbers.		
	a.	$\sqrt{36}$	6
	b.	$\sqrt{72}$	$6\sqrt{2}$
	c.	$\sqrt{18}$	3√2
	d.	$\sqrt{9x^3}$	$3x\sqrt{x}$







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	e.	$\sqrt{27x^2}$	$3x\sqrt{3}$
	f.	$\sqrt[3]{16}$	2 ³ √2
	g.	$\sqrt[3]{24a}$	$2\sqrt[3]{3a}$
	h.	$\sqrt{9a^2+9b^2}$	$3\sqrt{a^2+b^2}$
2.	Expre	ss each of the following in simplest radical form, c	ombining terms where possible.
	a.	$\sqrt{25} + \sqrt{45} - \sqrt{20}$	$5 + \sqrt{5}$
	b.	$3\sqrt{3} - \sqrt{\frac{3}{4}} + \sqrt{\frac{1}{3}}$	$\frac{17\sqrt{3}}{6}$
	c.	$\sqrt[3]{54} - \sqrt[3]{8} + 7\sqrt[3]{\frac{1}{4}}$	$\frac{13\sqrt[3]{2}}{2} - 2$
	d.	$\sqrt[3]{\frac{5}{8}} + \sqrt[3]{40} - \sqrt[3]{\frac{8}{9}}$	$\frac{5\sqrt[3]{5}}{2} - \frac{2\sqrt[3]{3}}{3}$
3.	Evalua	ate $\sqrt{x^2-y^2}$ when $x=33$ and $y=15$.	12√6
4.	Evalua	ate $\sqrt{x^2+y^2}$ when $x=20$ and $y=10$.	10 √5
5.	Expres	ss each of the following as a rational expression or	r in simplest radical form. Assume that the symbols x and y
	a.	$\sqrt{3}(\sqrt{7}-\sqrt{3})$	$\sqrt{21} - 3$
	b.	$\left(3+\sqrt{2}\right)^2$	$11 + 6\sqrt{2}$
	c.	$(2+\sqrt{3})(2-\sqrt{3})$	1
	d.	$(2+2\sqrt{5})(2-2\sqrt{5})$	-16
	e.	$(\sqrt{7}-3)(\sqrt{7}+3)$	-2
	f.	$(3\sqrt{2}+\sqrt{7})(3\sqrt{2}-\sqrt{7})$	11
	g.	$(x-\sqrt{3})(x+\sqrt{3})$	$x^2 - 3$
	h.	$(2x\sqrt{2}+y)(2x\sqrt{2}-y)$	$8x^2 - y^2$

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6. Simplify each of the following quotients as far as possible. a. $(\sqrt{21} - \sqrt{3}) \div \sqrt{3}$ $\sqrt{7} - 1$ $\frac{1}{4} \big(1 + 3\sqrt{5} \big)$ **b.** $\left(\sqrt{5}+4\right) \div \left(\sqrt{5}+1\right)$ $-\frac{1}{7}(9+4\sqrt{2})$ c. $(3-\sqrt{2})\div(3\sqrt{2}-5)$ $\frac{1}{12}(30-3\sqrt{15}+8\sqrt{10}-4\sqrt{6})$ d. $(2\sqrt{5}-\sqrt{3})\div(3\sqrt{5}-4\sqrt{2})$ 7. If $x = 2 + \sqrt{3}$, show that $x + \frac{1}{x}$ has a rational value. $x + \frac{1}{x} = 4$ Evaluate $5x^2 - 10x$ when the value of x is $rac{2-\sqrt{5}}{2}$. 8. 5 4 Write the factors of $a^4 - b^4$. Express $(\sqrt{3} + \sqrt{2})^4 - (\sqrt{3} - \sqrt{2})^4$ in a simpler form. 9. **Factors:** $(a^2 + b^2)(a + b)(a - b)$ Simplified form: $40\sqrt{6}$ 10. The converse of the Pythagorean theorem is also a theorem: If the square of one side of a triangle is equal to the sum of the squares of the other two sides, then the triangle is a right triangle. Use the converse of the Pythagorean theorem to show that for A, B, C > 0, if A + B = C, then $\sqrt{A} + \sqrt{B} > \sqrt{C}$, so that $\sqrt{A} + \sqrt{B} > \sqrt{A + B}$. Solution 1: Since A, B, C > 0, we can interpret these quantities as the areas of three squares whose sides have lengths \sqrt{A} , \sqrt{B} , and \sqrt{C} . Because A + B = C, then by the converse of the Pythagorean theorem, \sqrt{A} , \sqrt{B} , and \sqrt{C} are the lengths of the legs and hypotenuse of a right triangle. In a triangle, the sum of any two sides is greater than the third side. Therefore, $\sqrt{A} + \sqrt{B} > \sqrt{C}$, so $\sqrt{A} + \sqrt{B} > \sqrt{A+B}$. Solution 2: Since A, B, C > 0, we can interpret these quantities as the areas of three squares whose sides have lengths \sqrt{A} , \sqrt{B} , and \sqrt{C} . Because A + B = C, then by the converse of the Pythagorean theorem, $\sqrt{A} = a$, $\sqrt{B} = b$, and $c = \sqrt{C}$ are the lengths of the legs and hypotenuse of a right triangle, so a, b, c > 0. Therefore, 2ab > 0. Adding equal positive quantities to each side of that inequality, we get $a^2 + b^2 + 2ab > c^2$, which we can rewrite as $(a + b)^2 > c^2$. Taking the positive square root of each side, we get a + b > c, or equivalently, $\sqrt{A} + \sqrt{B} > \sqrt{C}$. We then have $\sqrt{A} + \sqrt{B} > \sqrt{A + B}$.








Lesson 10: The Power of Algebra—Finding Pythagorean Triples

Student Outcomes

• Students explore the difference of two squares identity $x^2 - y^2 = (x - y)(x + y)$ in the context of finding Pythagorean triples.

Lesson Notes

This lesson addresses standards **A-SSE.A.2** and **A-APR.C.4**, and MP.7 directly. In particular, this lesson investigates the example suggested by **A-APR.C.4**: Show how "the polynomial identity $(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2$ can be used to generate Pythagorean triples." This polynomial identity is proven in this lesson using the difference of two squares identity by

$$(x^{2} + y^{2})^{2} - (x^{2} - y^{2})^{2} = ((x^{2} + y^{2}) - (x^{2} - y^{2}))((x^{2} + y^{2}) + (x^{2} - y^{2})) = (2y^{2})(2x^{2}) = (2xy)^{2}.$$

However, students are first asked to prove the identity on their own in the case when y = 1. Very few (or likely none) of the students will use the difference of two squares identity, offering an opportunity to surprise them with the quick solution presented here.

The lesson starts with a quick review of the most important theorem in all of geometry and arguably in all of mathematics: the Pythagorean theorem. Students have already studied the Pythagorean theorem in Grade 8 and high school Geometry, have proven the theorem in numerous ways, and have used it in a wide variety of situations. Students are asked to prove it in yet a different way in the Problem Set to this lesson. The Pythagorean theorem plays an important role in both this module and the next.

Classwork

Opening Exercise (10 minutes)

This exercise is meant to help students recall facts about the Pythagorean theorem. Because it is not the main point of this lesson, feel free to move through this exercise quickly. After they have worked the problem, summarize with a statement of the Pythagorean theorem and its converse, and then move on.

Have students work in groups of two on this problem. Suggest immediately that they draw a diagram to represent the problem.

Consider starting by showing a simple example of the Pythagorean theorem. $5 \text{ cm} \qquad \frac{x \text{ cm}}{12 \text{ cm}}$

 $5^2 + 12^2 = x^2 \implies x = 13$

Scaffolding:



D: The Power of Algebra—Finding Pythagorean Triples







Students may have a question about what the problem is asking: Does distance mean, "Who walked the farthest?", or "Who is the farthest (as the crow flies) from the starting point?" This question boils down to the difference between the definitions of path length versus distance. While Sam's path length is 8 city blocks and Jill's is 10 city blocks, the question asks for the distance between the starting point and their final destinations. To calculate distance, students need to use the Pythagorean theorem.

The problem is designed so that answers cannot be guessed easily from precisely drawn pictures.

Another (valid) issue that a student may bring up is whether the streets are considered to have width or not. Discuss this possibility with the class (again, it is a valid point). Suggest that for the purposes of this problem, the assumption is that the streets have no width (or, as some may point out, Sam and Jill could walk down the center of the streets—but this is not advisable).

Try to get students to conclude that $\sqrt{18} < \sqrt{20}$ simply because 18 < 20 and the square root function increases.

- Why must the side length of a square with area 18 square units be smaller than the side length of a square with area 20 square units?
- Can you state the Pythagorean theorem?
 - If a right triangle has legs of length a and b units and a hypotenuse of length c units, then $a^2 + b^2 = c^2$.
- What is the converse of the Pythagorean theorem? Can you state it as an *if-then* statement?
 - If the lengths a, b, c of the sides of a triangle are related by $a^2 + b^2 = c^2$, then the angle opposite the side of length c is a right angle.
- We will need the converse of the Pythagorean theorem for this lesson.



MP.6

The Power of Algebra—Finding Pythagorean Triples



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To make this example more

generate (or ask students to

concrete and accessible,

Lesson 10

Scaffolding:

D

Example 1 (15 minutes)

In this example, students explore a specific case of the general method of generating Pythagorean triples, that is triples of positive integers (a, b, c) that satisfy $a^2 + b^2 = c^2$. The general form that students explore in the Problem Set is $(x^2 - y^2, 2xy, x^2 - y^2)$ for x > y.



Proving that $(x^2 - 1)^2 + (2x)^2 = (x^2 + 1)^2$ can be done in different ways. Consider asking students to try their own method first, and then show the method above. Very few students will use the identity $a^2 - b^2 = (a - b)(a + b)$. Most will use $(x^2 - 1)^2 + 4x^2 = x^4 - 2x^2 + 1 + 4x^2 = x^4 + 2x^2 + 1 = (x^2 + 1)^2$. This is an excellent exercise as well, since it gets students to wrestle with squares of quadratic polynomials and requires factoring. After they have tried it on their own, they will be surprised by the use of the difference of squares identity.

- A Pythagorean triple is a triple of positive integers (a, b, c) such that $a^2 + b^2 = c^2$. So, while (3, 4, 5) is a Pythagorean triple, the triple $(1, 1, \sqrt{2})$ is not, even though 1, 1, and $\sqrt{2}$ are side lengths of a 45°-45°-90° triangle and $1^2 + 1^2 = (\sqrt{2})^2$. While the triangle from Example 1 can have non-integer side lengths, notice that a Pythagorean triple must comprise positive integers by definition.
- Note that any multiple of a Pythagorean triple is also a Pythagorean triple: if (a, b, c) is a Pythagorean triple, then so is (na, nb, nc) for any positive integer n (discuss why). Thus, (6, 8, 10), (9, 12, 15), (12, 16, 20), (15, 20, 25) are all Pythagorean triples because they are multiples of (3, 4, 5).



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- Also note that if (a, b, c) is a Pythagorean triple, then (b, a, c) is also a Pythagorean triple. To reduce redundancy, we often write the smaller number of a and b first. Although (3, 4, 5) and (4, 3, 5) are both Pythagorean triples, they represent the same triple, and we refer to it as (3, 4, 5).
- One way to generate Pythagorean triples is to use the expressions from Example 1: $(x^2 1, 2x, x^2 + 1)$.

Have students try a few as mental math exercises: x = 2 gives (4, 3, 5), x = 3 gives (8, 6, 10), x = 4 gives (15, 8, 17), and so on.

One of the Problem Set questions asks students to generalize triples from $(x^2 - 1, 2x, x^2 + 1)$ to show that triples generated by $(x^2 - y^2, 2xy, x^2 + y^2)$ also form Pythagorean triples for x > y > 0. The next example helps students see the general pattern.

Example 2 (12 minutes)

This example shows a clever way for students to remember that $x^2 - 1$, 2x, and $x^2 + 1$ can be used to find Pythagorean triples.





The Power of Algebra—Finding Pythagorean Triples





Lesson 10

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In the Problem Set, students prove that if x and y are positive integers with x > y, then $(x^2 - y^2, 2xy, x^2 + y^2)$ is a Pythagorean triple, mimicking the proof of Example 1.

Closing (3 minutes)

- Pythagorean triples are triples of positive integers (a, b, c) that satisfy the relationship $a^2 + b^2 = c^2$. Such a triple is called a Pythagorean triple because a right triangle with legs of length a and b will have a hypotenuse of length c by the Pythagorean theorem.
- To generate a Pythagorean triple, take any two positive integers x and y with x > y, and compute $(x^2 y^2, 2xy, x^2 + y^2)$.

Relevant Facts and Vocabulary

PYTHAGOREAN THEOREM: If a right triangle has legs of length a and b units and hypotenuse of length c units, then $a^2 + b^2 = c^2$.

CONVERSE TO THE PYTHAGOREAN THEOREM: If the lengths a, b, c of the sides of a triangle are related by $a^2 + b^2 = c^2$, then the angle opposite the side of length c is a right angle.

PYTHAGOREAN TRIPLE: A *Pythagorean triple* is a triple of positive integers (a, b, c) such that $a^2 + b^2 = c^2$. The triple (3, 4, 5) is a Pythagorean triple but $(1, 1, \sqrt{2})$ is not, even though the numbers are side lengths of an isosceles right triangle.

Exit Ticket (5 minutes)



D: The Power of Algebra—Finding Pythagorean Triples





Name

Date

Lesson 10: The Power of Algebra—Finding Pythagorean Triples

Exit Ticket

Generate six Pythagorean triples using any method discussed during class. Explain each method you use.



0: The Power of Algebra—Finding Pythagorean Triples







Exit Ticket Sample Solutions

Generate six Pythagorean triples using any method discussed during class. Explain each method you use.

Answers will vary. One example should use either $(x^2 - 1, 2x, x^2 + 1)$ or $(x^2 - y^2, 2xy, x^2 + y^2)$, but after that students can use the fact that a multiple of a Pythagorean triple is again a Pythagorean triple.

Problem Set Sample Solutions

Rewrite each expression as a sum or difference of terms. 1. b. $(x^2 - 3)(x^2 + 3)$ (x-3)(x+3)a. $x^4 - 9$ $x^2 - 9$ c. $(x^{15}+3)(x^{15}-3)$ d. $(x-3)(x^2+9)(x+3)$ $x^{30} - 9$ $x^4 - 81$ f. $(x^2 + y^2)^2$ e. $(x^2 + y^2)(x^2 - y^2)$ $x^4 - v^4$ $x^4 + 2x^2y^2 + y^4$ g. $(x-y)^2(x+y)^2$ h. $(x-y)^2(x^2+y^2)^2(x+y)^2$ $x^4 - 2x^2y^2 + y^4$ $x^8 - 2x^4y^4 + y^8$ Tasha used a clever method to expand (a + b + c)(a + b - c). She grouped the addends together like this 2. [(a + b) + c][(a + b) - c] and then expanded them to get the difference of two squares: $(a+b+c)(a+b-c) = [(a+b)+c][(a+b)-c] = (a+b)^2 - c^2 = a^2 + 2ab + b^2 - c^2.$ Is Tasha's method correct? Explain why or why not. a. Yes, Tasha is correct. Expanding in the traditional way gives the same result. (a + b + c)(a + b - c) = (a + b + c)a + (a + b + c)b - (a + b + c)c= a² + ba + ca + ab + b² + cb - ac - bc - c² $= a^{2} + 2ab + b^{2} - c^{2}$ b. Use a version of her method to find (a + b + c)(a - b - c). $(a + (b + c))(a - (b + c)) = a^2 - (b + c)^2 = a^2 - b^2 - 2bc - c^2$ Use a version of her method to find (a + b - c)(a - b + c). c. $(a + (b - c))(a - (b - c)) = a^{2} - (b - c)^{2} = a^{2} - b^{2} + 2bc - c^{2}$ 3. Use the difference of two squares identity to factor each of the following expressions. a. $x^2 - 81$ b. $(3x + y)^2 - (2y)^2$ (x-9)(x+9)(3x - y)(3x + 3y) = 3(3x - y)(x + y)c. $4 - (x - 1)^2$ d. $(x+2)^2 - (y+2)^2$ (3-x)(1+x)(x - y)(x + y + 4)



Lesson 10: The Power of Algebra—Finding Pythagorean Triples



4. Show that the expression (x + y)(x - y) - 6x + 9 may be written as the difference of two squares, and then factor the expression. $(x + y)(x - y) - 6x + 9 = x^{2} - y^{2} - 6x + 9 = (x^{2} - 6x + 9) - y^{2} = (x - 3)^{2} - y^{2} = (x - 3 - y)(x - 3 + y)$ 5. Show that $(x + y)^2 - (x - y)^2 = 4xy$ for all real numbers x and y. $(x + y)^{2} - (x - y)^{2} = [(x + y) - (x - y)][(x + y) + (x - y)] = (2y)(2x) = 4x$ Prove that a triangle with side lengths $x^2 - y^2$, 2xy, and $x^2 + y^2$ with x > y > 0 is a right triangle. 6. The proof should look like the proof in Example 1 but with y instead of 1. Complete the table below to find Pythagorean triples (the first row is done for you). 7. Check: Is it a Pythagorean Triple? $x^2 - y^2$ 2xy $x^2 + y^2$ y х Yes: $3^2 + 4^2 = 25 = 5^2$ 2 1 3 4 5 *Yes:* $8^2 + 6^2 = 100 = 10^2$ 3 1 8 6 10 *Yes:* $5^2 + 12^2 = 169 = 13^2$ 3 2 5 12 13 *Yes:* $15^2 + 8^2 = 289 = 17^2$ 4 1 15 8 17 4 2 12 16 20 *Yes:* $12^2 + 16^2 = 400 = 20^2$ Yes: $7^2 + 24^2 = 625 = 25^2$ 7 4 24 25 3 5 1 24 10 26 Yes: $24^2 + 10^2 = 676 = 26^2$ Answer the following parts about the triple (9, 12, 15). 8. Show that (9, 12, 15) is a Pythagorean triple. а. We see that $9^2 + 12^2 = 81 + 144 = 225$, and $15^2 = 225$ so $9^2 + 12^2 = 15^2$. Prove that neither (9, 12, 15) nor (12, 9, 15) can be found by choosing a pair of integers x and y with x > yb. and computing $(x^2 - y^2, 2xy, x^2 + y^2)$. (Hint: What are the possible values of x and y if 2xy = 12? What about if 2xy = 9?) **PROOF:** Since 9 is odd and 2xy is even, there are no integer values of x and y that satisfy 2xy = 9. Thus, our formula cannot generate the triple (12, 9, 15). Now suppose x and y are integers such that 2xy = 12. Thus xy = 6 and x > y. There are only two possibilities: either x = 6 and y = 1, or x = 3 and y = 2. In the first case, our formula generates the triple $(6^2 - 1, 2 \cdot 6 \cdot 1, 6^2 + 1) = (35, 12, 37)$. In the second case, our formula generates the triple $(3^2 - 2^2, 2 \cdot 3 \cdot 2, 3^2 + 2^2) = (5, 12, 13)$. Thus, there is no way to generate the triple (9, 12, 15) using this method, even though it is a Pythagorean triple. Wouldn't it be nice if all Pythagorean triples were generated by $(x^2 - y^2, 2xy, x^2 + y^2)$? Research c. Pythagorean triples on the Internet to discover what is known to be true about generating all Pythagorean triples using this formula. All Pythagorean triples are some multiple of a Pythagorean triple generated using this formula. For example, while (9, 12, 15) is not generated by the formula, it is a multiple of a Pythagorean triple (3, 4, 5), which is generated by the formula. Follow the steps below to prove the identity $(a^2 + b^2)(x^2 + y^2) = (ax - by)^2 + (bx + ay)^2$. 9. Multiply $(a^2 + b^2)(x^2 + y^2)$. а. $(a^{2}+b^{2})(x^{2}+y^{2}) = a^{2}x^{2}+a^{2}y^{2}+b^{2}x^{2}+b^{2}y^{2}$



Lesson 10:

The Power of Algebra—Finding Pythagorean Triples











Lesson 10:

0: The Power of Algebra—Finding Pythagorean Triples







Lesson 11: The Special Role of Zero in Factoring

Student Outcomes

- Students find solutions to polynomial equations where the polynomial expression is not factored into linear factors.
- Students construct a polynomial function that has a specified set of zeros with stated multiplicity.

Lesson Notes

This lesson focuses on the first part of standard **A-APR.B.3**, identifying zeros of polynomials presented in factored form. Although the terms *root* and *zero* are interchangeable, for consistency only the term *zero* is used throughout this lesson and in later lessons. The second part of the standard, using the zeros to construct a rough graph of a polynomial function, is delayed until Lesson 14. The ideas that begin in this lesson continue in Lesson 19, in which students will be able to associate a zero of a polynomial function to a factor in the factored form of the associated polynomial as a consequence of the remainder theorem, and culminate in Lesson 39, in which students apply the fundamental theorem of algebra to factor polynomial expressions completely over the complex numbers.

Classwork

Opening Exercise (12 minutes)

Opening Exercise

Find all solutions to the equation $(x^2 + 5x + 6)(x^2 - 3x - 4) = 0$.

The main point of this opening exercise is for students to recognize and then formalize that the statement "If ab = 0, then a = 0 or b = 0" applies not only when a and b are numbers or linear functions (which we used when solving a quadratic equation), but also applies to cases where a and b are polynomial functions of any degree.

In small groups, let students discuss ways to solve this equation. Walk around the room and offer advice such as, "Have you considered factoring each quadratic expression? What do you get?" As soon as one group factors both quadratic expressions, or when three minutes have passed, show, or let that group show, the factorization on the board.

$$\underbrace{(x+2)(x+3)}_{x^2+5x+6} \cdot \underbrace{(x-4)(x+1)}_{x^2-3x-4} = 0$$

The Special Role of Zero in Factoring

What are the solutions to this equation?

Lesson 11:

□ -2, -3, 4, -1

EUREKA



Scaffolding:

and v.

others?

zero.

Here is an alternative opening

For each equation, list some possible values for x

xy = 10, xy = 1,

xy = -1, xy = 0

What do you notice? Does one equation tell you

more information than

activity that may better illuminate the special role of





- □ If x is any number other than -2, -3, 4, or -1, then each factor is a nonzero number that is $x + 2 \neq 0$, $x + 3 \neq 0$, etc. However, the multiplication of four nonzero numbers is nonzero, so that value of x cannot be a solution. Therefore, the only possible solutions are -2, -3, 4, and -1. It is easy to confirm that these are indeed solutions by substituting them each into the equation individually.
- Why are these numbers also solutions to the original equation?
 - Because the expression (x + 2)(x + 3)(x 4)(x + 1) is equivalent to $(x^2 + 5x + 6)(x^2 3x 4)$.

Now let's study the solutions to $x^2 + 5x + 6 = 0$ and $x^2 - 3x - 4 = 0$ separately.

- What are the solutions to $x^2 + 5x + 6 = 0$?
- What are the solutions to $x^2 3x 4 = 0$?
 - 4, -1
- Relate the solutions of the equation $(x^2 + 5x + 6)(x^2 3x 4) = 0$ to the solutions of the compound statement, " $x^2 + 5x + 6 = 0$ or $x^2 3x 4 = 0$."
 - They are the same.
- Given two polynomial functions p and q of any degree, the solution set of the equation p(x)q(x) = 0 is the union of the solution set of p(x) = 0 and the solution set of q(x) = 0. Let's think about why.

Lead students in a discussion of the following proof:

- Suppose *a* is a solution to the equation p(x)q(x) = 0; that is, it is a number that satisfies p(a)q(a) = 0. Since p(a) is a number and q(a) is a number, one or both of them must be zero, by the zero product property that states, "If the product of two numbers is zero, then at least one of the numbers is zero." Therefore, p(a) = 0 or q(a) = 0, which means *a* is a solution to the compound statement, "p(x) = 0 or q(x) = 0."
- Now let's prove the other direction and show that if *a* is a solution to the compound statement, then it is a solution to the equation p(x)q(x) = 0. This direction is also easy: Suppose *a* is a number such that either p(a) = 0 or q(a) = 0. In the first case, $p(a)q(a) = 0 \cdot q(a) = 0$. In the second case, $p(a)q(a) = p(a) \cdot 0 = 0$. Hence, in either case, *a* is a solution to the equation p(x)q(x) = 0.

Students may have difficulty understanding the distinction between the equations p(x)q(x) = 0 and p(a)q(a) = 0. Help students understand that p(x)q(x) = 0 is an equation in a variable x, while p(a) is the value of the function p when it is evaluated at the number a. Thus, p(a)q(a) is a number. For example, if p and q are the quadratic polynomials $p(x) = x^2 + 5x + 6$ and $q(x) = x^2 - 3x - 4$ from the Opening Exercise, and students are considering the case when a is 5, then p(5) = 56 and q(5) = 6. Therefore, 5 cannot be a solution to the equation p(x)q(x) = 0.

Communicate to students that they can use the statement below to break problems into simpler parts: Given any two polynomial functions p and q, the set of solutions to the equation p(x)q(x) = 0 can be found by solving p(x) = 0, solving q(x) = 0, and combining the solutions into one set.

Ask students to try the following exercise on their own.





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Exercise 1 (2 minutes)

Exercise 1 1. Find the solutions of $(x^2 - 9)(x^2 - 16) = 0$. The solutions to $(x^2 - 9)(x^2 - 16)$ are the solutions of $x^2 - 9 = 0$ combined with the solutions of $x^2 - 16 = 0$. These solutions are -3, 3, -4, and 4.

The next example looks at a polynomial equation for which the solution is already known. The goal of this example and the discussion that follows is to use a solution to the equation f(x) = 0 to further factor the polynomial f. In doing so, the class ends with a description of the zeros of a function, a concept first introduced in Algebra I, Module 4.

Example 1 (8 minutes)

Example 1 Suppose we know that the polynomial equation $4x^3 - 12x^2 + 3x + 5 = 0$ has three real solutions and that one of the factors of $4x^3 - 12x^2 + 3x + 5$ is (x - 1). How can we find all three solutions to the given equation?

Steer the discussion to help students conjecture that the polynomial $4x^3 - 12x^2 + 3x + 5$ must be the product of (x - 1) and some quadratic polynomial.

Since (x - 1) is a factor, and we know how to divide polynomials, we can find the quadratic polynomial by dividing:

$$\frac{4x^3 - 12x^2 + 3x + 5}{x - 1} = 4x^2 - 8x - 5.$$

Now we know that $4x^3 - 12x^2 + 3x + 5 = (x - 1)(4x^2 - 8x - 5)$, and we also know that $4x^2 - 8x - 5$ is a quadratic polynomial that has linear factors (2x + 1) and (2x - 5).

Therefore, $4x^3 - 12x^2 + 3x + 5 = 0$ has the same solutions as $(x - 1)(4x^2 - 8x - 5) = 0$, which has the same solutions as

$$(x-1)(2x+1)(2x-5) = 0.$$

In this factored form, the solutions of f(x) = 0 are readily apparent: $-\frac{1}{2}$, 1, and $\frac{5}{2}$.

Scaffolding:

Allow students to generate ideas about how the linear factors affect the behavior of the graph so that they can use the graph of a function to identify zeros.









Discussion (8 minutes)

- In Example 1 above, we saw that factoring the polynomial into linear factors helped us to find solutions to the original polynomial equation
 4x³ 12x² + 3x + 5 = 0.
- There is a corresponding notion for the zeros of a function. Let *f* be a function whose domain is a subset of the real numbers and whose range is a subset of the real numbers. A zero (or root) of the function *f* is a number *c* such that *f*(*c*) = 0.
- The zeros of the function $f(x) = 4x^3 12x^2 + 3x + 5$ are the *x*-intercepts of the graph of *f*: these are $-\frac{1}{2}$, 1, 5
 - and $\frac{5}{2}$.
- By definition, a zero of a polynomial function f is a solution to the equation f(x) = 0. If (x a) is a factor of a polynomial function f, then f(a) = 0 and a is a zero of f.



However, consider the polynomial functions $p(x) = (x - 2)(x + 3)^2$, $q(x) = (x - 2)^2(x + 3)^4$, and $r(x) = (x - 2)^4(x - 3)^5$. Because p(2) = 0, q(2) = 0, and r(2) = 0, the number 2 is a zero of p, q, and r. Likewise, -3 is also a zero of p, q, and r. Even though these polynomial functions have the same zeros, they are not the same function; they do not even have the same degree!



We would like to be able to distinguish between the zeros of these two polynomial functions. If we write out all of the factors for p, q, and r, we see that

$$p(x) = (x - 2)(x + 3)(x + 3)$$

$$q(x) = (x - 2)(x - 2)(x + 3)(x + 3)(x + 3)(x + 3)$$

$$r(x) = (x - 2)(x - 2)(x - 2)(x - 2)(x + 3)(x + 3)(x + 3)(x + 3)(x + 3).$$





- We notice that (x 2) is a factor of p once, and (x + 3) is a factor of p twice. Thus, we say that 2 is a zero of p of multiplicity 1, and -3 is a zero of p of multiplicity 2. Zeros of multiplicity 1 are usually just referred to as zeros, without mentioning the multiplicity.
- What are the zeros of *q*, with their multiplicities?
 - For q, 2 is a zero of multiplicity 2, and -3 is a zero of multiplicity 4.
- What are the zeros of r, with their multiplicities?
 - For r, 2 is a zero of multiplicity 4, and -3 is a zero of multiplicity 5.
- Can you look at the factored form of a polynomial equation and identify the zeros with their multiplicities? Explain how you know.
 - Yes. Each linear factor $(ax b)^m$ of the polynomial will produce a zero $\frac{b}{a}$ with multiplicity m.
- Can multiplicity be negative? Can it be zero? Can it be a fraction?
 - No. Multiplicity is the count of the number of times a factor appears in a factored polynomial expression. Polynomials can only have positive integer exponents, so a factor must have positive integer exponents. Thus, the multiplicity of a zero must be a positive integer.

Note: In Lesson 14, students use the zeros of a polynomial function together with their multiplicities to create a graph of the function, and in Lesson 19, students use the zeros of a polynomial function with their multiplicities to construct the equation of the function.

Exercises 2–5 (8 minutes)

```
Exercises 2-5
     Find the zeros of the following polynomial functions, with their multiplicities.
2.
            f(x) = (x+1)(x-1)(x^2+1)
      а.
            -1 with multiplicity 1
            1 with multiplicity 1
           g(x) = (x-4)^3(x-2)^8
      b.
            4 with multiplicity 3
            2 with multiplicity 8
            h(x) = (2x - 3)^5
      c.
            \frac{3}{2} with multiplicity 5
            k(x) = (3x+4)^{100}(x-17)^4
      d.
            -\frac{4}{2} with multiplicity 100
            17 with multiplicity 4
```





Lesson 11

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Closing (2 minutes)

Ask students to summarize the key ideas of the lesson, either in writing or with a neighbor. Consider posing the questions below.

- Part of the lesson today has been that given two polynomials, p and q, we can determine solutions to p(x)q(x) = 0 by solving both p(x) = 0 and q(x) = 0, even if they are high-degree polynomials. If p and q are polynomial functions that do not have any real number zeros, do you think the equation p(x)q(x) = 0 still has real number solutions? Can you give an example of two such functions?
 - □ If $p(x) \neq 0$ for all real numbers x and $q(x) \neq 0$ for all real numbers x, then there is no possible way to have p(x)q(x) = 0.
 - For example: If $p(x) = x^2 + 1$ and $q(x) = x^4 + 1$, then the equation $(x^2 + 1)(x^4 + 1) = 0$ has no real solutions.

The following vocabulary was introduced in Algebra I (please see Module 3 and Module 4 in Algebra I). While the teacher should not have to teach these terms explicitly, it may still be a good idea to go through them with the class.

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Relevant Vocabulary Terms

In the definitions below, the symbol \mathbb{R} stands for the set of real numbers.

FUNCTION: A function is a correspondence between two sets, X and Y, in which each element of X is assigned to one and

only one element of Y.

The set X in the definition above is called the domain of the function. The range (or image) of the function is the subset

of Y, denoted f(X), that is defined by the following property: y is an element of f(X) if and only if there is an x in X such

that f(x) = y.

If f(x) = x^2 where x can be any real number, then the domain is all real numbers (denoted \mathbb{R}), and the range is the set of

nonnegative real numbers.
```







POLYNOMIAL FUNCTION: Given a polynomial expression in one variable, a *polynomial function in one variable* is a function $f: \mathbb{R} \to \mathbb{R}$ such that for each real number x in the domain, f(x) is the value found by substituting the number x into all instances of the variable symbol in the polynomial expression and evaluating.

It can be shown that if a function $f: \mathbb{R} \to \mathbb{R}$ is a polynomial function, then there is some nonnegative integer n and collection of real numbers $a_0, a_1, a_2, \dots, a_n$ with $a_n \neq 0$ such that the function satisfies the equation

 $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0,$

for every real number x in the domain, which is called the *standard form of the polynomial function*. The function $f(x) = 3x^3 + 4x^2 + 4x + 7$, where x can be any real number, is an example of a function written in standard form.

DEGREE OF A POLYNOMIAL FUNCTION: The *degree of a polynomial function* is the degree of the polynomial expression used to define the polynomial function.

The degree of $f(x) = 8x^3 + 4x^2 + 7x + 6$ is 3, but the degree of $g(x) = (x + 1)^2 - (x - 1)^2$ is 1 because when g is put into standard form, it is g(x) = 4x.

CONSTANT FUNCTION: A constant function is a polynomial function of degree 0. A constant function is of the form f(x) = c, for a constant c.

LINEAR FUNCTION: A *linear function* is a polynomial function of degree 1. A linear function is of the form f(x) = ax + b, for constants a and b with $a \neq 0$.

QUADRATIC FUNCTION: A *quadratic function* is a polynomial function of degree 2. A quadratic function is in *standard form* if it is written in the form $f(x) = ax^2 + bx + c$, for constants a, b, c with $a \neq 0$ and any real number x.

CUBIC FUNCTION: A *cubic function* is a polynomial function of degree 3. A cubic function is of the form $f(x) = ax^3 + bx^2 + cx + d$, for constants *a*, *b*, *c*, *d* with $a \neq 0$.

ZEROS OR ROOTS OF A FUNCTION: A zero (or root) of a function $f: \mathbb{R} \to \mathbb{R}$ is a number x of the domain such that f(x) = 0. A zero of a function is an element in the solution set of the equation f(x) = 0.

Lesson Summary

Given any two polynomial functions p and q, the solution set of the equation p(x)q(x) = 0 can be quickly found by solving the two equations p(x) = 0 and q(x) = 0 and combining the solutions into one set.

The number *a* is a zero of a polynomial function *p* with multiplicity *m* if the factored form of *p* contains $(x - a)^m$.

Exit Ticket (5 minutes)







Lesson 11

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Name

Date _____

Lesson 11: The Special Role of Zero in Factoring

Exit Ticket

Suppose that a polynomial function p can be factored into seven factors: (x - 3), (x + 1), and 5 factors of (x - 2). What are its zeros with multiplicity, and what is the degree of the polynomial? Explain how you know.









Exit Ticket Sample Solutions

Suppose that a polynomial function p can be factored into seven factors: (x - 3), (x + 1), and 5 factors of (x - 2). What are its zeros with multiplicity, and what is the degree of the polynomial? Explain how you know. Zeros: 3 with multiplicity 1; -1 with multiplicity 1; 2 with multiplicity 5 The polynomial has degree seven. There are seven linear factors as given above, so $p(x) = (x - 3)(x + 1)(x - 2)^5$. If the factors were multiplied out, the leading term would be x^7 , so the degree of p is 7.

Problem Set Sample Solutions

For Problems 1–4, find all solutions to the given equations. 1. (x-3)(x+2) = 03, -2 2. (x-5)(x+2)(x+3) = 05, -2, -3 3. (2x-4)(x+5) = 02, -5 4. (2x-2)(3x+1)(x-1) = 0 $1, -\frac{1}{2}, 1$ 5. Find four solutions to the equation $(x^2 - 9)(x^4 - 16) = 0$. 2, -2, 3, -36. Find the zeros with multiplicity for the function $p(x) = (x^3 - 8)(x^5 - 4x^3)$. We can factor p to give $p(x) = x^3(x-2)(x^2+2x+4)(x-2)(x+2) = x^3(x-2)^2(x+2)(x^2+2x+4)$. Then, 0 is a zero of multiplicity 3, -2 is a zero of multiplicity 1, and 2 is a zero of multiplicity 2. 7. Find two different polynomial functions that have zeros at 1, 3, and 5 of multiplicity 1. p(x) = (x-1)(x-3)(x-5) and $q(x) = (x^2+1)(x-1)(x-3)(x-5)$ 8. Find two different polynomial functions that have a zero at 2 of multiplicity 5 and a zero at -4 of multiplicity 3. $p(x) = (x-2)^5(x+4)^3$ and $q(x) = (x^2+1)(x-2)^5(x+4)^3$ 9. Find three solutions to the equation $(x^2 - 9)(x^3 - 8) = 0$. From Lesson 6, we know that (x - 2) is a factor of $(x^3 - 8)$, so three solutions are 3,-3, and 2.



Lesson 11: The Special Role of Zero in Factoring





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17. Consider the polynomial function S(x) = x⁷ + x⁶ - x⁵ - x⁴ + x³ + x² - x - 1.
a. Divide S by the divisor (x + 1) and rewrite in the form S(x) = (divisor)(quotient) + remainder. S(x) = (x + 1)(x⁶ - x⁴ + x² - 1) + 0
b. Evaluate S(-1). S(-1) = 0
18. Make a conjecture based on the results of Problems 14-17. It seems that the zeros a of a polynomial function correspond to factors (x - a) in the equation of the polynomial.











C Lesson 12: Overcoming Obstacles in Factoring

Student Outcomes

Students factor certain forms of polynomial expressions by using the structure of the polynomials.

Lesson Notes

Students have factored polynomial expressions in earlier lessons and in earlier courses. In this lesson, students explore further techniques for factoring polynomial expressions, including factoring by completing the square, by applying the quadratic formula, and by grouping. They apply these techniques to solve polynomial equations.

The idea of the greatest common factor (GCF) is important to this lesson. The teacher may want to consider displaying a GCF poster on the classroom wall for reference. Consider using some of the problem set exercises during the lesson to supplement the examples included here.

Classwork

Opening (4 minutes)

Consider the following polynomial equation.

$$(x^2 - 4x + 3)(x^2 + 4x - 5) = 0$$

Discuss the following questions in pairs or small groups:

- 1. What is the degree of this polynomial? How do you know?
- 2. How many solutions to this equation should there be? How do you know?
- 3. How might you begin to solve this equation?

We can solve this equation by factoring because we can solve each of the equations

$$x^2 - 4x + 3 = 0$$
$$x^2 + 4x - 5 = 0$$

There is no need to solve the whole way through; students completed a problem like this in Lesson 11. The idea is that students see that this can be done relatively quickly. The factored form of the original equation is

$$(x^{2} - 4x + 3)(x^{2} + 4x - 5) = (x - 1)(x - 3)(x - 1)(x + 5) = (x - 1)^{2}(x - 3)(x + 5) = 0,$$

and the three solutions are 1, 3, and -5.

However, consider the next example.

Example 1 (8 minutes)

Example 1

Find all real solutions to the equation $(x^2 - 6x + 3)(2x^2 - 4x - 7) = 0$.

Overcoming Obstacles in Factoring



Lesson 12:



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Scaffolding:

Ask struggling students to first solve the equation (x-3)(x-1) = 0. Point out that they have used the zero product property, and ask how that property applies to the given problem.



Allow students the opportunity to struggle with factoring these expressions, discuss with their neighbors, and reach the conclusion that neither expression can be factored with integer coefficients.

• We have discovered an obstacle to factoring. The expressions $x^2 - 6x + 3$ and $2x^2 - 4x - 7$ do not factor as readily as the examples from the previous lesson. Does anybody recall how we might factor them?

Students have completed the square in both Geometry and Algebra I, so give them an opportunity to recall the process.

- When a quadratic expression is not easily factorable, we can either apply a technique called *completing the square*, or we can use the *quadratic formula*. Let's factor the first expression by completing the square.
- We first create some space between the *x* term and the constant term:

$$x^2 - 6x + ___ - __ + 3 = 0.$$

• The next step is the key step. Take half of the coefficient of the *x* term, square that number, and add and subtract it in the space we created:

$$x^{2} - 6x + (-3)^{2} - (-3)^{2} + 3 = 0$$

$$x^{2} - 6x + 9 - 9 + 3 = 0$$

Discuss the following questions with the class, and give them the opportunity to justify this step.

- Why did we choose 9? Why did we both add and subtract 9? How does this help us solve the equation?
 - Adding 9 creates a perfect square trinomial in the first three terms.
 - Adding and subtracting 9 means that we have not changed the value of the expression on the left side of the equation.
 - Adding and subtracting 9 creates a perfect square trinomial $x^2 6x + 9 = (x 3)^2$.
- We cannot just add a number to an expression without changing its value. By adding 9 and subtracting 9, we have essentially added 0 using the additive identity property, which leads to an equivalent expression on the left-hand side of the equation and thus preserves solutions of the equation.
- This process creates a structure that allows us to factor the first three terms of the expression on the left side of the equation and then solve for the variable.

$$x^2 - 6x + 9 - 9 + 3 = 0$$
Perfect square trinomial

$$(x-3)^2 - 6 = 0$$

• Solving for *x*:

$$(x-3)^2 = 6$$

 $x-3 = \sqrt{6}$ or $x-3 = -\sqrt{6}$
 $x = 3 + \sqrt{6}$ or $x = 3 - \sqrt{6}$.

• Thus, we have found two solutions by setting the first quadratic expression equal to zero, completing the square, and solving the factored equation. Since the leading coefficient of $x^2 - 6x + 3$ is 1, we know from our work in Algebra I that the factored form is

$$x^{2} - 6x + 3 = \left(x - (3 + \sqrt{6})\right)\left(x - (3 - \sqrt{6})\right).$$



We can use the tabular method to give a visual representation of the process of completing the square. For example, the polynomial $x^2 - 6x$ can be represented as follows to illustrate the "missing" term.







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Let's repeat the process with the second equation. What is the first step to completing the square?

$$2x^2 - 4x - 7 = 0$$

Allow students an opportunity to suggest the first step to completing the square.

• We can only complete the square when the leading coefficient is 1, so our first step is to factor out the 2.

$$2\left(x^2 - 2x - \frac{7}{2}\right) = 0$$

Now we can complete the square with the expression inside the parentheses.

$$2\left(x^{2} - 2x + \underline{\qquad} - \underline{\qquad} + \frac{7}{2}\right) = 0$$

$$2\left(x^{2} - 2x + (-1)^{2} - (-1)^{2} - \frac{7}{2}\right) = 0$$

$$2\left(x^{2} - 2x + 1 - \frac{9}{2}\right) = 0$$

$$2\left((x - 1)^{2} - \frac{9}{2}\right) = 0$$

• Next, we divide both sides by 2.

$$(x-1)^2 - \frac{9}{2} = 0$$

~

• Finally, we solve for *x*.

$$(x-1)^{2} = \frac{9}{2}$$

$$x = 1 + \sqrt{\frac{9}{2}} \text{ or } x = 1 - \sqrt{\frac{9}{2}}$$

$$x = 1 + \frac{3\sqrt{2}}{2} \text{ or } x = 1 - \frac{3\sqrt{2}}{2}$$

Thus, we have found two more solutions to our original fourth-degree equation. We then have

$$2x^{2} - 4x - 7 = 2\left(x^{2} - 2x - \frac{7}{2}\right)$$
$$= 2\left(x - \left(1 + \frac{3\sqrt{2}}{2}\right)\right)\left(x - \left(1 - \frac{3\sqrt{2}}{2}\right)\right).$$

- Notice that we needed to multiply the factors by 2 to make the leading coefficients match.
- Finally, we have the factored form of our original polynomial equation:

$$(x^{2} - 6x + 3)(2x^{2} - 4x - 7) = 0 \text{ in factored form}$$
$$2(x - (3 + \sqrt{6}))(x - (3 - \sqrt{6}))\left(x - \left(1 + \frac{3\sqrt{2}}{2}\right)\right)\left(x - \left(1 - \frac{3\sqrt{2}}{2}\right)\right) = 0.$$







Lesson 12

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Lesson 12

- Thus, the solutions to the equation $(x^2 6x + 3)(2x^2 4x 7) = 0$ are the four values $3 + \sqrt{6}$, $3 \sqrt{6}$, $1 + \frac{3\sqrt{2}}{2}$, and $1 \frac{3\sqrt{2}}{2}$.
- Similarly, we could have applied the quadratic formula to find the solutions to each quadratic equation in the previous example. Recall the quadratic formula.

The two solutions to the quadratic equation $ax^2 + bx + c = 0$ are $\frac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $\frac{-b - \sqrt{b^2 - 4ac}}{2a}$.

Exercise 1 (6 minutes)

Exercise 1

Factor and find all real solutions to the equation $(x^2 - 2x - 4)(3x^2 + 8x - 3) = 0$.

Ask half of the students to apply the quadratic formula to solve $x^2 - 2x - 4 = 0$ and the other half to apply the quadratic formula to solve $3x^2 + 8x - 3 = 0$.

The quadratic formula gives solutions $1 + \sqrt{5}$ and $1 - \sqrt{5}$ for the first equation and -3 and $\frac{1}{3}$ for the second equation.

Since $1 + \sqrt{5}$ and $1 - \sqrt{5}$ are the two solutions to $x^2 - 2x - 4 = 0$ found by the quadratic formula, we know from work in Algebra I that $\left(x - (1 + \sqrt{5})\right)\left(x - (1 - \sqrt{5})\right) = x^2 - 2x - 4$. However, we need to be more careful when using the solutions to factor the second quadratic expression. The leading coefficient of $(x + 3)\left(x - \frac{1}{3}\right) = x^2 + \frac{8}{3}x - 1$ is 1, and the leading coefficient of $3x^2 + 8x - 3$ is 3, so we need to multiply our factors by 3:

$$3x^{2} + 8x - 3 = 3(x+3)\left(x - \frac{1}{3}\right)$$

Thus, the factored form of the original equation is

$$(x^{2} - 2x - 4)(3x^{2} + 8x - 3) = 3\left(x - \left(1 + \sqrt{5}\right)\right)\left(x - \left(1 - \sqrt{5}\right)\right)(x + 3)\left(x - \frac{1}{3}\right) = 0,$$

r solutions to $(x^{2} - 2x - 4)(3x^{2} + 8x - 3) = 0$ are $1 + \sqrt{5}$, $1 - \sqrt{5}$, -3 , and $\frac{1}{3}$.

To summarize, if we have a fourth-degree polynomial already factored into two quadratic expressions, we can try to factor the entire polynomial by completing the square on one or both quadratic expressions, or by using the quadratic formula to find the roots of the quadratic polynomials and then constructing the factored form of each quadratic polynomial.

Discussion (6 minutes)

and the fou

 We have overcome the obstacle of difficult-to-factor quadratic expressions. Let's look next at the obstacles encountered when attempting to solve a third-degree polynomial equation such as the following:

$$x^3 + 3x^2 - 9x - 27 = 0.$$

Scaffolding:

Allow students to generate ideas. Advanced students may benefit from significant independent time to attempt to solve the equation (MP.1).

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Lesson 12: Overcoming Obstacles in Factoring





How might we begin to solve this equation?

Allow students an opportunity to brainstorm as a class, in pairs, or in table groups. Students may note that coefficients are powers of 3 but may not be sure how that helps. Let them know they are seeing something important that they may be able to use. Stronger students might even try to group the components.

While we have made some interesting observations, we have not quite found a way to factor this expression. What if we know that x + 3 is one factor?

If students do not come up with polynomial division, point them in that direction through a numerical example: Suppose we want the factors of 210, and we know that one factor is 3. How do we find the other factors?

Have students perform the polynomial division

$$x+3)x^3+3x^2-9x-27$$

to find additional factors. Students may also use the tabular method discussed in earlier lessons in this module.

$$\frac{\chi^{2} + 0\chi - 9}{\chi + 3} \frac{\chi^{3} + 3\chi^{2} - 9\chi - 27}{\chi^{3} + 3\chi^{2}} - 9\chi - 27$$

$$- 9\chi - 27$$

$$- 9\chi - 27$$

$$0$$
So $\frac{\chi^{3} + 3\chi^{2} - 9\chi - 27}{\chi + 3} = \chi^{2} - 9$
and $\chi^{3} + 3\chi^{2} - 9\chi - 27 = (\chi + 3)(\chi^{2} - 9)$

• Since $x^3 + 3x^3 - 9x - 27 = (x + 3)(x^2 - 9)$, we know that

$$x^{3} + 3x^{2} - 9x - 27 = (x+3)(x-3)(x+3) = (x+3)^{2}(x-3).$$

- By the zero product property, the solutions to $x^3 + 3x^2 9x 27 = 0$ are -3 and 3.
- But, how do we start if we don't know any of the factors in advance?

Example 2 (6 minutes)

	Scaffolding:
Example 2	If needed, have students compute the GCF for some
Find all solutions to $x^3 + 3x^2 - 9x - 27 = 0$ by factoring the equation.	numerical examples such as
	GCF 75, 100 is 25
Let's start with our original equation $x^3 + 3x^2 - 9x - 27 = 0$. Is there a	GCF 72, 16, 24 is 8
greatest common factor (CCE) for all four terms on the left hand side we can	GCF 10, 35, 63 is 1.

greatest common factor (GCF) for all four terms on the left-hand side we can factor out?

 \square No, the GCF is 1.



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$$x^{3} + 3x^{2} - 9x - 27 = (x^{3} + 3x^{2}) - (9x + 27).$$

Lesson 12

ALGEBRA II

- Can we factor out a GCF from each set of parentheses independently?
 - Yes, x^2 can be factored out of the first piece and 9 out of the second.

Factor the GCF out of each part. Have students do as much of this work as possible.

$$x^{3} + 3x^{2} - 9x - 27 = x^{2}(x+3) - 9(x+3)$$

Do you notice anything interesting about the right side of the above equation?

- I noticed that x + 3 is a common factor.
- Since both terms have a factor of (x + 3), we have found a quantity that can be factored out.

$$x^3 + 3x^2 - 9x - 27 = (x+3)(x^2 - 9)$$

And as we saw above, we can take this one step further.

$$x^{3} + 3x^{2} - 9x - 27 = (x + 3)(x + 3)(x - 3)$$

- Because of the zero property, the original problem is now easy to solve because $x^3 + 3x^2 9x 27 = 0$ exactly when $(x + 3)^2(x - 3) = 0$. What are the solutions to the original equation?
 - The solutions to $x^3 + 3x^2 9x 27 = 0$ are x = -3 and x = 3.
- The process you just completed is often called *factoring by grouping*, and it works only on certain 3^{rd} degree polynomial expressions, such as $x^3 + 3x^2 9x 27$.

Exercise 2 (4 minutes)

Allow students to work in pairs or small groups on these exercises. Realize that there are two ways to group the terms that result in the same factored expression. Circulate around the room while students are working, and take note of any groups that are using a different approach. At the end of these exercises, ask students who grouped differently to share their method, and discuss as a class.

Γ	Exercise 2		
	Find all real solutions to $x^3 - 5x^2 - 4x + 20 = 0$.		
	$x^3 - 5x^2 - 4x + 20 = 0$	Alternate approach:	
	$x^2(x-5) - 4(x-5) = 0$	$x^3 - 5x^2 - 4x + 20 = 0$	
	$(x-5)(x^2-4)=0$	$x(x^2 - 4) - 5(x^2 - 4) = 0$	
	(x-5)(x-2)(x+2) = 0	$(x-5)(x^2-4)=0$	
	Thus, the solutions are 5, 2, and -2 .	(x-5)(x-2)(x+2) = 0	





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Exercise 3 (4 minutes)

Exercise 3	
Find all real solutions to $x^3 - 8x^2 - 2x + 16 = 0$.	
$x^3 - 8x^2 - 2x + 16 = 0$	$x^3 - 8x^2 - 2x + 16 = 0$
$x^2(x-8) - 2(x-8) = 0$	$x(x^2-2)-8(x^2-2)=0$
$(x-8)(x^2-2)=0$	$(x-8)(x^2-2)=0$
Thus, the solutions are $8,\sqrt{2},$ and $-\sqrt{2}.$	

Closing (2 minutes)

Ask students to summarize the important parts of the lesson in writing, to a partner, or as a class. Use this as an opportunity to informally assess understanding of the lesson. The following are some important summary elements.



Exit Ticket (5 minutes)









Name

Date _____

Lesson 12: Overcoming Obstacles in Factoring

Exit Ticket

Solve the following equation, and explain your solution method.

 $x^3 + 7x^2 - x - 7 = 0$







Exit Ticket Sample Solutions

Solve the following equation, and explain your solution method.

$$x^{3} + 7x^{2} - x - 7 = 0$$

$$x^{2}(x + 7) - (x + 7) = 0$$

$$(x + 7)(x^{2} - 1) = 0$$

$$(x + 7)(x - 1)(x + 1) = 0$$

The solutions are -7, 1, and -1. The equation was solved by factoring and by grouping. I grouped the four terms into two groups, and then factored the GCF from each group. I then factored out the common term (x + 7) from each group to find the factored form of the equation. I then applied the zero product property to find the solutions to the equation.

Problem Set Sample Solutions

1.	Solve	lve each of the following equations by completing the square.				
	a.	$x^2-6x+2=0$	$3+\sqrt{7},\ 3-\sqrt{7}$			
	b.	$x^2 - 4x = -1$	$2+\sqrt{3}, \ 2-\sqrt{3}$			
	c.	$x^2 + x - \frac{3}{4} = 0$	$\frac{1}{2'} - \frac{3}{2}$			
	d.	$3x^2 - 9x = -6$	2, 1			
	e.	$(2x^2 - 5x + 2)(3x^2 - 4x + 1) = 0$	$2, \frac{1}{2}, 1, \frac{1}{3}$			
	f.	$x^4 - 4x^2 + 2 = 0$	$\sqrt{2+\sqrt{2}}, -\sqrt{2+\sqrt{2}}, \sqrt{2-\sqrt{2}}, -\sqrt{2-\sqrt{2}}$			
2.	2. Solve each of the following equations using the quadratic formula.					
	a.	$x^2-5x-3=0$	$\frac{5}{2} + \frac{\sqrt{37}}{2}, \ \frac{5}{2} - \frac{\sqrt{37}}{2}$			
	b.	$(6x^2 - 7x + 2)(x^2 - 5x + 5) = 0$	$\frac{1}{2}, \frac{2}{3}, \frac{1}{2}(5+\sqrt{5}), \frac{1}{2}(5-\sqrt{5})$			
	c.	$(3x^2 - 13x + 14)(x^2 - 4x + 1) = 0$	2, $\frac{7}{3}$, 2 + $\sqrt{3}$, 2 - $\sqrt{3}$			

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ALGEBRA II

Not all of the expressions in the equations below can be factored using the techniques discussed so far in this 3. course. First, determine if the expression can be factored with real coefficients. If so, factor the expression, and find all real solutions to the equation. b. $3x^2 + 5x - 2 = 0$ $x^2 - 5x - 24 = 0$ а. *Can be factored:* (x - 8)(x + 3) = 0. *Can be factored:* (3x - 1)(x + 2) = 0. Solutions: 8, -3 Solutions: $\frac{1}{2}$, -2 $x^3 + 3x^2 - 2x + 6 = 0$ $x^2 + 2x + 4 = 0$ d. c. Cannot be factored with real number coefficients. Cannot be factored with real number coefficients. $2x^3 + x^2 - 6x - 3 = 0$ $x^3 + 3x^2 + 2x + 6 = 0$ f. e. *Can be factored:* $(x + 3)(x^2 + 2) = 0$. *Can be factored:* $(2x + 1)(x - \sqrt{3})(x + \sqrt{3}) =$ 0. Solution: -3 Solutions: $-\frac{1}{2}$, $\sqrt{3}$, $-\sqrt{3}$ $8x^3 - 12x^2 + 2x - 3 = 0$ h. $6x^3 + 8x^2 + 15x + 20 = 0$ g. Can be factored: $(2x - 3)(4x^2 + 1) = 0$. Can be factored: $(3x + 4)(2x^2 + 5) = 0$. Solution: $-\frac{4}{2}$ Solution: $\frac{3}{2}$ $4x^3 + 2x^2 - 36x - 18 = 0$ j. $x^2 - \frac{1}{2}x - \frac{15}{2} = 0$ i. Can be factored: 2(2x+1)(x-3)(x+3) = 0. Can be factored: $(x + \frac{5}{2})(x - 3) = 0.$ Solutions: $-\frac{1}{2}$, 3, -3 Solutions: $-\frac{5}{2}$, 3 Solve the following equations by bringing all terms to one side of the equation and factoring out the greatest 4. common factor. (x-2)(x-1) = (x-2)(x+1)a. (x-2)(x+1) - (x-2)(x-1) = 0(x-2)(x+1-(x-1))=0(x-2)(2) = 0*x* = 2 So, the only solution to (x - 2)(x - 1) = (x - 2)(x + 1) is 2. (2x+3)(x-4) = (2x+3)(x+5)b. (2x+3)(x-4) - (2x+3)(x+5) = 0(2x+3)(x-4-(x+5))=0(2x+3)(-9) = 0 $x = -\frac{3}{2}$ So, the only solution to (2x+3)(x-4) = (2x+3)(x+5) is $-\frac{3}{2}$.



Lesson 12: Overcoming Obstacles in Factoring

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(x-1)(2x+3) = (x-1)(x+2)c. (x-1)(2x+3) - (x-1)(x+2) = 0(x-1)(2x+3-(x+2))=0(x-1)(x+1) = 0x = 1 or x = -1The solutions to (x - 1)(2x + 3) = (x - 1)(x + 2) are 1 and -1. d. $(x^2 + 1)(3x - 7) = (x^2 + 1)(3x + 2)$ $(x^{2}+1)(3x-7) - (x^{2}+1)(3x+2) = 0$ $(x^{2}+1)(3x-7-(3x+2))=0$ $(x^2 + 1)(-9) = 0$ $x^2 + 1 = 0$ *There are no real number solutions to* $(x^2 + 1)(3x - 7) = (x^2 + 1)(3x + 2).$ $(x+3)(2x^2+7) = (x+3)(x^2+8)$ e. $(x+3)(2x^2+7) - (x+3)(x^2+8) = 0$ $(x+3)(2x^2+7-(x^2+8))=0$ $(x+3)(x^2-1)=0$ (x+3)(x-1)(x+1) = 0The three solutions to $(x + 3)(2x^2 + 7) = (x + 3)(x^2 + 8)$ are -3, -1, and 1. Consider the expression $x^4 + 1$. Since $x^2 + 1$ does not factor with real number coefficients, we might expect that 5. $x^4 + 1$ also does not factor with real number coefficients. In this exercise, we investigate the possibility of factoring $x^4 + 1$. Simplify the expression $(x^2 + 1)^2 - 2x^2$. а. $(x^2 + 1)^2 - 2x^2 = x^4 + 1$ Factor $(x^2 + 1)^2 - 2x^2$ as a difference of squares. b. $(x^{2}+1)^{2}-2x^{2}=((x^{2}+1)-\sqrt{2}x)((x^{2}+1)+\sqrt{2}x)$ Is it possible to factor $x^4 + 1$ with real number coefficients? Explain. c. Yes. $x^4 + 1 = ((x^2 + 1) - \sqrt{2}x)((x^2 + 1) + \sqrt{2}x)$ In an equivalent but more conventional form, we have $x^4 + 1 = (x^2 - \sqrt{2}x + 1)(x^2 + \sqrt{2}x + 1).$







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Student Outcomes

Students use the structure of polynomials to identify factors.

Lesson Notes

In previous lessons in this module, students practiced the techniques of factoring by completing the square, by applying the quadratic formula, and by grouping. In this lesson, students look for structure in more complicated polynomial expressions that allow factorization. But first, students review several factoring techniques; some they learned about in the last lesson, and others they learned about in previous classes.

Opening Exercise (8 minutes)

MP.7

In this exercise, students should begin to factor polynomial expressions by first analyzing their structure, a skill that is
developed throughout the lesson. Suggest that students work on their own for five minutes and then compare answers
with a neighbor; allow students to help each other out for an additional three minutes, if needed.

Оре	Opening Exercise			
Fac tho	Factor each of the following expressions. What similarities do you notice between the examples in the left column and those on the right?			
a.	$x^2 - 1$	b.	$9x^2 - 1$	
	(x-1)(x+1)		(3x-1)(3x+1)	
с.	$x^2 + 8x + 15$	d.	$4x^2 + 16x + 15$	
	(x+5)(x+3)		(2x+5)(2x+3)	
е.	$x^2 - y^2$	f.	$x^4 - y^4$	
	(x-y)(x+y)		$(x^2 - y^2)(x^2 + y^2)$	

Students should notice that the structure of each of the factored polynomials is the same; for example, the factored forms of part (a) and part (b) are nearly the same, except that part (b) contains 3x in place of the x in part (a). In parts (c) and (d), the factored form of part (d) contains 2x, where there is only an x in part (c). The factored form of part (f) is nearly the same as the factored form of part (e), with x^2 replacing x and y^2 replacing y.





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If students are unfamiliar with the difference of squares formula, work

examples, and ask them to look for

a - b

2

3

3

2

Students may benefit from teacher

 $a^2 - b^2$

8

15

21

20

through a table of numeric

a + b

4

5

7

10

Scaffolding:

patterns.

a 3 1

4

5

6 4

modeling.

b

1

2

Discussion (2 minutes)

The difference of two squares formula,

$$a^2 - b^2 = (a+b)(a-b),$$

can be used to factor an expression even when the two squares are not obvious.

Consider the following examples.

Example 1 (3 minutes)

Example 1 Write $9 - 16x^4$ as the product of two factors. $9-16x^4 = (3)^2 - (4x^2)^2$ $= (3 - 4x^2)(3 + 4x^2)$

Example 2 (3 minutes)

Example 2 Factor $4x^2y^4 - 25x^4z^6$. $4x^2y^4 - 25x^4z^6 = (2xy^2)^2 - (5x^2z^3)^2$ $= (2xy^2 + 5x^2z^3)(2xy^2 - 5x^2z^3)$ $= [x(2y^2 + 5xz^3)][x(2y^2 - 5xz^3)]$ $= x^2(2y^2 + 5xz^3)(2y^2 - 5xz^3)$

Have students discuss with each other the structure of each polynomial expression in the previous two examples and how it helps to factor the expressions.

There are two terms that are subtracted, and each term can be written as the square of an expression.

Example 3 (3 minutes)

Consider the quadratic polynomial expression $9x^2 + 12x - 5$. We can factor this expression by considering 3x as a single quantity as follows:

$$9x^2 + 12x - 5 = (3x)^2 + 4(3x) - 5.$$

Ask students to suggest the next step in factoring this expression.

Now, if we rename u = 3x, we have a quadratic expression of the form $u^2 + 4u - 5$, which we can factor

 $u^{2} + 4u - 5 = (u - 1)(u + 5).$

Replacing u by 3x, we have the following form of our original expression:

$$9x^2 + 12x - 5 = (3x - 1)(3x - 5).$$









Exercise 1 (4 minutes)

Allow students to work in pairs or small groups on the following exercises.

Exercise 1 1. Factor the following expressions:

```
a. 4x^2 + 4x - 63

4x^2 + 4x - 63 = (2x)^2 + 2(2x) - 63

= (2x + 9)(2x - 7)

b. 12y^2 - 24y - 15

12y^2 - 24y - 15 = 3(4y^2 - 8y - 5)

= 3((2y)^2 - 4(2y) - 5)

= 3(2y + 1)(2y - 5)
```

Example 4 (10 minutes)

Use the example of factoring $x^3 - 8$ to scaffold the discussion of factoring $x^3 + 8$. Students should be pretty familiar by now with factors of $x^3 - 8$. Let them try the problem on their own to check their understanding.

- Suppose we want to factor $x^3 8$.
- Do you see anything interesting about this expression?

If they do not notice it, guide them toward both terms being perfect cubes.

- We can rewrite $x^3 8$ as $x^3 2^3$.
- Guess a factor.
 - Anticipate that they will suggest x 2 and x + 2 as possible factors, or guide them to these suggestions.

Ask half of the students to divide $\frac{x^3-8}{x-2}$ and the other half to divide $\frac{x^3-8}{x+2}$. They should discover that x-2 is a factor of x^3-8 , but x+2 is not.





 $\frac{\chi^3 + 2\chi^2}{-2\chi^2 + 0\chi + 8}$

 $\frac{-2\chi^2 - 4\chi}{4\chi + 8}$

4x+8



ALGEBRA II

• Use the results of the previous step to factor $x^3 - 8$.

$$x^{3} - 8 = (x - 2)(x^{2} + 2x + 4)$$

- Repeat the above process for $x^3 27$.
 - $x^3 27 = (x 3)(x^2 + 3x + 9)$
- Make a conjecture about a rule for factoring $x^3 a^3$.

•
$$x^3 - a^3 = (x - a)(x^2 + ax + a^2)$$

- Verify the conjecture: Multiply out $(x a)(x^2 + ax + a^2)$ to establish the identity for factoring a difference of cubes.
- While we can factor a difference of squares such as the expression $x^2 9$, we cannot similarly factor a sum of squares such as $x^2 + 9$. Do we run into a similar problem when trying to factor a sum of cubes such as $x^3 + 8$?

Again, ask students to propose potential factors of $x^3 + 8$. Lead students to x + 2 if they do not guess it automatically.

Work through the polynomial long division for $\frac{x^3+8}{x+2}$ as shown.	$\chi^2 - 2\chi + 4$
Conclude that $x^3 + 8 = (x + 2)(x^2 - 2x + 4)$.	$\chi_{+2})\chi^{3} + 0\chi^{2} + 0\chi + 8$

• Make a conjecture about a rule for factoring $x^3 + a^3$.

$$x^3 + a^3 = (x + a)(x^2 - ax + a^2)$$

• Verify the conjecture: Multiply out the expression $(x + a)(x^2 - ax + a^2)$ to establish the identity for factoring a sum of cubes.

Exercises 2–4 (5 minutes)

Exe	rcises 2–4		
Factor each of the following, and show that the factored form is equivalent to the original expression.			
2.	$a^3 + 27$		
3.	$(a+3)(a^2-3a+9)$ x^3-64 $(x-4)(x^2+4x+16)$	Scaffolding: Ask advanced students to generate their own factoring problems using the structure of $a^3 + b^3$ or $a^3 - b^3$.	
4.	$2x^3 + 128$		
	$2(x^3 + 64) = 2(x + 4)(x^2 - 4x + 16)$		

Closing (2 minutes)

Ask students to summarize the important parts of the lesson in writing, to a partner, or as a class. Use this as an opportunity to informally assess understanding of the lesson. The following are some important summary elements.









Exit Ticket (5 minutes)





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Name

Date _____

Lesson 13: Mastering Factoring

Exit Ticket

1. Factor the following expression, and verify that the factored expression is equivalent to the original: $4x^2 - 9a^6$

2. Factor the following expression, and verify that the factored expression is equivalent to the original: $16x^2 - 8x - 3$





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Exit Ticket Sample Solutions

Factor the following expression, and verify that the factored expression is equivalent to the original: 4x² - 9a⁶
 (2x - 3a³)(2x + 3a³) = 4x² + 6a³x - 6a³x - 9a⁶
 = 4x² - 9a⁶
 = 4x² - 9a⁶

 Factor the following expression, and verify that the factored expression is equivalent to the original: 16x² - 8x - 3
 (4x - 3)(4x + 1) = 16x² + 4x - 12x - 3
 = 16x² - 8x - 3

Problem Set Sample Solutions

1.	lf po	If possible, factor the following expressions using the techniques discussed in this lesson.					
	a.	$25x^2 - 25x - 14$	g.	$9x^2 - 25y^4z^6$			
		(5x-7)(5x+2)		$(3x-5y^2z^3)(3x+5y^2z^3)$			
	b.	$9x^2y^2 - 18xy + 8$	h.	$36x^6y^4z^2 - 25x^2z^{10}$			
		(3xy-4)(3xy-2)		$x^2 z^2 (6x^2 y^2 - 5z^4) (6x^2 y^2 + 5z^4)$			
	c.	$45y^2 + 15y - 10$	i.	$4x^2 + 9$			
		5(3y+2)(3y-1)		Cannot be factored.			
	d.	$y^6 - y^3 - 6$	j.	$x^4 - 36$			
		$(y^3 - 3)(y^3 + 2)$		$(x-\sqrt{6})(x+\sqrt{6})(x^2+6)$			
	e.	$x^3 - 125$	k.	$1+27x^9$			
		$(x-5)(x^2+5x+25)$		$(1+3x^3)(1-3x^3+9x^6)$			
	f.	$2x^4 - 16x$	I.	$x^3y^6 + 8z^3$			
		$2x(x-2)(x^2+2x+4)$		$(xy^2 + 2z)(x^2y^4 - 2xy^2z + 4z^2)$			





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Consider the polynomial expression $y^4 + 4y^2 + 16$. 2. Is $y^4 + 4y^2 + 16$ factorable using the methods we have seen so far? а. No. This will not factor into the form $(y^2 + a)(y^2 + b)$ using any of our previous methods. Factor $y^6 - 64$ first as a difference of cubes, and then factor completely: $(y^2)^3 - 4^3$. b. $y^6 - 64 = (y^2 - 4)(y^4 + 4y^2 + 16)$ $= (y-2)(y+2)(y^4+4y^2+16)$ Factor $y^6 - 64$ first as a difference of squares, and then factor completely: $(y^3)^2 - 8^2$. c. $v^6 - 64 = (v^3 - 8)(v^3 + 8)$ $= (y-2)(y^2+2y+4)(y+2)(y^2-2y+4)$ $= (y-2)(y+2)(y^2-2y+4)(y^2+2y+4)$ Explain how your answers to parts (b) and (c) provide a factorization of $y^4 + 4y^2 + 16$. d. Since $y^6 - 64$ can be factored two different ways, those factorizations are equal. Thus we have $(y-2)(y+2)(y^4+4y^2+16) = (y-2)(y+2)(y^2-2y+4)(y^2+2y+4).$ If we specify that $y \neq 2$ and $y \neq -2$, we can cancel the common terms from both sides: $(y^4 + 4y^2 + 16) = (y^2 - 2y + 4)(y^2 + 2y + 4).$ Multiplying this out, we see that $(y^2 - 2y + 4)(y^2 + 2y + 4) = y^4 + 2y^3 + 4y^2 - 2y^3 - 4y^2 - 8y + 4y^2 + 8y + 16$ = $y^4 + 4y^2 + 16$ for every value of y. If a polynomial can be factored as either a difference of squares or a difference of cubes, which formula e. should you apply first, and why? Based on this example, a polynomial should first be factored as a difference of squares and then as a difference of cubes. This will produce factors of lower degree. 3. Create expressions that have a structure that allows them to be factored using the specified identity. Be creative, and produce challenging problems! Difference of squares а. $x^{14}y^4 - 225z^{10}$ Difference of cubes b. $27x^9y^6 - 1$ Sum of cubes c. $x^{6}z^{3} + 64y^{12}$





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Student Outcomes

- Students use the factored forms of polynomials to find zeros of a function.
- Students use the factored forms of polynomials to sketch the components of graphs between zeros.

Lesson Notes

In this lesson, students use the factored form of polynomials to identify important aspects of the graphs of polynomial functions and, therefore, important aspects of the situations they model. Using the factored form, students identify zeros of the polynomial (and thus *x*-intercepts of the graph of the polynomial function) and see how to sketch a graph of the polynomial functions by examining what happens between the *x*-intercepts. They are also introduced to the concepts of relative minima and maxima and determining the possible degree of the polynomial by noting the number of relative extrema by looking at the graph of a function. A relative maximum (or minimum) is a property of a function that is visible in its graph. A relative maximum occurs at an *x*-value, *c*, in the domain of the function, and the relative maximum value is the corresponding function value at *c*. If a relative maximum of a function *f* occurs at *c*, then (c, f(c)) is a relative maximum point. As an example, if (10,300) is a relative extrema, however, *relative maximum* is often used informally to refer to either a relative maximum at *c*, a relative maximum value, or a relative maximum point when the context is clear. Definitions of relevant vocabulary are included at the end of the lesson

MP.5 & MP.7

MP.3

The use of a graphing utility is recommended for some examples in this lesson to encourage students to focus on understanding the structure of the polynomials without the tedium of repeated graphing by hand.

Opening Exercise (10 minutes)

Prompt students to answer part (a) of the Opening Exercise independently or in pairs before continuing with the scaffolded questions.

Opening Exercise

An engineer is designing a roller coaster for younger children and has tried some functions to model the height of the roller coaster during the first 300 yards. She came up with the following function to describe what she believes would make a fun start to the ride:

$$H(x) = -3x^4 + 21x^3 - 48x^2 + 36x,$$

where H(x) is the height of the roller coaster (in yards) when the roller coaster is 100x yards from the beginning of the ride. Answer the following questions to help determine at which distances from the beginning of the ride the roller coaster is at its lowest height.

a. Does this function describe a roller coaster that would be fun to ride? Explain.

Yes, the roller coaster quickly goes to the top and then drops you down. This looks like a fun ride.

No, I don't like roller coasters that climb steeply, and this one goes nearly straight up.

Scaffolding:

- Consider beginning the class by reviewing graphs of simpler functions modeling simple roller coasters, such as G(x) = -x² + 4x.
- A more visual approach may be taken by first describing and analyzing the graph of *H* before connecting each concept to the algebra associated with the function. Pose questions such as *When is* the roller coaster going up? Going down? How many times does the roller coaster touch the bottom?



Lesson 14: Graphing Factored Polynomials















Discussion (4 minutes)

By manipulating a polynomial function into its factored form, we can identify the zeros of the function as well as identify the general shape of the graph. Thinking about the Opening Exercise, what else can we say about the polynomial function and its graph?

- The degree of the polynomial function *H* is 4. How can you find the degree of the function from its factored form?
 - Add the highest degree term from each factor:
 - −3 is a degree 0 factor
 - x is degree 1 factor
 - x 3 is degree 1 factor
 - $(x-2)^2$ is a degree 2 factor, since $(x-2)^2 = (x-2)(x-2)$.
 - Thus, 0 + 1 + 1 + 2 = 4.
- How many *x*-intercepts does the graph of the polynomial function have?
 - For this graph, there are three: (0,0), (2,0), and (3,0).

You may want to include a discussion that the zeros of a function correspond to the x-intercepts of the graph of the function.

- Note that there are four factors, but only three *x*-intercepts. Why is that?
 - Two of the factors are the same.

Remind students that the x-intercepts of the graph of y = f(x) are solutions to the equation f(x) = 0. Values of r that satisfy f(r) = 0 are called zeros (or roots) of the function. Some of these zeros may be repeated.

- Can you make one change to the polynomial function such that the new graph would have four *x*-intercepts?
 - Change one of the (x 2) factors to (x 1), for example.

Example 1 (10 minutes)

Students are now going to examine a few polynomial functions in factored form and compare the zeros of the function to the graph of the function on the calculator. Help students with part (a), and ask them to do part (b) on their own.

Scaffolding:

Encourage struggling learners to graph the original and the factored forms using a graphing utility to confirm that they are the same.

Scaffolding:

- For advanced learners, consider challenging students to construct a variety of functions to meet different criteria such as three factors and no *x*-intercepts or four factors with two *x*-intercepts.
- Students may enjoy challenging each other by trying to guess the equation that goes with the graph of their classmates.











Graph each of the following polynomial functions. What are the function's zeros (counting multiplicities)? What are the solutions to f(x) = 0? What are the *x*-intercepts to the graph of the function? How does the degree of the polynomial function compare to the *x*-intercepts of the graph of the function?



Before graphing the next equation, ask students where they think the graph of f will cross the x-axis and how the repeated factor will affect the graph. After graphing, students may need to trace near x = -3 depending on the graphing window to obtain a clear picture of the x-intercept.



By now, students should have an idea of what to expect in part (c). It may be worth noting the differences in the end behavior of the graphs, which will be explored further in Lesson 15. Discuss the degree of each polynomial.









• Why is the factor $x^2 + 1$ never zero and how does this affect the graph of f?

(At this point in the module, all polynomial functions are defined from the real numbers to the real numbers; hence, the functions can have only real number zeros. We will extend polynomial functions to the domain of complex numbers later, and then it will be possible to consider complex solutions to a polynomial equation.)

For real numbers x, the value of x^2 is always greater than or equal to zero, so $x^2 + 1$ will always be strictly greater than zero. Thus, $x^2 + 1 \neq 0$ for all real numbers x. Since there can be no x-intercept from this factor, the graph of f can have at most two x-intercepts.

If there is time, consider graphing the functions for parts (e)–(h) on the board and asking students to match the functions to the graphs. Encourage students to use a graphing utility to graph their guesses, talk about the differences between guesses and the actual graph, and what may cause them in each case.















Discussion (1 minutes)

Ask students to summarize what they have learned so far, either in writing or with a partner. Check for understanding of the concepts, and help students reach the following conclusions if they do not do so on their own.

- The x-intercepts in the graph of a function correspond to the solutions to the equation f(x) = 0 and correspond to the number of distinct zeros of the function (but the x-intercepts do not help us to determine the multiplicity of a given zero).
- The graph of a polynomial function of degree *n* has at most *n x*-intercepts but may have fewer.
- A polynomial function whose graph has *m x*-intercepts is at least a degree *m* polynomial.

Example 2 (8 minutes)

Lead students through the questions in order to arrive at a sketch of the final graph. The main point of this exercise is that if students know the *x*-intercepts of a polynomial function, then they can sketch a fairly accurate graph of the function by just checking to see if the function is positive or negative at a few points. They are not graphing by plotting points and connecting the dots but by applying properties of polynomial functions.

Give time for students to work through parts (a) and (b) in pairs or small groups before continuing with the discussion in parts (c)-(i). When sketching the graph in part (j), it is important to let students know that we cannot pinpoint exactly the high and low points on the graph—the relative maximum and relative minimum points. For this reason, omit a scale on the *y*-axis in the sketch.

Example 2

Consider the function $f(x) = x^3 - 13x^2 + 44x - 32$.

- a. Use the fact that x 4 is a factor of f to factor this polynomial.
 - Using polynomial division and then factoring, $f(x) = (x-4)(x^2-9x+8) = (x-4)(x-8)(x-1)$.
 - b. Find the *x*-intercepts for the graph of *f*.
 - The x-intercepts are 1, 4, and 8.



















Discussion (6 minutes)

- Let's examine the graph of $f(x) = x^3 13x^2 + 44x 32$ for $1 \le x \le 4$. Is there a number *c* in that interval where the value f(c) is greater than or equal to any other value of the function on that interval? Do we know exactly where that is?
 - There is a value of c such that f(c) that is greater than or equal to the other values. It seems that 2 < c < 2.5, but we do not know its exact value.

It could be mentioned that the exact value of c can be found exactly using calculus, but this is a topic for another class. For now, point out that the relative maximum or relative minimum point of a quadratic function can always be found— the only one is the vertex of the parabola.

If such a number *c* exists, then the function has a *relative maximum* at *c*. The *relative maximum value*, f(c), may not be the greatest overall value of the function, but there is an open interval around *c* so that for every *x* in that interval, $f(x) \le f(c)$. That is, for values of *x* near *c* (where "near" is a relative term), the point (x, f(x)) on the graph of *f* is not higher than (c, f(c)).

Scaffolding:

For English language learners, the term *relative* may need some additional instruction and practice to help differentiate it from other uses of this word.

It may help to think of the other points in the interval containing the relative maximum as all being related, and of all the relatives present, *c* is the value that gives the highest function value.

- Similarly, a function f has a *relative minimum* at d if there is an open interval around d so that for every x in that interval, $f(x) \ge f(d)$. That is, for values of x near d, the point (x, f(x)) on the graph of f is not lower than the point (d, f(d)). In this case, the *relative minimum value* is f(d).
- Show the relative maxima and relative minima on the graph. The image below clarifies the distinction between the relative maximum point and the relative minimum value. Point out that there are values of the function that are larger than f(c), such as f(9), but that f(c) is the highest value among the "neighbors" of c.



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The precise definitions of relative maxima and relative minima are listed in the glossary of terms for this lesson. These definitions are new to students, so it is worth going over them at the end of the lesson. Reiterate to students that if a relative maximum occurs at a value c, then that the relative maximum point is the point (c, f(c)) on the graph, and the relative maximum value is the y-value of the function at that point, f(c). Analogous definitions hold for relative minimum, relative minimum value, and relative minimum point.









What observations can we make from this information?

The observations made in the previous examples do not hold for these examples, so it is difficult to determine from the degree of the polynomial function the number of relative maximum and minimum points in the graph of the function. What we can say is that for a degree n polynomial function, there are at most n-1 relative maxima and minima. You can also think about the information you can get from a graph. If a graph of a polynomial function has n relative maximum and minimum points, you can say that the degree of the polynomial is at least n + 1.

Closing (1 minute)

- By looking at the factored form of a polynomial, we can identify important characteristics of the graph such as x-intercepts and degree of the function, which in turn allow us to develop a sketch of the graph.
- A polynomial function of degree *n* may have up to *n x*-intercepts.
- A polynomial function of degree n may have up to n-1 relative maxima and minima.

Relevant Vocabulary

INCREASING/DECREASING: Given a function f whose domain and range are subsets of the real numbers and I is an interval contained within the domain, the function is called increasing on the interval I if

 $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ in *I*.

It is called *decreasing on the interval I* if

 $f(x_1) > f(x_2)$ whenever $x_1 < x_2$ in *I*.

RELATIVE MAXIMUM: Let f be a function whose domain and range are subsets of the real numbers. The function has a relative maximum at c if there exists an open interval I of the domain that contains c such that

 $f(x) \le f(c)$ for all x in the interval I.

If f has a relative maximum at c, then the value f(c) is called the *relative maximum value*.

RELATIVE MINIMUM: Let f be a function whose domain and range are subsets of the real numbers. The function has a relative minimum at c if there exists an open interval I of the domain that contains c such that

 $f(x) \ge f(c)$ for all x in the interval I.

If f has a relative minimum at c, then the value f(c) is called the *relative minimum value*.

GRAPH OF f: Given a function f whose domain D and the range are subsets of the real numbers, the graph of f is the set of ordered pairs in the Cartesian plane given by

 $\{(x,f(x))\mid x\in D\}.$

GRAPH OF y = f(x): Given a function f whose domain D and the range are subsets of the real numbers, the graph of y = f(x) is the set of ordered pairs (x, y) in the Cartesian plane given by

 $\{(x, y) \mid x \in D \text{ and } y = f(x)\}.$









A polynomial of degree n may have up to n x-intercepts and up to n-1 relative maximum/minimum points.

The function f has a relative maximum at c if there is an open interval around c so that for all x in that interval, $f(x) \le f(c)$. That is, looking near the point (c, f(c)) on the graph of f, there is no point higher than (c, f(c)) in that region. The value f(c) is a relative maximum value.

The function f has a relative minimum at d if there is an open interval around d so that for all x in that interval, $f(x) \ge f(d)$. That is, looking near the point (d, f(d)) on the graph of f, there is no point lower than (d, f(d)) in that region. The value f(d) is a relative minimum value.

The plural of maximum is maxima, and the plural of minimum is minima.

Exit Ticket (5 minutes)









Name

Date _____

Lesson 14: Graphing Factored Polynomials

Exit Ticket

Sketch a graph of the function $f(x) = x^3 + x^2 - 4x - 4$ by finding the zeros and determining the sign of the function between zeros. Explain how the structure of the equation helps guide your sketch.









Exit Ticket Sample Solutions

Sketch a graph of the function $f(x) = x^3 + x^2 - 4x - 4$ by finding the zeros and determining the sign of the function between zeros. Explain how the structure of the equation helps guide your sketch. f(x) = (x+1)(x+2)(x-2)Zeros: -1, -2, 2 For x < -2: f(-3) = -10, so the graph is below the x-axis on this interval. For -2 < x < -1: f(-1.5) = 0.875, so the graph is above the *x*-axis on this interval. *For* – 1 < *x* < 2: f(0) = -4, so the graph is below the x-axis on 3.75 this interval. *For x* > 2*:* f(3) = 20, so the graph is above the x-axis on this interval. -7.5

Problem Set Sample Solutions

1. For each function below, identify the largest possible number of *x*-intercepts and the largest possible number of relative maxima and minima based on the degree of the polynomial. Then use a calculator or graphing utility to graph the function and find the actual number of *x*-intercepts and relative maxima and minima.

a.
$$f(x) = 4x^3 - 2x + 1$$

b.
$$g(x) = x^7 - 4x^5 - x^3 + 4x$$

c.
$$h(x) = x^4 + 4x^3 + 2x^2 - 4x + 2$$

Function	Largest number of <i>x</i> -intercepts	Largest number of relative max/min	Actual number of <i>x</i> -intercepts	Actual number of relative max/min
a. <i>f</i>	3	2	1	2
b. <i>g</i>	7	6	5	4
c. <i>h</i>	4	3	0	3







Lesson 14 M1

























Lesson 15: Structure in Graphs of Polynomial Functions

Student Outcomes

Students graph polynomial functions and describe end behavior based upon the degree of the polynomial.

Lesson Notes

So far in this module, students have practiced factoring polynomials using several techniques and examined how they can use the factored form of the polynomial to identify interesting characteristics of the graphs of these functions. In this lesson, students continue exploring graphs of polynomial functions in order to identify how the degree of the polynomial influences the end behavior of these graphs. They also discuss how to identify *y*-intercepts of the graphs of polynomial functions and are given an opportunity to construct viable arguments and critique the reasoning of others in the Opening Exercise (MP.3).

Opening Exercise (8 minutes)

Opening Exercise

Sketch the graph of $f(x) = x^2$. What will the graph of $g(x) = x^4$ look like? Sketch it on the same coordinate plane. What will the graph of $h(x) = x^6$ look like?

Have students recall and sketch the graph of $f(x) = x^2$. Discuss the characteristics of the graph, where the x-intercept is, and why the graph stays above the x-axis on either side of the x-intercept.

In pairs or in groups, have them discuss or write what they think the graph of $g(x) = x^4$ will look like and how they think it compares to the graph of $f(x) = x^2$. Once they do so, they should sketch their idea of the graph of g on top of the graph of f. Discuss with students what they have sketched, and emphasize the similarities between the two graphs.

Since $g(x) = (x^2)^2$, g(x) will increase faster as x increases than f(x) does. Both graphs pass through (0, 0). The basic shapes are the same, but near the origin the graph of g is flatter than the graph of f.

MP.3

Finally, in pairs or in groups, have students discuss or write what they think the graph of $h(x) = x^6$ will look like and how they think it will compare to graphs of f and g. Once they do so, students should sketch on the same graph the previous two graphs. Again, discuss graphs with students, and emphasize the similarities between graphs.

Since $h(x) = x^2 \cdot x^2 \cdot x^2$, the graph of h again passes through the origin. Since we are squaring and multiplying by squares, the graph of h should look about the same as the graphs of f and g but increase even faster and be even flatter near the origin.



Structure in Graphs of Polynomial Functions







Using a graphing utility, have students graph all three functions simultaneously to confirm their sketches.



Discussion (5 minutes)

Use the graphs from the Opening Exercise to frame the following discussion about end behavior.

Ask students to compare and describe the behavior of the value f(x) as the absolute value of x increases without bound. Introduce the term end behavior as a way to talk about the function and what happens to its graph beyond the bounded region of the coordinate plane that is drawn on paper. That is, the end behavior is a way to describe what happens to the function as x approaches positive and negative infinity without having to draw the graph.

Note to teacher: It is important to note that end behavior cannot be given a precise mathematical definition until the concept of a limit is introduced in calculus. To get around this difficulty, most high school textbooks draw pictures and state things like, "As $x \to \infty$, $f(x) \to \infty$." We do this also, but it is important to carefully describe to students the meaning of the phrase, "As x approaches positive infinity," before using the phrase (or its symbol version) to describe end behavior. That is because the phrase appears to mean that the symbol x is literally "moving along the number line to the right." Not true! Recall that a variable is just a placeholder for which a number can be substituted (think of a blank or box used in Grade 2 equations) and, therefore, does not actually move or vary.

The phrase, "As $x \to \infty$," can be profitably described as a *process* by which the user of the phrase thinks of repeatedly substituting larger and larger positive numbers in for x, each time performing whatever calculation is required by the problem for that number (which in this lesson is finding the value of the function).

This is how mathematicians often use the phrase even though the precise definition of limit removes any need to think of a limit as a process.

- **END BEHAVIOR (description):** Let f be a function whose domain and range are subsets of the real numbers. The *end behavior of a function* f is a description of what happens to the values of the function
 - as x approaches positive infinity, and
 - as x approaches negative infinity.



Structure in Graphs of Polynomial Functions



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Help students understand the description of end behavior using the following picture.



Ask students to make a generalization about the end behavior of polynomials of even degree in writing individually or with a partner. They should conclude that an even degree polynomial function has the same end behavior as $x \to \infty$ and as $x \to -\infty$. After students have generalized the end behavior, have them create their own graphic organizer like the following.



L5: Structure in Graphs of Polynomial Functions









If students suspect that end behavior of a polynomial function with even degree will always increase, then suggest examining the graphs of $f(x) = 1 - x^2$ and $g(x) = -x^4$.

Example 1 (8 minutes)

Students are now going to look at a new set of functions but ask similar questions to those asked in the Opening Exercise.

Example 1

Sketch the graph of $f(x) = x^3$. What will the graph of $g(x) = x^5$ look like? Sketch this on the same coordinate plane. What will the graph of $h(x) = x^7$ look like? Sketch this on the same coordinate plane.

Have students recall and sketch the graph of $f(x) = x^3$. Discuss the characteristics of the graph, where the x-intercept is, and why the graph is above the x-axis for x > 0 and below the x-axis for x < 0.

In pairs or in groups, have students discuss or write what they think the graph of $g(x) = x^5$ will look like and how it will relate to the graph of $f(x) = x^3$. They should sketch their results on top of the original graph of f. Discuss with students what they have sketched, and emphasize the similarities to the graph of $f(x) = x^3$.

Finally, in pairs or in groups, have students discuss or write what they think the graph of $h(x) = x^7$ will look like and how it will relate to the graphs of f and g. They should sketch on the same graph they used with the previous two graphs. Again, discuss the graphs with students, and emphasize the similarities between graphs.



MP.3







Using a graphing utility, have students graph all three functions simultaneously to confirm their sketches.



Ask students to compare and describe the behavior of the value of f(x) as the absolute value of x increases without bound. Guide them to use the terminology of the end behavior of the function.

Ask students to make a generalization about the end behavior of polynomials of odd degree individually or with a partner. After students have generalized the end behavior, have them create their own graphic organizer like the following.





Lesson 15:

: Structure in Graphs of Polynomial Functions





If students suspect that polynomial functions with odd degree always have the value of the function increase as x increases, then suggest examining a function with a negative leading coefficient, such as f(x) = 4 - x or $g(x) = -x^3$.

- How do these graphs differ from those in the Opening Exercise? Why are they different?
 - ^a Students may talk about how the Opening Exercise graphs stay above the *x*-axis while in this example the graphs cut through the *x*-axis. Guide students as necessary to concluding that the end behavior of even-degree polynomial functions is that both ends both approach positive infinity or both approach negative infinity while the end behavior of odd-degree polynomial functions is that the behavior as $x \to \infty$ is opposite of the behavior as $x \to -\infty$.

Exercise 1 (8 minutes)

Keeping the results of the examples above in mind, have students work with partners or in groups to answer the following questions.

Exercise 1	
a.	Consider the following function, $f(x) = 2x^4 + x^3 - x^2 + 5x + 3$, with a mixture of odd and even degree terms. Predict whether its end behavior will be like the functions in the Opening Exercise or more like the functions from Example 1. Graph the function f using a graphing utility to check your prediction.
	Students see that this function acts more like the even-degree monomial functions from the Opening Exercise.
b.	Consider the following function, $f(x) = 2x^5 - x^4 - 2x^3 + 4x^2 + x + 3$, with a mixture of odd and even degree terms. Predict whether its end behavior will be like the functions in the Opening Exercise or more like the functions from Example 1. Graph the function f using a graphing utility to check your prediction.
	Students see that this function acts more like odd-degree monomial functions from Example 1. They can draw a conclusion such as that the function behaves like the highest degree term.
c.	Thinking back to our discussion of x -intercepts of graphs of polynomial functions from the previous lesson, sketch a graph of an even-degree polynomial function that has no x -intercepts.
	Students may draw the graph of a quadratic function that stays above the x-axis such as the graph of $f(x) = x^2 + 1$.
d.	Similarly, can you sketch a graph of an odd-degree polynomial function with no x -intercepts?
	Have students work in pairs or groups and discover that because of the "cut through" nature of graphs of odd-degree polynomial function there is always an x -intercept.
	Conclusion: Graphs of odd-powered polynomial functions always have an x-intercept, which means that odd-degree polynomial functions always have at least one zero (or root) and that polynomial functions of odd-degree always have opposite end behaviors as $x \to \infty$ and $x \to -\infty$.

Have students conclude that the graphs of odd-degree polynomial functions always have at least one x-intercept and so the functions always have at least one zero. The graphs of even-degree polynomial functions may or may not have x-intercepts.



Structure in Graphs of Polynomial Functions



Exercise 2 (8 minutes)

In this exercise, students use what they learned today about end behavior to determine whether or not the polynomial function used to model the data has an even or odd degree.

Exercise 2 The Center for Transportation Analysis (CTA) studies all aspects of transportation in the United States, from energy and environmental concerns to safety and security challenges. A 1997 study compiled the following data of the fuel economy in miles per gallon (mpg) of a car or light truck at various speeds measured in miles per hour (mph). The data are compiled in the table below. Fuel Economy by Speed Speed (mph) Fuel Economy (mpg) 15 24.4 27.9 20 25 30.5 30 31.7 35 31.2 40 31.0 45 31.6 32.4 50 55 32.4 60 31.4 29.2 65 70 26.8 75 24.8 Source: Transportation Energy Data Book, Table 4.28. http://cta.ornl.gov/data/chapter4.shtml Plot the data using a graphing utility. Which variable is the independent variable? а. Speed is the independent variable. • • • This data can be modeled by a polynomial function. Determine if the function that models the data would b. have an even or odd degree. It seems we could model this data by an even-degree polynomial function. c. Is the leading coefficient of the polynomial that can be used to model this data positive or negative? The leading coefficient would be negative since the end behavior of this function is to approach negative infinity on both sides. List two possible reasons the data might have the shape that it does. d. Possible responses: Fuel economy improves up to a certain speed, but then wind resistance at higher speeds



Lesson 15:

: Structure in Graphs of Polynomial Functions

reduces fuel economy; the increased gas needed to go higher speeds reduces fuel economy.





Closing (3 minutes)

- In this lesson, students explored the characteristics of the graphs of polynomial functions of even and odd-degree. Graphs of even-degree polynomials demonstrate the same end behavior as x → ∞ as it does as x → -∞, while graphs of odd-degree polynomials demonstrate opposite end behavior as x → ∞ as it does as x → -∞. Because of this fact, graphs of odd-degree polynomial functions always intersect the x-axis; therefore, odd-degree polynomial functions have at least one zero or root.
- Students also learned that it is the highest degree term of the polynomial that determines if the graph exhibits odd-degree end behavior or even-degree end behavior. This makes sense because the highest degree term of a polynomial determines the degree of the polynomial.

Have students summarize the lesson either with a graphic organizer or a written summary. A graphic organizer is included below.



Exit Ticket (5 minutes)

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Structure in Graphs of Polynomial Functions









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Lesson 15: Structure in Graphs of Polynomial Functions

Exit Ticket

Without using a graphing utility, match each graph below in column 1 with the function in column 2 that it represents.







Exit Ticket Sample Solutions



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Lesson 15:

Structure in Graphs of Polynomial Functions







Problem Set Sample Solutions





: Structure in Graphs of Polynomial Functions







c. List two possible reasons the data might have such a shape.

Possible responses: Bad weather in 1992 and 1993; shifts in demand for fresh foods vs. processed.

3. The U.S. Energy Information Administration (EIA) is responsible for collecting and analyzing information about energy production and use in the United States and for informing policy makers and the public about issues of energy, the economy, and the environment. The following table contains data from the EIA about natural gas consumption from 1950–2010, measured in millions of cubic feet.

Year	U.S. natural gas total consumption
	(millions of cubic feet)
1950	5.77
1955	8.69
1960	11.97
1965	15.28
1970	21.14
1975	19.54
1980	19.88
1985	17.28
1990	19.17
1995	22.21
2000	23.33
2005	22.01
2010	24.09

U.S. Natural Gas Consumption by Year

Source: U.S. Energy Information Administration. http://www.eia.gov/dnav/ng/hist/n9140us2a.htm

a. Plot the data using a graphing utility.



- Determine if the data display the characteristics of an odd- or even-degree polynomial function.
 Looking at the end behavior, the data show the characteristics of an odd-degree polynomial function.
- c. List two possible reasons the data might have such a shape.

Possible responses: changes in supply, new sources and technology created new supplies, weather may impact usage.



Structure in Graphs of Polynomial Functions











Structure in Graphs of Polynomial Functions











5: Structure in Graphs of Polynomial Functions







Lesson 16: Modeling with Polynomials—An Introduction

Student Outcomes

 Students transition between verbal, numerical, algebraic, and graphical thinking in analyzing applied polynomial problems.

Lesson Notes

Creating an open-topped box of maximum volume is a very common problem seen in calculus. The goal is to optimize resources by enclosing the most volume possible given the constraint of the size of the construction material; here, students use paper. The dimensions given can be adjusted depending on the size of the paper chosen; hence, the dimensions are omitted from the figure on the student pages. This is the first part of a two-day lesson on modeling. Lesson 16 focuses more on students writing equations to model a situation.

Classwork

Opening (5 minutes)

Each group has a piece of construction paper that measures $45.7 \text{ cm} \times 30.5 \text{ cm}$. Other sizes of paper may be used if necessary, but ensure that each group is using the same-sized paper. Cut out congruent squares from each corner, and fold the sides in order to create an open-topped box. The goal is to create a box with the maximum possible volume.

Ask students to make conjectures about what size cut will create the box with the largest volume. Demonstrate if desired using the applet <u>http://mste.illinois.edu/carvell/3dbox/</u>.

Mathematical Modeling Exercise (30 minutes)

While students work on their boxes, put the following table on the board. As students measure their boxes and calculate the volume, they should be recording the values in the table. Stop students once each group has recorded its values, and have the discussion below before allowing them to continue working.

Mathematical Modeling Exercise

You will be assigned to a group, which will create a box from a piece of construction paper. Each group will record its box's measurements and use said measurement values to calculate and record the volume of its box. Each group will contribute to the following class table on the board.

Group	Length	Width	Height	Volume
1				
2				
3				
4				



6: Modeling with Polynomials—An Introduction









Discuss the results, and compare them with the conjectures made before cutting the paper.

- Who was able to enclose the most volume?
- Why would our goal be to enclose the most volume?
 - We are optimizing our resources by enclosing more volume than the other groups using the same-size paper.

Have students continue with the exercise.



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Modeling with Polynomials-An Introduction






- What are the possible values for the height of the box?
 - *From* 0 *to* 15.25 cm
- What is the domain of the volume function?
 - The domain is the interval 0 < x < 15.25.

Closing (5 minutes)

Use the applet <u>http://www.mathopenref.com/calcboxproblem.html</u> to summarize what the students discovered.

Revisit your original conjecture either in writing or with a neighbor. Was it accurate? How would you change it now?

Have students share responses.

- Why would our goal be to maximize the volume?
 - Maximizing resources, enclosing as much volume as possible using the least amount of material
- Is constructing a box in such a way that its volume is maximized always the best option?
 - No, a box may need to have particular dimensions (such as a shoe box). In some cases, the base of the box may need to be stronger, so the material is more expensive. Minimizing cost may be different than maximizing the volume.

Exit Ticket (5 minutes)







Name

Date

Lesson 16: Modeling with Polynomials—An Introduction

Exit Ticket

Jeannie wishes to construct a cylinder closed at both ends. The figure below shows the graph of a cubic polynomial function used to model the volume of the cylinder as a function of the radius if the cylinder is constructed using 150π cm² of material. Use the graph to answer the questions below. Estimate values to the nearest half unit on the horizontal axis and to the nearest 50 units on the vertical axis.



- 1. What is the domain of the volume function? Explain.
- 2. What is the most volume that Jeannie's cylinder can enclose?
- 3. What radius yields the maximum volume?
- 4. The volume of a cylinder is given by the formula $V = \pi r^2 h$. Calculate the height of the cylinder that maximizes the volume.









Exit Ticket Sample Solutions









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For a bonus, ask students what is meant by the caption on the t-shirt. (Hint that they can do a web search to find out.)





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Student Outcomes

• Students interpret and represent relationships between two types of quantities with polynomial functions.

Lesson Notes

In this lesson, students delve more deeply into modeling by writing polynomial equations that can be used to model a particular situation. Students are asked to interpret key features from a graph or table within a contextual situation (F-IF.B.4) and select the domain that corresponds to the appropriate graph or table (F-IF.B.5).

Classwork

Opening Exercise (8 minutes)

Give students time to work independently on the Opening Exercise before discussing as a class.



- How does this compare with the regression function found yesterday?
 - Answers will vary. Compare each parameter in the function.
- Which function is more accurate? Why?
 - The one found today is more accurate. The one found yesterday depended on measurements that may not have been exact.



17: Modeling with Polynomials—An Introduction





Mathematical Modeling Exercises 1–13 (30 minutes)

Allow students to work through the exercises in groups. Circulate the room to monitor students' progress. Then, discuss results.





Lesson 17:





7.	Use the relative maximum to find the leading coefficient of f . Explain your reasoning.										
	Since we have estimated $f(5.5) = 372$, we can plug 5.5 into the function above, and we find that $f(5.5) = 1058.75c = 372$. Solving for c, we find that $c \approx 0.35$. The function that could model the data is then given by $f(t) = 0.35t(t - 12.5)(t - 33)$.										
8.	What would be a reasonable domain for your function f ? Why?										
	A reasonable domain for f would be $0 \le x \le 12.5$ because the opening of the park corresponds to $t = 0$, and after 12.5 hours the park closes, so there are no people waiting in line.										
9.	Use your function f to calculate the number of people in line ${f 10}$ hours after the park opens.										
The formula developed in Exercise 7 gives $f(10) = 201$. After the park has been open for 10 hours, 201 people in line.						10 hours, th	nere will be				
10. Comparing the value calculated above to the actual value in the table, is your function <i>f</i> an accurat data? Explain.						n accurate m	odel for th				
	The value of the function differs from the value from the table by about 21 people, so it is not a perfect fit fo data, but it is otherwise very close. It appears to overestimate the number of people in line.							t fit for the			
11.	Use the regression featu	ure of a gr	aphing calc	ulator to fir	id a cubic fu	unction g to	model the	data.			
The calculator gives $g(t) = 0.43t^3 - 17.78t^2 + 156.63t - 24.16$.											
12.	Graph the function <i>f</i> yo to complete the table.	ou found a Round yo	nd the fund ur answers	tion g prod to the near	luced by the est integer.	e graphing c	alculator. L	Jse the grap	hing calcula		
Γ	t (hours)	0	1	2	4	7	8	10	12		
	P (people in line)	0	75	225	345	355	310	180	45		
	f(t) (your equation)	0	129	228	345	350	315	201	44		
	g(t) (regression eqn.)	-24	115	221	345	349	311	194	38		
13.	Based on the results fro At $t = 2$ hours, the fund	m the tab	lle, which m d by hand w	nodel was m was more ac	ore accurat	te at $t=2$ h vas off by 3	ours? t =	10 hours? creas the cal	culator		
	function was off by 4 pe 14 people whereas the	cople. At function f	t = 10 hou ound by ha	rs, the grap nd was off l	hing calculo by 21 peopl	ntor function le.	was more	accurate. It	was off by		

Closing (2 minutes)

MP.2

- What type of functions were used to model the data? Were they good models?
 - Cubic polynomial functions; yes, both functions were reasonably accurate.
 - What information did we use to find the function by hand?
 - We used the *x*-intercepts and the relative maximum.
- Did we have to use the relative maximum specifically to find the leading coefficient?
 - No, we could have chosen a different point on the curve.









- How did polynomials help us solve a real-world problem?
 - We were able to model the data using a polynomial function. The function allows us to estimate the number of people in line at any time t and also to estimate the time when the line is the longest and the maximum number of people are in line.

Exit Ticket (5 minutes)









Name

Date

Lesson 17: Modeling with Polynomials—An Introduction

Exit Ticket

Jeannie wishes to construct a cylinder closed at both ends. The figure at right shows the graph of a cubic polynomial function, V, used to model the volume of the cylinder as a function of the radius if the cylinder is constructed using 150π cm³ of material. Use the graph to answer the questions below. Estimate values to the nearest half unit on the horizontal axis and to the nearest 50 units on the vertical axis.

1. What are the zeros of the function V?



- 2. What is the relative maximum value of V, and where does it occur?
- 3. The equation of this function is $V(r) = c(r^3 72.25r)$ for some real number *c*. Find the value of *c* so that this formula fits the graph.
- 4. Use the graph to estimate the volume of the cylinder with r = 2 cm.
- 5. Use your formula for V to find the volume of the cylinder when r = 2 cm. How close is the value from the formula to the value on the graph?







Exit Ticket Sample Solutions



Problem Set Sample Solutions

Problem 2 requires the use of a graphing calculator. If students do not have the means to complete this, the last two parts could be done in class.





Lesson 17:









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engage



Lesson 18: Overcoming a Second Obstacle in Factoring— What If There Is a Remainder?

Student Outcomes

 Students rewrite simple rational expressions in different forms, including representing remainders when dividing.

Lesson Notes

Students have worked on dividing two polynomials using both the reverse tabular method and long division. In this lesson, they continue that work but with quotients that have a remainder. In addition to the two methods of division already presented in this module, students also use the method of inspection as stated in standard **A-APR.D.6**. The method of inspection is an opportunity to emphasize the mathematical practice of making use of structure (MP.7). The purpose of the Opening Exercise is to get students thinking about this idea of structure by leading them from writing rational numbers as mixed numbers to writing rational expressions as "mixed expressions."

Classwork

Opening Exercise (3 minutes)

Have students work through the Opening Exercise briefly by themselves, and then summarize the exercise as a whole class by displaying all three methods. This starts students thinking about the different ways of rewriting an improper fraction as a mixed number. Students use methods 2 and 3 later today to write rational expressions as "mixed expressions."





Lesson 18:





Example 1 (8 minutes)

Work through the example as a class. Relate the process of inspection used in part (b) below to method 2 used in the Opening Exercise. Then, demonstrate how the quotient could have been found using the reverse tabular method or long division.



Solutions for part (b):

Method 1: Inspection

- We already know that $\frac{x^2+3x+2}{x+2} = x + 1$, as long as $x \neq -2$. How could we use this fact to find $\frac{x^2+3x+5}{x+2}$?
 - Since 3 + 2 = 5, there must be 3 left over after performing division.
- How could we rewrite the problem in a way that is more convenient?

$$\frac{x^2+3x+5}{x+2} = \frac{(x^2+3x+2)+3}{x+2} = \frac{x^2+3x+2}{x+2} + \frac{3}{x+2}$$

MP.7

- So, what are the quotient and remainder?
 - The quotient is x + 1 with a remainder of 3.
- Since the 3 is left over and has not been divided by the x + 2, it is still written as a quotient.

$$\frac{x^2 + 3x + 5}{x + 2} = \frac{x^2 + 3x + 2 + 3}{x + 2} = \frac{x^2 + 3x + 2}{x + 2} + \frac{3}{x + 2} = (x + 1) + \frac{3}{x + 2}$$

Method 2: Reverse Tabular Method



$$\frac{x^2 + 3x + 5}{x + 2} = (x + 1) + \frac{3}{x + 2}$$



Lesson 18:

Method 3: Long division

Work the problem as a class using long division. Remind the students that we used the same process in method 3 of the Opening Exercise. T

$$\begin{array}{r} x+1 \\ x+2) \hline x^2 + 3x + 5 \\ -x^2 - 2x \\ \hline x+5 \\ -x-2 \\ \hline 3 \end{array}$$

Example 2 (7 minutes)

Repeat the process for this example. Work through the process of inspection as a pair/share exercise. Then, ask the students to repeat the process using either the reverse tabular method or long division. Share the work from both methods.

Example 2							
a.	Find the quotient by factoring the numerator. $\frac{x^3-8}{x-2}$	b.	Find the quotient. $\frac{x^3-4}{x-2}$				
	$= \frac{(x-2)(x^2+2x+4)}{x-2}$ = x ² +2x+4		$= \frac{x^3 - 8 + 4}{x - 2} = \frac{x^3 - 8}{x - 2} + \frac{4}{x - 2}$ $= x^2 + 2x + 4 + \frac{4}{x - 2}$				

In pairs, see if you can determine how to use the quotient from (a) to find the quotient of $\frac{x^3-4}{x-2}$

Give students a couple of minutes to discuss, and then elicit responses.

How did you rewrite the numerator?

Why is this a useful way to rewrite the problem?

• We know that
$$\frac{x^3-8}{x-2} = x^2 + 2x + 4$$
.

So what are the quotient and remainder of
$$\frac{x^3-4}{x-2}$$
?

• The quotient is $x^2 + 2x + 4$ with a remainder of 4.

Therefore,
$$\frac{x^3-4}{x-2} = (x^2 + 2x + 4) + \frac{4}{x-2}$$
.



MP.7





Give students a couple of minutes to rework the problem using either the reverse tabular method or long division and then share student work.



• Which method is easier? Allow students to discuss advantages and disadvantages of the three methods.

Exercise (5 minutes)

Have students practice finding quotients by factoring the numerator using the cards on the page after next. Cut out the cards and hand each student a card. The students must move around the room and match their card with the same quotient. Then have students stay in these pairs to work on the exercises.

Exercises 1–10 (15 minutes)

Allow students to work through the exercises either independently or in pairs. Some students may need to be reminded how to complete the square.



Scaffolding:

- Some students may have difficulty with inspection. Encourage them to use the reverse tabular method first, and then see if they can use that to rewrite the numerator.
- Early finishers can be given more challenging inspection problems such as the following.

$$\frac{x^2 - 5x + 9}{x - 1} = (x - 4) + \frac{5}{x - 1}$$
$$\frac{2x^2 - 5}{x - 3} = 2(x + 3) + \frac{13}{x - 3}$$

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Lesson 18:

Overcoming a Second Obstacle in Factoring—What If There Is a Remainder?









The mental math exercises on the next page can be used for building fluency. All numerators factor nicely so that there are no remainders. The exercise can be timed and restrictions can be imposed (such as, "Only write your answer in the box next to the expression.").

It is always a good idea to keep fluency exercises quick and stress-free for students. Here is one way to do that: Tell them that the activity will not be turned in for a grade, but they will be timed. Give them 2 minutes to write down as many answers as possible (using a stopwatch or a cell phone stopwatch feature.). Afterward, go through the solutions with them quickly; allow them only to mark the ones they did correctly/incorrectly—do not let them copy the correct answers down. Celebrate the student who got the greatest number correct, and then provide another 2–4 minutes for students to work on the remaining problems that they did not get right.

$\frac{x^2 - 9}{2}$	$\frac{x^2-4x+3}{2}$	$x^2 - 16$	$\frac{x^2 - 3x - 4}{2}$
x + 3	x-1	x + 4	x + 1
x - 3	<i>x</i> – 3	x-4	x-4
$x^3 - 3x^2$	$x^4 - x^2$	$x^2 + x - 6$	$x^2 - 4$
x - 3	$x^2 - 1$	<i>x</i> + 3	x + 2
<i>x</i> ²	<i>x</i> ²	x - 2	x-2
$x^2 - 8x + 12$	$x^2 - 36$	$x^2 + 6x + 8$	$x^2 - 4$
x-2	<i>x</i> + 6	x+4	$\overline{x-2}$
<i>x</i> – 6	<i>x</i> – 6	<i>x</i> + 2	<i>x</i> + 2
$x^2 - x - 20$	$x^2 - 25$	$x^2 - 2x + 1$	$x^2 - 3x + 2$
x + 4	x + 5	x - 1	x-2
x-5	x-5	<i>x</i> – 1	x-1
$x^2 + 4x - 5$	$x^2 - 25$	$x^2 - 10x$	$x^2 - 12x + 20$
x-1	x-5	x	x-2
x + 5	x + 5	x - 10	x - 10



Lesson 18:

Overcoming a Second Obstacle in Factoring—What If There Is a Remainder?





$\frac{x^2+5x+4}{x+4}$	$\frac{x^2-1}{x-1}$	$\frac{x^2+16x+64}{x+8}$	$\frac{x^2+9x+8}{x+1}$
<i>x</i> + 1	<i>x</i> + 1	<i>x</i> + 8	<i>x</i> + 8

Closing (2 minutes)

Consider asking students to respond in pairs or in writing.

- In the pair/share exercise, how did we use the structure of the expressions to help us to simplify them?
 - We were able to factor the numerator. Since the numerator and denominator contained a common factor, we were able to simplify the expression.
- How did we use structure in Exercise 10?
 - ^a We rewrote the expression by completing the square and then used inspection.
- How does this use of structure help us when working with algebraic expressions?
 - We can rewrite expressions into equivalent forms that may be more convenient.
- What methods were used to find the quotients?
 - Inspection, reverse tabular method, long division
- What are some pros and cons of the methods?
 - You may not see the answer when trying to divide by inspection, but if you do see the structure of the expression, the quotient can often be quickly found. Using the long division or tabular method can be time consuming, but both rely on a known process instead of insight.

Exit Ticket (5 minutes)









Name

Date _____

Lesson 18: Overcoming a Second Obstacle in Factoring—What If

There Is a Remainder?

Exit Ticket

1. Find the quotient of $\frac{x-6}{x-8}$ by inspection.

2. Find the quotient of $\frac{9x^3 - 12x^2 + 4}{x - 2}$ by using either long division or the reverse tabular method.









Exit Ticket Sample Solutions

1. Find the quotient of
$$\frac{x-6}{x-8}$$
 by inspection.
 $1 + \frac{2}{x-8}$
2. Find the quotient of $\frac{9x^3-12x^2+4}{x-2}$ by using either long division or the reverse tabular method.
 $(9x^2 + 6x + 12) + \frac{28}{x-2}$

Problem Set Sample Solutions

1.	For each pair of problems, find the first quotient by factoring the numerator. Then, find the second quotient by using the first quotient.					
	a.	$\frac{3x-6}{x-2}$		$\frac{3x-9}{x-2}$		
		3		$3 - \frac{3}{x-2}$		
	b.	$\frac{x^2-5x-14}{x-7}$		$\frac{x^2-5x+2}{x-7}$		
		<i>x</i> + 2		$(x+2)+\frac{16}{x-7}$		
	c.	$\frac{x^3+1}{x+1}$		$\frac{x^3}{x+1}$		
		$x^2 - x + 1$		$(x^2 - x + 1) - \frac{1}{x + 1}$		
	d.	$\frac{x^2-13x+36}{x-4}$		$\frac{x^2 - 13x + 30}{x - 4}$		
		<i>x</i> – 9		$(x-9)-\frac{6}{x-4}$		
Find	each e	quotient by using the reverse tabular method.				
2.	<u>x³-9</u>	$\frac{2x^2+5x+2}{x-1}$	3.	$\frac{x^2+x+10}{x+12}$		
	(<i>x</i> ² -	$-8x-3)-\frac{1}{x-1}$		$(x-11) + \frac{142}{x+12}$		

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Lesson 18:















Student Outcomes

Students know and apply the remainder theorem and understand the role zeros play in the theorem.

Lesson Notes

In this lesson, students are primarily working on exercises that lead them to the concept of the remainder theorem, the connection between factors and zeros of a polynomial, and how this relates to the graph of a polynomial function. Students should understand that for a polynomial function P and a number a, the remainder on division by x - a is the value P(a) and extend this to the idea that P(a) = 0 if and only if (x - a) is a factor of the polynomial (A-APR.B.2). There should be plenty of discussion after each exercise.

h.

Find f(2).

b. Find g(-1).

g(-1) = -2

f(2) = 24

Classwork

Exercises 1–3 (5 minutes)

Exercises 1–3

a.

1.

Assign different groups of students one of the three problems from this exercise. Have them complete their assigned problem, and then have a student from each group put their solution on the board. Having the solutions readily available allows students to start looking for a pattern without making the lesson too tedious.

Consider the polynomial function $f(x) = 3x^2 + 8x - 4$.

2. Consider the polynomial function $g(x) = x^3 - 3x^2 + 6x + 8$.

 $\frac{g(x)}{x+1} = \frac{x^3 - 3x^2 + 6x + 8}{x+1}$ $= (x^2 - 4x + 10) - \frac{2}{x+1}$

Divide f by x - 2.

Divide g by x + 1.

 $\frac{f(x)}{x-2} = \frac{3x^2 + 8x - 4}{x-2}$ $= (3x+14) + \frac{24}{x-2}$

Scaffolding:

If students are struggling, replace the polynomials in Exercises 2 and 3 with easier polynomial functions.

Examples:

 $g(x) = x^2 - 7x - 11$ a. Divide by x + 1. b. Find g(-1). $(x - 8) - \frac{3}{x + 1}$ g(-1) = -3 $h(x) = 2x^2 + 9$

a. Divide by
$$x - 3$$
. b. Find $h(3)$.
 $(2x + 6) + \frac{27}{2 - 3 - 2}$ $h(3) = 27$

$$(x+6) + \frac{1}{2x^2 + 9} \qquad h(3) =$$

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3. Consider the polynomial function $h(x) = x^3 + 2x - 3$. a. Divide h by x - 3. b. Find h(3). $\frac{h(x)}{x - 3} = \frac{x^3 + 2x - 3}{x - 3}$ $= (x^2 + 3x + 11) + \frac{30}{x - 3}$

Discussion (7 minutes)

MP.8

• What is f(2)? What is g(-1)? What is h(3)?

• f(2) = 24; g(-1) = -2; h(3) = 30

- Looking at the results of the quotient, what pattern do we see?
 - The remainder is the value of the function.
- Stating this in more general terms, what do we suspect about the connection between the remainder from dividing a polynomial P by x a and the value of P(a)?
 - The remainder found after dividing P by x a will be the same value as P(a).
- Why would this be? Think about the quotient $\frac{13}{3}$. We could write this as $13 = 4 \cdot 3 + 1$, where 4 is the quotient and 1 is the remainder.
- Apply this same principle to Exercise 1. Write the following on the board, and talk through it:

$$\frac{f(x)}{x-2} = \frac{3x^2 + 8x - 4}{x-2} = (3x+14) + \frac{24}{x-2}$$

- How can we rewrite f using the equation above?
 - Multiply both sides of the equation by x 2 to get f(x) = (3x + 14)(x 2) + 24.
- In general we can say that if you divide polynomial P by x a, then the remainder must be a number; in fact, there is a (possibly non-zero degree) polynomial function q such that the equation,

P(x)	=	q(x) ·	(x-a)	+	r
		↑			Ŷ
		quotient		re	mainder

is true for all x.

What is P(a)?

$$P(a) = q(a)(a - a) + r = q(a) \cdot 0 + r = 0 + r = r$$

We have just proven the remainder theorem, which is formally stated in the box below.

REMAINDER THEOREM: Let P be a polynomial function in x, and let a be any real number. Then there exists a unique polynomial function q such that the equation

$$P(x) = q(x)(x - a) + P(a)$$

is true for all x. That is, when a polynomial is divided by (x - a), the remainder is the value of the polynomial evaluated at a.

• Restate the remainder theorem in your own words to your partner.









While students are doing this, circulate and informally assess student understanding before asking students to share their responses as a class.

Exercise 4 (5 minutes)

Students may need more guidance through this exercise, but allow them to struggle with it first. After a few students have found k, share various methods used.

Exercise 4-6
4. Consider the polynomial P(x) = x³ + kx² + x + 6.
a. Find the value of k so that x + 1 is a factor of P.
In order for x + 1 to be a factor of P, the remainder must be zero. Hence, since x + 1 = x - (-1), we must have P(-1) = 0 so that 0 = -1 + k - 1 + 6. Then k = -4.
b. Find the other two factors of P for the value of k found in part (a). P(x) = (x + 1)(x² - 5x + 6) = (x + 1)(x - 2)(x - 3)

Scaffolding: Challenge early finishers with this problem: Given that x + 1 and x - 1 are factors of $P(x) = x^4 + 2x^3 - 49x^2 - 2x + 48$, write P in factored form. Answer: (x + 1)(x - 1)(x + 8)(x - 6)

Discussion (7 minutes)

- Remember that for any polynomial function P and real number a, the remainder theorem says that there exists a polynomial q so that P(x) = q(x)(x a) + P(a).
- What does it mean if a is a zero of a polynomial P?

$$\square \quad P(a) = 0$$

- So what does the remainder theorem say if a is a zero of P?
 - There is a polynomial q so that P(x) = q(x)(x a) + 0.
- How does (x a) relate to *P* if *a* is a zero of *P*?
 - If a is a zero of P, then (x a) is a factor of P.
- How does the graph of a polynomial function y = P(x) correspond to the equation of the polynomial P?
 - The zeros are the x-intercepts of the graph of P. If we know a zero of P, then we know a factor of P.
- If we know all of the zeros of a polynomial function, and their multiplicities, do we know the equation of the function?
 - Not necessarily. It is possible that the equation of the function contains some factors that cannot factor into linear terms.

We have just proven the factor theorem, which is a direct consequence of the remainder theorem.

FACTOR THEOREM: Let *P* be a polynomial function in *x*, and let *a* be any real number. If *a* is a zero of *P* then (x - a) is a factor of *P*.

- Give an example of a polynomial function with zeros of multiplicity 2 at 1 and 3.
 - $P(x) = (x 1)^2 (x 3)^2$







• Give another example of a polynomial function with zeros of multiplicity 2 at 1 and 3.

•
$$Q(x) = (x-1)^2(x-3)^2(x^2+1)$$
 or $R(x) = 4(x-1)^2(x-3)^2$

- If we know the zeros of a polynomial, does the factor theorem tell us the exact formula for the polynomial?
 - No. But, if we know the degree of the polynomial and the leading coefficient, we can often deduce the equation of the polynomial.

Exercise 5 (8 minutes)

5.

а.

b.

c.

d.

No

As students work through this exercise, circulate the room to make sure students have made the connection between zeros, factors, and *x*-intercepts. Question students to see if they can verbalize the ideas discussed in the prior exercise.

Consider the polynomial $P(x) = x^4 + 3x^3 - 28x^2 - 36x + 144$.

Yes; P(-3) = 81 - 81 - 252 + 108 + 144 = 0.

The graph of P is shown to the right. What are the zeros of P?

Is 1 a zero of the polynomial P?

Is x + 3 one of the factors of P?

Approximately -6, -3, 2, and 4.

Write the equation of P in factored form.

P(x) = (x+6)(x+3)(x-2)(x-4)

• by dividing P by x - 1. This helps to reinforce the

Scaffolding:

ideas discussed in Exercises 1 and 2.

Encourage students who are

struggling to work on part (a)

by finding P(1), and

using two methods:



Is 1 a zero of the polynomial P? How do you know?

• No. $P(1) \neq 0$.

- What are two ways to determine the value of P(1)?
 - Substitute 1 for x into the function or divide P by x 1. The remainder will be P(1).
- Is x + 3 a factor of *P*? How do you know?
 - Yes. Because P(-3) = 0, then when P is divided by x + 3, the remainder is 0, which means that x + 3 is a factor of the polynomial P.
- How do you find the zeros of *P* from the graph?
 - The zeros are the *x*-intercepts of the graph.
- How do you find the factors?
 - By using the zeros: if x = a is a zero, then x a is a factor of P.
- Expand the expression in part (d) to see that it is indeed the original polynomial function.





Exercise 6 (6 minutes)

Allow students a few minutes to work on the problem and then share results.



- What information from the graph was needed to write the equation?
 - The *x*-intercepts were needed to write the factors.
- Why would there be more than one polynomial function possible?
 - Because the factors could be multiplied by any constant and still produce a graph with the same x-intercepts.
- Why can't we find the constant factor *c* by just knowing the zeros of the polynomial?
 - The zeros only determine where the graph crosses the *x*-axis, not how the graph is stretched vertically. The constant factor can be used to vertically scale the graph of the polynomial function that we found to fit the depicted graph.

Closing (2 minutes)

Have students summarize the results of the remainder theorem and the factor theorem.

- What is the connection between the remainder when a polynomial *P* is divided by x a and the value of P(a)?
 - They are the same.
 - If x a is factor, then _____
 - The number a is a zero of P.
- If P(a) = 0, then _____
 - (x-a) is a factor of *P*.







Lesson Summary

REMAINDER THEOREM: Let P be a polynomial function in x, and let a be any real number. Then there exists a unique polynomial function q such that the equation

P(x) = q(x)(x - a) + P(a)

is true for all x. That is, when a polynomial is divided by (x - a), the remainder is the value of the polynomial evaluated at a.

FACTOR THEOREM: Let P be a polynomial function in x, and let a be any real number. If a is a zero of P, then (x - a) is a factor of P.

Example: If $P(x) = x^2 - 3$ and a = 4, then P(x) = (x + 4)(x - 4) + 13 where q(x) = x + 4 and P(4) = 13.

Example: If $P(x) = x^3 - 5x^2 + 3x + 9$, then P(3) = 27 - 45 + 9 + 9 = 0, so (x - 3) is a factor of P.

Exit Ticket (5 minutes)







Name_____

Date _____

Lesson 19: The Remainder Theorem

Exit Ticket

Consider the polynomial $P(x) = x^3 + x^2 - 10x - 10$.

1. Is x + 1 one of the factors of *P*? Explain.

2. The graph shown has x-intercepts at $\sqrt{10}$, -1, and $-\sqrt{10}$. Could this be the graph of $P(x) = x^3 + x^2 - 10x - 10$? Explain how you know.





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Exit Ticket Sample Solutions



Problem Set Sample Solutions





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Lesson 19

12. Find the value of k so that $\frac{x^3 - kx^2 + 2}{x - 1}$ has remainder 8. k = -513. Find the value k so that $\frac{kx^3 + x - k}{x + 2}$ has remainder 16. k = -214. Show that $x^{51} - 21x + 20$ is divisible by x - 1. Let $P(x) = x^{51} - 21x + 20$. Then $P(1) = 1^{51} - 21(1) + 20 = 1 - 21 + 20 = 0$. Since P(1) = 0, the remainder of the quotient $(x^{51} - 21x + 20) \div (x - 1)$ is 0. Therefore, $x^{51} - 21x + 20$ is divisible by x - 1. 15. Show that x + 1 is a factor of $19x^{42} + 18x - 1$. Let $P(x) = 19x^{42} + 18x - 1$. Then $P(-1) = 19(-1)^{42} + 18(-1) - 1 = 19 - 18 - 1 = 0$. Since P(-1) = 0, x + 1 must be a factor of P.

Note to Teacher: The following problems have multiple correct solutions. The answers provided here are polynomials with leading coefficient 1 and the lowest degree that meet the specified criteria. As an example, the answer to Exercise 16 is given as p(x) = (x + 2)(x - 1), but the following are also correct responses: q(x) = 14(x + 2)(x - 1), $r(x) = (x + 2)^4(x - 1)^8$, and $s(x) = (x^2 + 1)(x + 2)(x - 1)$.

Write a polynomial function that meets the stated conditions. 16. The zeros are -2 and 1. p(x) = (x + 2)(x - 1) or, equivalently, $p(x) = x^2 + x - 2$ 17. The zeros are -1, 2, and 7. p(x) = (x + 1)(x - 2)(x - 7) or, equivalently, $p(x) = x^3 - 8x^2 + 5x + 14$ 18. The zeros are $-\frac{1}{2}$ and $\frac{3}{4}$. $p(x) = (x + \frac{1}{2})(x - \frac{3}{4})$ or, equivalently, $p(x) = x^2 - \frac{x}{4} - \frac{3}{8}$ 19. The zeros are $-\frac{2}{3}$ and 5, and the constant term of the polynomial is -10. p(x) = (x - 5)(3x + 2) or, equivalently, $p(x) = 3x^2 - 13x - 10$ 20. The zeros are 2 and $-\frac{3}{2}$, the polynomial has degree 3, and there are no other zeros. $p(x) = (x - 2)^2(2x + 3)$ or, equivalently, $p(x) = (x - 2)(2x + 3)^2$





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Student Outcomes

Students learn to fit polynomial functions to data values.

Lesson Notes

In previous modeling lessons, students relied on the graphing calculator to find polynomial functions that fit a set of data. Lessons 20 and 21 comprise a two-lesson modeling exercise in which students model the shape of a riverbed in order to calculate flow rate of the river; this can be used to determine if the river is vulnerable to flooding. In this first lesson, students use the remainder theorem as a way to fit a polynomial function to data points without relying on technology. In the second lesson, students use technology, particularly Wolfram Alpha, to find the equation of the interpolating polynomial that best fits the data. The online calculations through Wolfram Alpha can be done as a demonstration for the class or individually if students have access to computers. They can also be replaced by polynomial regression on a graphing calculator.

Students have some experience with modeling with polynomials from previous lessons in this module, but this is the first lesson sequence that takes them through the full modeling cycle as outlined in the modeling standard and as shown in the figure below. Students must first formulate a plan and develop a method for using the remainder theorem to fit a polynomial function to data. They must then use the model to compute the cross-sectional area of the riverbed and then volumetric flow of the water in the river. Students interpret their results and discuss how the results would be useful to the EPA.



Classwork

Mathematical Modeling Exercise (7 minutes): Discussion

Have students read through the Mathematical Modeling Exercise. Have them discuss with a partner what questions need to be answered and brainstorm some ideas of how to answer them before having the discussion below. Tell the students that at the end of the next lesson, they will be expected to write a short report about the volumetric flow of the river. They will need to include their data and calculations in their report and should be preparing the report throughout the modeling exercise. Some pertinent vocabulary for this lesson includes the following:

Flow Rate (or volumetric flow rate): The volume of fluid that passes through a given surface per unit time.

Riverbed: The channel in which a river flows.

Cross-Section: A two-dimensional view of a slice through an object.









- Read through the statement of the Mathematical Modeling Exercise.
 What is our goal?
 - We need to find the flow rate of water through the riverbed.
- How can we estimate the flow rate of water?
 - We need the area of the cross-section and the current speed of the water. Then the flow rate is the product of the cross-sectional area (in square feet) and the speed of the water (in feet per second).
- To estimate the area of the cross-section, what information do we need?
 - We need enough data points to make a good estimate.
- To gather enough data points, we will need to make an assumption that the riverbed changes smoothly between the data points we have. With that assumption, we could estimate other data points by approximating the curve of the cross-section using a polynomial function.
- What is our first task?

- We need to find a polynomial function to fit the data.
- Draw the graph of a polynomial function that passes through the 5 given data points.





Lesson 20: Modeling Riverbeds with Polynomials



Lesson 20

M1

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- We have five data points. What information are we missing?
 - The depths at x = 40 and x = 80
- How can we find the missing values?
 - Find a polynomial function P that fits the given data and then use its equation to find P(40) and P(80), which should approximate the depths at x = 40 and x = 80.
- What is the lowest-degree polynomial that could be used to model this data?
 - Degree four, because we have three relative maximum and minimum points.
- What constraints must our polynomial meet?
 - P(0) = 0, P(20) = -20, P(60) = -15, P(100) = -25, and P(120) = 0

Example 1 (12 minutes)

MP.1

Work this example out two ways. First, let students find the equation by setting up a system of equations (as in Lesson 1). Then, demonstrate how the remainder theorem can be used to find the polynomial.

Before trying to find a polynomial function whose graph goes through all five points, let's try a simpler problem and find a polynomial whose graph goes through just three of the points: (0,28), (2,0), and (8, 12).

Display these three points on a coordinate grid to use as reference through this discussion.

- Will the polynomial function that we find through these points be linear?
 - No, looking at the graph, we see that the points do not lie on a line.
- What do you think is the lowest-degree polynomial we can find that passes through the three points (0,28), (2,0), and (8, 12)? (Remind students to look at the locations of these points in the plane as they decide on the lowest possible degree.)
 - It appears that we could fit a quadratic polynomial to these points; that is, they seem to lie on a parabola.
- Try to find a quadratic polynomial whose graph passes through these points. (Give students a hint to find a system of equations by plugging in x and P(x) values into the equation $P(x) = ax^2 + bx + c$ as they did in Lesson 1.)







Allow students time to work, and then discuss the process as a class.



Have students check their equation by finding P(0), P(2), and P(8) and showing that P(0) = 28, P(2) = 0, and P(8) = 12. Now work the problem again using the remainder theorem.

- We can also use the remainder theorem to find the quadratic polynomial P that satisfies P(0) = 28, P(2) = 0, and P(8) = 12. We have already seen a technique that finds this quadratic polynomial, but the method using the remainder theorem generalizes to higher degree polynomials.
- <u>Use the first data point</u>:

Since P(0) = 28, the remainder theorem tells us that for some polynomial *a*,

$$P(x) = (x - 0)a(x) + 28$$

= $x a(x) + 28$.

• What do we know about the degree of the polynomial *a*? How do we know?

• Since *P* will be a quadratic polynomial, a must be a linear polynomial.

Use the second data point:

Since P(2) = 0, we have

$$P(2) = 2 a(2) + 28 = 0$$

2 a(2) = -28
a(2) = -14.

Thus, the remainder theorem tells us that for some polynomial b,

$$a(x) = (x - 2)b(x) - 14.$$

- What do we know about the degree of the polynomial b? How do we know?
 - Since a is a linear polynomial, b must be a constant polynomial.
- Rewrite *P* in terms of *b*:

$$P(x) = x a(x) + 28$$

= x[(x - 2)b(x) - 14] + 28.

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Use the third data point:

Since P(8) = 12, we have

$$P(8) = 8[(8-2)b(x) - 14] + 28 = 12$$

$$48b(x) - 104 + 28 = 12$$

$$48b(x) = 96$$

$$b(x) = 2.$$

- Remember that we said the function b(x) would be constant, and we found that b(x) = 2 for all values of x.
- Now we can rewrite the function *P*:

$$P(x) = x[(x-2)b(x) - 14] + 28$$

= x[(x-2)2 - 14] + 28.

- Do the three points satisfy this equation?
 - $P(0) = 0[(0-2) \cdot 2 14] + 28 = 28$
 - □ $P(2) = 2[(2-2) \cdot 2 14] + 28 = 0$ ✓
 - $P(8) = 8[(8-2) \cdot 2 14] + 28 = 12$
- For a final check, write the polynomial we just found in standard form.

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$$(x) = x[(x-2)2 - 14] + 28$$
$$= 2x^{2} - 18x - 28$$

- What if we wanted to fit a polynomial function to two data points? What is the lowest degree of a polynomial that fits two data points?
 - We know from geometry that two points determine a unique line, so a polynomial of degree one fits two data points.
- In the next example, we fit a polynomial to four data points. What is the lowest degree of a polynomial that fits four data points?
 - Since two data points can be fit by a degree 1 polynomial, and three data points can be fit by a degree 2 polynomial, it makes sense that four data points can be fit by a degree 3 polynomial.

Example 2 (15 minutes)

The table below helps students organize their work as they proceed to find the cubic polynomial that fits four data points. Work through the first line of the table with students, and then have students work in groups to complete the table to determine the equation of the polynomial *P*. Point out to students that this would be a difficult task if we used systems of equations as we did first in Example 1. Have students present their work on the board.



Modeling Riverbeds with Polynomials





Example 2					
Find a degree 3	Find a degree 3 polynomial P such that $P(-1) = -3$, $P(0) = -2$, $P(1) = -1$, and $P(2) = 6$.				
Function Value	Substitute the data point into the current form of the equation for <i>P</i> .	Apply the remainder theorem to <i>a</i> , <i>b</i> , or <i>c</i> .	Rewrite the equation for P in terms of a , b , or c .		
P(-1) = -3			P(x) = (x+1)a(x) - 3		
P(0) = -2	(0+1)a(0) - 3 = -2	a(x) = xb(x) + 1	P(x) = (x+1)(xb(x)+1) - 3		
P(1) = -1	(1+1)(b(1)+1) - 3 = -1	$\boldsymbol{b}(\boldsymbol{x}) = (\boldsymbol{x} - 1)\boldsymbol{c}(\boldsymbol{x})$	P(x) = (x+1)(x(x-1)c(x)+1) - 3		
P(2) = 6	(2+1)(2(2-1)c(2)+1) - 3 = 6	c(x) = 1	$P(x) = (x+1)(x(x-1) \cdot 1 + 1) - 3$		

- Write the polynomial *P* in standard form.
 - $P(x) = (x+1)(x^2 x + 1) 3$
 - $P(x) = x^3 x^2 + x + x^2 x + 1 3$
 - $P(x) = x^3 2$
- Verify that *P* satisfies the four given constraints.
 - $P(-1) = (-1)^3 2 = -3$
 - $P(0) = 0^3 2 = -2$
 - $P(1) = 1^3 2 = -1$
 - ^D $P(2) = 2^3 2 = 6$

Closing (5 minutes)

- What methods have we used to find polynomial functions that fit given data?
 - We plugged in data values to obtain a system of equations and used the remainder theorem. In previous lessons, we used the graphing calculator.
- If we want a polynomial to perfectly fit data points, how does the degree of the polynomial relate to the number of data points?
 - The degree of the polynomial whose graph passes through a set of data points is one less than the number of data points.

Ask students to summarize the important parts of the lesson in writing, to a partner, or as a class. Use this as an opportunity to informally assess understanding of the lesson. The following are some important summary elements.



Exit Ticket (6 minutes)



0: Modeling Riverbeds with Polynomials





MP.7



Name

Date _____

Lesson 20: Modeling Riverbeds with Polynomials

Exit Ticket

Use the remainder theorem to find a quadratic polynomial P so that P(1) = 5, P(2) = 12, and P(3) = 25. Give your answer in standard form.



D: Modeling Riverbeds with Polynomials







Exit Ticket Sample Solutions

Use the remainder theorem to find a quadratic polynomial P so that P(1) = 5, P(2) = 12, and P(3) = 25. Give your answer in standard form. Since P(1) = 5, there is a linear polynomial a so that P(x) = (x - 1)a(x) + 5. Since P(2) = 12, we have P(2) = (2 - 1)a(2) + 5 = 12, so a(2) = 7. Because a(2) = 7, there exists a constant b so that a(x) = (x - 2)b + 7. Substituting into P(x) gives P(x) = (x-1)[(x-2)b+7]. Since P(3) = 25, we have P(3) = (3-1)[(3-2)b+7] = 25. Solving for b gives b = 3. It follows that P(x) = (x-1)[(x-2)3+7], and in standard form we have $P(x) = 3x^2 - 2x + 4$.

Problem Set Sample Solutions

1.	Suppose a polynomial function P is such that $P(2) = 5$ and $P(3) = 12$.		
	a.	What is the largest-degree polynomial that can be uniquely determined given the information?	
		Degree one	
	b.	Is this the only polynomial that satisfies $P(2) = 5$ and $P(3) = 12$?	
		No, there are an infinite number of polynomials that pass through those two points. However, two points will determine a unique equation for a linear function.	
	c.	Use the remainder theorem to find the polynomial P of least degree that satisfies the two points given.	
		P(x)=7x-9	
	d.	Verify that your equation is correct by demonstrating that it satisfies the given points.	
		P(2) = 7(2) - 9 = 5	
		P(3) = 7(3) - 9 = 12	
2.	Write	e a quadratic function P such that $P(0) = -10$, $P(5) = 0$, and $P(7) = 18$ using the specified method.	
	a.	Setting up a system of equations b. Using the remainder theorem	
		$P(x) = x^2 - 3x - 10 \qquad P(x) = x^2 - 3x - 10$	



Modeling Riverbeds with Polynomials





3. Find a degree-three polynomial function P such that P(-1) = 0, P(0) = 2, P(2) = 12, and P(3) = 32. Use the table below to organize your work. Write your answer in standard form, and verify by showing that each point satisfies the equation.

Function Value	Substitute the data point into the current form of the equation for <i>P</i> .	Apply the remainder theorem to <i>a</i> , <i>b</i> , or <i>c</i> .	Rewrite the equation for P in terms of a , b , or c .	
P(-1) = 0			P(x) = (x+1)a(x)	
P(0) = 2	<i>a</i> (0) = 2	a(x) = xb(x) + 2	P(x) = (x+1)(xb(x)+2)	
<i>P</i> (2) = 12	3(2b(2)) + 2) = 12 b(2) = 1	b(x) = (x-2)c(x) + 1	P(x) = (x+1)[x[(x-2)c(x)+1]+2]	
P(3) = 32	4[3(c(3) + 1)] + 2 = 32 c(3) = 1	<i>c</i> (3) = 1	P(x) = (x+1)[x[(x-2)+1]+2]	
$P(x) = x^3 + x + 2$				
P(-1)=-1	-1+2=0			
P(0)=0+0	+ 2 = 2			
P(2) = 8 + 2	+ 2 = 12 ✓			

4. The method used in Problem 3 is based on the Lagrange interpolation method. Research Joseph-Louis Lagrange, and write a paragraph about his mathematical work.

~

Answers will vary.

P(3) = 27 + 3 + 2 = 32











Student Outcomes

 Students model a cross-section of a riverbed with a polynomial function and estimate fluid flow with their algebraic model.

Lesson Notes

This is the second half of a two-day modeling lesson. In previous modeling lessons, students relied on the graphing calculator to find polynomial functions that fit a set of data, and in the previous lesson the polynomial was found algebraically. In this lesson, we use the website Wolfram Alpha (<u>www.wolframalpha.com</u>) to find the polynomial to fit the data. If students do not have access to computers or the Internet during class, the polynomial can be found using quartic regression on a graphing calculator.

The previous lesson introduced the problem of modeling the shape of a riverbed and computing the volumetric flow. In this lesson, students can actually do the modeling, with the help of technology, and then interpret the results and create a report detailing their findings, thus completing the modeling cycle.



Classwork

Opening (3 minutes)

Review the Mathematical Modeling Exercise with students. In this lesson, students:

- 1. Find a function that models the shape of the riverbed based on the five data points given in the graph below.
- 2. Approximate the area of the cross-sectional region using triangles and trapezoids.
- 3. Calculate the volumetric flow rate of the water in gallons per minute.
- 4. Create a report of the findings.



Modeling Riverbeds with Polynomials





Mathematical Modeling Exercise

The Environmental Protection Agency (EPA) is studying the flow of a river in order to establish flood zones. The EPA hired a surveying company to determine the flow rate of the river, measured as volume of water per minute. The firm set up a coordinate system and found the depths of the river at five locations as shown on the graph below. After studying the data, the firm decided to model the riverbed with a polynomial function and divide the area into six regions that are either trapezoidal or triangular so that the overall area can be easily estimated. The firm needs to approximate the depth of the river at two more data points in order to do this.



Mathematical Modeling Exercises 1–5 (32 minutes)

Now return to the Mathematical Modeling Exercise.

- How many data points are we given? What is the lowest degree of a polynomial that passes through these points?
 - We were given five data points, so we can model the data using a fourth-degree polynomial function.
- We are going to use Wolfram Alpha to find the fourth-degree polynomial that fits the data. It follows a procedure based on the remainder theorem that we used in the previous two examples to find the equation.

Go to <u>www.wolframalpha.com</u>. Type in the following command:

Interpolating polynomial [$\{(0,0), (20, -20), (60, -15), (100, -25), (120, 0)\}, x$].

Students can find P(40) and P(80) either by substituting into the function displayed or by editing the command on Wolfram Alpha as follows:

Interpolating polynomial [$\{(0,0), (20, -20), (60, -15), (100, -25), (120,0)\}, 40$] *Interpolating polynomial* [$\{(0,0), (20, -20), (60, -15), (100, -25), (120,0)\}, 80$].

Allow students time to work through the remainder of the exercise either individually or in groups. The conversion from cubic feet per minute to gallons per minute can also be done using Wolfram Alpha.



If students are overwhelmed with the new technology along with new content, most graphing calculators perform polynomial regression up to degree 4, so consider having students use that more familiar technological tool.



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Lesson 21: Modeling Riverbeds with Polynomials









- They could time an object (like a ball) floating over a set distance.
- What factors would need to be considered when measuring the flow?
 - Water may flow faster below the surface. The flow rate may vary from the edge of the river to the middle of the river and around obstacles such as rocks.
- Remember that the EPA is interested in identifying flood-prone areas. How might the information you have gathered help the EPA?
 - If the normal volumetric flow of the river has been recorded by the EPA, then they can record the normal water levels in the river and use that knowledge to predict what level of volumetric flow would result in flooding. In cases of heavy rain, the EPA could identify if an area is likely to flood due to the increased volumetric flow.



Modeling Riverbeds with Polynomials





Closing (5 minutes)

Ask students to summarize the important parts of the lesson in writing, to a partner, or as a class. Use this as an opportunity to informally assess understanding of the lesson.

Exit Ticket (5 minutes)



L: Modeling Riverbeds with Polynomials







Name

Date _____

Lesson 21: Modeling Riverbeds with Polynomials

Exit Ticket

Explain the process you used to estimate the volumetric flow of the river, from accumulating the data to calculating the flow of water.



: Modeling Riverbeds with Polynomials







Exit Ticket Sample Solutions

Explain the process you used to estimate the volumetric flow of the river, from accumulating the data to calculating the flow of water.

We were given data points that represented the depth of the river at various points in the cross-section. We used Wolfram Alpha to find a polynomial P that passed through those given points. We used the polynomial P to find more data points. Then we approximated the area of the cross-section using triangles and trapezoids; figures whose area we know how to calculate. Once we had an approximate area of the cross-section, we multiplied that area by the speed of the water across the cross-section to find the amount of volumetric flow in units of cubic feet per minute. The last step was to convert cubic feet to gallons to get the amount of volumetric flow in units of gallons per minute.

Problem Set Sample Solutions

Problem 2 requires the use of a computer. This could be completed in class if students do not have access to computers.

1.	As the leader of the surveying team, write a short report to the EPA on your findings from the in-class exercises. Be sure to include data and calculations.				
	After collecting data at the site, we decided that the cross-sectional area could be approximated using trapezoids. In order to increase the accuracy of our area approximation, more data points were needed. We chose to use the data collected to model the riverbed using a degree 4 polynomial function. We used the computational knowledge engine Wolfram Alpha to find the polynomial that fit the data, which is $P(x) = \frac{17}{3840000}x^4 - \frac{33}{32000}x^3 + \frac{751}{9600}x^2 - \frac{35}{16}x$. Using the polynomial P, we were able to estimate enough data points to calculate the cross-sectional area and determined that it was 1,960 ft ² . Using this information and the average speed of the water at the cross-section, which was $\frac{\text{ft}}{\text{min}}$, we were able to compute the volumetric flow of the river at that cross-section. We determined that the volumetric flow is approximately 344,960 $\frac{\text{ft}^3}{\text{min}}$, which is 2,580,480 $\frac{\text{gallons}}{\text{min}}$.				
2.	Suppose that depths of the riverbed were measured for a different cross-section of the river.				
	u.	O(0) = 0	O(16.5) = -27.4	$O(44 \ 4) = -19 \ 6$	O(77.6) = -25.1
		Q(0) = 0,	Q(10.5) = 27.1,	Q(11.1) = 10.0,	Q(11.0) = 20.1
		Q(123.3) = -15.0,	Q(131.1) = -15.1,	Q(150) = 0.	
		$Q(x) = \frac{7}{8789062500}$	$x^6 - \frac{11}{29296875}x^5 + \frac{1}{23}x^5 +$	$\frac{191}{812500}x^4 - \frac{11}{1875}x^3 + $	$\frac{2759}{11250}x^2 - \frac{329}{75}x$











Modeling Riverbeds with Polynomials







Student Outcomes

Students define rational expressions and write them in equivalent forms.

Lesson Notes

In this module, students have been working with polynomial expressions and polynomial functions. In elementary school, students mastered arithmetic operations with integers before advancing to performing arithmetic operations with rational numbers. Just as a rational number is built from integers, a rational expression is built from polynomial expressions. A precise definition of a rational expression is included at the end of the lesson.

Informally, a rational expression is any expression that is made by a finite sequence of addition, subtraction, multiplication, and division operations on polynomials. After algebraic manipulation, a rational expression can always be

written as $\frac{P}{Q}$, where P is any polynomial, and Q is any polynomial except the zero polynomial. Remember that constants,

such as 2, and variables, such as x, count as polynomials, so the rational numbers are also considered to be rational expressions. Standard A-APR.C.6 focuses on rewriting rational expressions in equivalent forms, and the next three

lessons apply that standard to write complicated rational expressions in the simplified form $\frac{r}{\rho}$. However, the prompt

simplify the rational expression does not only mean putting expressions in the form $\frac{P}{Q}$ but also any form that is conducive

to solving the problem at hand. The skills developed in Lessons 22–25 are necessary prerequisites for addressing standard A-REI.A.2, solving rational equations, which is the focus of Lessons 26 and 27.

Classwork

Opening Exercise (8 minutes)

The Opening Exercise serves two purposes: (1) to reactivate prior knowledge of equivalent fractions, and (2) as a review for students who struggle with fractions. The goal is for students to see that the process they use to reduce a fraction to lowest terms is the same process they will use to reduce a rational expression to lowest terms. To begin, pass out 2–3 notecard-sized slips of paper to each student or pair of students.

We are going to start with a review of how to visualize equivalent fractions.

Opening Exercise

On your own or with a partner, write two fractions that are equivalent to $\frac{1}{3}$, and use the slips of paper to create visual models to justify your response.







Use the following to either walk through the exercise for scaffolding or as an example of likely student responses.

• We can use the following area model to represent the fraction $\frac{1}{3}$. Because the three boxes have the same area, shading one of the three boxes shows that $\frac{1}{3}$ of the area in the figure is shaded.



• Now, if we draw a horizontal line dividing the columns in half, we have six congruent rectangles, two of which are shaded so that $\frac{2}{6}$ of the area in the figure is shaded.



In the figure below, we have now divided the original rectangle into nine congruent sub-rectangles, three of which are shaded so that ³/₉ of the area in the figure is shaded.



Scaffolding:

Students who are already comfortable with fractions can instead reduce the following rational expressions to lowest terms.

$$\frac{5}{15'}\frac{27}{36'}\frac{\sqrt{75}}{5},\frac{\pi^4}{\pi^2}$$

In any case, do not spend too much time on these exercises; instead, use them as a bridge to reducing rational expressions that contain variables.

• Let's suppose that the area of the original rectangle is 1. In walking the class through the example, point out that the shaded area in the first figure is $\frac{1}{3}$, the shaded area in the second figure is $\frac{2}{6}$, and the shaded area in the third figure is $\frac{3}{9}$. Since the area of the shaded regions are the same in all three figures, we see that

$$\frac{1}{3} = \frac{2}{6} = \frac{3}{9}$$
. Thus, $\frac{1}{3}, \frac{2}{6}$, and $\frac{3}{9}$ are equivalent fractions.

If students come up with different equivalent fractions, then incorporate those into the discussion of equivalent areas, noting that the shaded regions are the same for every student.

Now, what if we were to choose any positive integer n and draw lines across our figure so that the columns are divided into n pieces of the same size? What is the area of the shaded region?









Give students time to think and write, and ask them to share their answers with a partner. Anticipate that students will express the generalization in words or suggest either $\frac{1}{3}$ or $\frac{n}{3n}$. Both are correct and, ideally, both will be suggested.

Thus, we have the rule:

If a, b, and n are integers with $n \neq 0$ and $h \neq 0$ then	
na = a	Scaffolding:
$\frac{na}{nb} = \frac{a}{b}$.	Students may also express the
	generalization in words.

The result summarized in the box above is also true for real numbers a, b, and n, as well as for polynomial and rational expressions.

- Then $\frac{2}{6} = \frac{2(1)}{2(3)} = \frac{1}{3}$ and $\frac{3}{9} = \frac{3(1)}{3(3)} = \frac{1}{3}$.
- We say that a rational number is simplified, or reduced to lowest terms, when the numerator and denominator do not have a factor in common. Thus, while $\frac{1}{3}$, $\frac{2}{6}$, and $\frac{3}{9}$ are equivalent, only $\frac{1}{3}$ is in lowest terms.

Discussion (10 minutes)

Which of the following are rational numbers, and which are not?

$$\frac{3}{4}$$
, 3.14, π , $\frac{5}{0}$, $-\sqrt{17}$, 23, $\frac{1+\sqrt{5}}{2}$, -1 , 6.022 × 10²³, 0

Rational:

$$\frac{3}{4}$$
, 3.14, 23, -1, 6.022 × 10²³, 0

- Not rational: $\pi, \frac{5}{0}, -\sqrt{17}, \frac{1+\sqrt{5}}{2}$
- Of the numbers that were not rational, were they all irrational numbers?
 - No. Since division by zero is undefined, $\frac{5}{0}$ is neither rational nor irrational.
- Today we learn about rational expressions, which are related to the polynomials we've been studying. Just as the integers are the foundational building blocks of rational numbers, polynomial expressions are the foundational building blocks for rational expressions. Based on what we know about rational numbers, give an example of what you think a rational expression is.

Ask students to write down an example and share it with their partner or small group. Allow groups to debate and present one of the group's examples to the class.

Scaffolding:

Relate the new ideas of rational expressions back to the more familiar ideas of rational numbers throughout this lesson.





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Recall that a rational number is a number that we can write as ^p/_q, where p and q are integers, and q is nonzero.
 We can consider a new type of expression, called a *rational expression*, which is made from polynomials by adding, subtracting, multiplying, and dividing them. Any rational expression can be expressed as ^P/_Q, where P and Q are polynomial expressions, and Q is not zero, even though it may not be presented in this form originally.

Remind students that numbers are also polynomial expressions, which means that rational numbers are included in the set of rational expressions.

• The following are examples of rational expressions. Notice that we need to exclude values of the variables that make the denominators zero so that we do not divide by zero.

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• The denominator is never zero, so we do not need to exclude any values.

$$\frac{ab^2}{3a-2b}$$

• We need $3a \neq 2b$.

 $3x^2+4$

^D The denominator is never zero, so we do not need to exclude any values.

$$\frac{3}{b^2-7}$$

• We need
$$b \neq \sqrt{7}$$
 and $b \neq -\sqrt{7}$.

Have students create a Frayer model in their notebooks, such as the one provided. Circulate around the classroom to informally assess student understanding. Since a formal definition of rational expressions has not yet been given, there is some leeway on the description and characteristics sections, but make sure that nothing they have written is incorrect. Ask students to share their characteristics, examples, and non-examples to populate a class model on the board.

Description	Characteristics			
An expression that can be written as $\frac{P}{Q}$, where P and Q are	Follows similar rules as rational numbers do.			
polynomials, and Q is not zero.				
Rational Expression				
$\frac{3}{5}, \frac{x^2 - 4x}{x + 1}$ with $x \neq -1$	$\frac{x+1}{x+1}$ (cannot divide by 0)			
$\frac{x^2y}{2}, \frac{a^2+b^2}{(a+2)(a-1)}, \text{ with } a \neq -2, 1$	$\frac{2^{x}}{3x}$ (2 ^x is not a polynomial)			
Examples	Non-examples			







Lesson 22

ALGEBRA II

Scaffolding:

Encourage struggling students to plug in various values of the variables to see that the expressions *are* equivalent for almost all values of the variables. But for values in which the denominator of one expression is equal to zero, they are not equivalent.

It is important to note that the excluded values of the variables remain even after simplification. This is because the two expressions would not be equal if the variables were allowed to take on these values. Discuss with a partner when the following are not equivalent and why:

• $\frac{2x}{3x}$ and $\frac{2}{3}$

• These are equivalent everywhere except at x = 0. At x = 0, $\frac{2x}{3x}$ is

undefined, whereas
$$\frac{2}{3}$$
 is equal to $\frac{2}{3}$

• $\frac{3x(x-5)}{4(x-5)}$ and $\frac{3x}{4}$

• At
$$x = 5$$
, $\frac{3x(x-5)}{4(x-5)}$ is undefined, whereas $\frac{3x}{4} = \frac{3(5)}{4} = \frac{15}{4}$.

$$\frac{x-3}{x^2-x-6} \text{ and } \frac{1}{x+2}$$

$$\therefore At \ x = 3, \frac{x-3}{x^2-x-6} \text{ is undefined, whereas } \frac{1}{x+2} = \frac{1}{3+2} = \frac{1}{5}$$

Summarize with your partner or in writing any conclusions you can draw about equivalent rational expressions.
 Circulate around the classroom to assess understanding.

Example (6 minutes)

Example Consider the following rational expression: $\frac{2(a-1)-2}{6(a-1)-3a}$. Turn to your neighbor, and discuss the following: For what values of *a* is the expression undefined?

Sample students' answers. When they suggest that the denominator cannot be zero, give the class a minute to work out that the denominator is zero when a = 2.

$\frac{2(a-1)-2}{6(a-1)-3a}$	
6(a-1) - 3a = 0	
3a-6=0	
<i>a</i> = 2	





Let's reduce the rational expression $\frac{2(a-1)-2}{6(a-1)-3a}$ with $a \neq 2$ to lowest terms. Since no common factor is visible in the given form of the expression, we first simplify the numerator and denominator by distributing and combining like

$$\frac{2(a-1)-2}{6(a-1)-3a} = \frac{2a-2-2}{6a-6-3a}$$
$$= \frac{2a-4}{3a-6}$$

• Next, we factor the numerator and denominator, and divide both by any common factors. This step shows clearly why we had to specify that $a \neq 2$.

$$\frac{2(a-1)-2}{6(a-1)-3a} = \frac{2a-4}{3a-6}$$
$$= \frac{2(a-2)}{3(a-2)}$$
$$= \frac{2}{3}$$

Scaffolding:

Students may need to be reminded that although (a - 1) appears in the numerator and denominator, it is not a common *factor* to the numerator and denominator, and thus, we cannot simplify the expression by dividing by (a - 1).

• As long as $a \neq 2$, we see that $\frac{2(a-1)-2}{6(a-1)-3a}$ and $\frac{2}{3}$ are equivalent rational expressions.

If we allow *a* to take on the value of 2, then $\frac{2(a-1)-2}{6(a-1)-3a}$ is undefined. However, the expression $\frac{2}{3}$ is always defined, so these expressions are not equivalent.

Exercise (10 minutes)

terms.

Allow students to work on the following exercises in pairs.

Exercise

а

Reduce the following rational expressions to lowest terms, and identify the values of the variable(s) that must be excluded to prevent division by zero.

$$\frac{2(x+1)+2}{(2x+3)(x+1)-1}$$

$$\frac{2(x+1)+2}{(2x+3)(x+1)-1} = \frac{2x+4}{2x^2+5x+2} = \frac{2(x+2)}{(2x+1)(x+2)} = \frac{2}{2x+1}, \text{ for } x \neq -2 \text{ and } x \neq -\frac{1}{2}.$$

b.
$$\frac{x^2 - x - 6}{5x^2 + 10x}$$
$$\frac{x^2 - x - 6}{5x^2 + 10x} = \frac{(x+2)(x-3)}{5x(x+2)} = \frac{x-3}{5x}, \text{ for } x \neq 0 \text{ and } x \neq -2.$$





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Closing (5 minutes)

The precise definition of a rational expression is presented here for teacher reference and may be shared with students. Discussion questions for closing this lesson follow the definition. Notice the similarity between the definition of a rational expression given here and the definition of a polynomial expression given in the closing of Lesson 5 earlier in this module.

RATIONAL EXPRESSION: A rational expression is either a numerical expression or a variable symbol or the result of placing two previously generated rational expressions into the blanks of the addition operator (__+__), the subtraction operator (__-__), the multiplication operator (__×__), or the division operator (__÷__).

Have students discuss the following questions with a partner and write down their conclusions. Circulate around the room to assess their understanding.

- How do you reduce a rational expression of the form $\frac{P}{Q}$ to lowest terms?
 - Factor the polynomial expressions in the numerator and denominator, and divide any common factors from both the numerator and denominator.
- How do you know which values of the variable(s) to exclude for a rational expression?
 - Any value of the variable(s) that makes the denominator zero at any point of the process must be excluded.



Exit Ticket (6 minutes)







Name

Date _____

Lesson 22: Equivalent Rational Expressions

Exit Ticket

1. Find an equivalent rational expression in lowest terms, and identify the value(s) of the variables that must be excluded to prevent division by zero.

$$\frac{x^2 - 7x + 12}{6 - 5x + x^2}$$

2. Determine whether or not the rational expressions $\frac{x+4}{(x+2)(x-3)}$ and $\frac{x^2+5x+4}{(x+1)(x+2)(x-3)}$ are equivalent for $x \neq -1$, $x \neq -2$, and $x \neq 3$. Explain how you know.





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Exit Ticket Sample Solutions



Problem Set Sample Solutions



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Lesson 22: Equivalent Rational Expressions







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calculation more quickly.



3. Remember that algebra is just a symbolic method for performing arithmetic.
a. Simplify the following rational expression: (x²y)²(xy)³z²/(xy²)²yz.
(x²y)²(xy)³z²/(xy²)²yz = x⁴y² ⋅ x³y³ ⋅ z²/x²y⁵ ⋅ z² = x⁵z
b. Simplify the following rational expression without using a calculator: 12²⋅6³⋅5²/18²⋅15.
12²⋅6³⋅5²/22⋅9²⋅3⋅5 = 2⁴⋅3²⋅2³⋅3³⋅5²/2²⋅3⁵⋅5 = 2⁵⋅5 = 32⋅5 = 160
c. How are the calculations in parts (a) and (b) similar? How are they different? Which expression was easier to simplify?
Both simplifications relied on using the rules of exponents. It was easier to simplify the algebraic expression

in part (a) because we did not have to factor any numbers, such as 18, 15, and 12. However, if we substitute x = 2, y = 3, and z = 5, these two expressions have the exact same structure. Algebra allows us to do this



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Student Outcomes

Students compare rational expressions by writing them in different but equivalent forms.

Lesson Notes

The skills developed in Lessons 22–25 are prerequisites for addressing standard **A-REI.A.2**, solving rational equations, which is the focus of Lessons 26 and 27. In this lesson, students extend comparisons of rational numbers to comparing rational expressions and using numerical, graphical, and algebraic analysis. Although students use graphing calculators to compare certain rational expressions, learning to graph rational functions is not the focus of this lesson.

Classwork

Opening Exercise (10 minutes)

The Opening Exercise serves two purposes: (1) to reactivate prior knowledge of comparing fractions and (2) as a review for students who struggle with fractions. The goal is for students to see that the same process is used to compare fractions and to compare rational expressions.

As done in the previous lesson, give students slips of notecard-sized paper on which to make visual arguments for which fraction is larger. Each student (or pair of students) should get at least two slips of paper. This exercise leads to the graphical analysis employed in the last example of the lesson.



Ask students to make visual arguments as to whether $\frac{1}{3}$ or $\frac{3}{8}$ is larger. Use the following as either scaffolding for struggling students or as an example of student work.

• We can use the following area models to represent the fraction $\frac{1}{3}$ and $\frac{3}{8}$ as we did in Lesson 22.



Scaffolding:

Students already comfortable with fractions may wish to only briefly review the visual representations. However, it is important for each student to be aware of the three methods of comparison: graphical (or visual), numerical, and algebraic (by finding a common denominator).

In any case, do not spend too much time on these exercises, but instead use them as a bridge to comparing rational expressions that contain variables.







Lesson 23

ALGEBRA II

- Discuss with your neighbor another way to make a comparison between the two fractions and why we might not want to always rely on visual representations.
 - Students should suggest finding decimal approximations of fractions and converting the fractions to equivalent fractions with common denominators. Reasons for not using visual representations may include the difficulty with fractions with large denominators.

Once students have had a chance to discuss alternative methods, ask them to choose one of the two methods to verify that the visual representations above are accurate.

- Decimal approximations: We have $\frac{1}{3} \approx 0.333$ and $\frac{3}{8} = 0.375$; thus, $\frac{1}{3} < \frac{3}{8}$.
- Common denominators: We have $\frac{1}{3} = \frac{8}{24}$ and $\frac{3}{8} = \frac{9}{24}$. Since $\frac{8}{24} < \frac{9}{24}$, we know that $\frac{1}{3} < \frac{3}{8}$.
- Discuss with your partner the pros and cons of both methods before discussing as a class.
 - Decimal approximations are quick with a calculator but may take a while if long division is needed. Many students prefer decimals to fractions, but they use approximations of the numbers instead of the exact values of the numbers. Common denominators use the actual numbers but require working with fractions.
- Just as we can determine whether two rational expressions are equivalent in a similar way as we can with
 rational numbers, we can extend our ideas of comparing rational numbers to comparing rational expressions.

Exercises (11 minutes)

As students work on Exercises 1–5, circulate through the class to assess their understanding.









2. Do you expect $\frac{1}{n}$ to be larger or smaller than $\frac{1}{n+1}$? Do you expect $\frac{1}{n}$ to be larger or smaller than $\frac{1}{n+2}$? Explain why. From the table, as n increases, $\frac{1}{n}$ decreases. This means that since 1 + n > n, we will have $\frac{1}{1+n} < \frac{1}{n}$. That is, $\frac{1}{n} > \frac{1}{n+1} > \frac{1}{n+2}$ 3. Compare the rational expressions $\frac{1}{n}$, $\frac{1}{n+1}$, and $\frac{1}{n+2}$ for n = 1, 2, and 3. Do your results support your conjecture For n = 1, $\frac{1}{n} = 1$, $\frac{1}{n+1} = \frac{1}{2}$ and $\frac{1}{n+2} = \frac{1}{3}$ and we know $1 > \frac{1}{2} > \frac{1}{3}$ For n=2, we have $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$, and we know $\frac{1}{2} > \frac{1}{2} > \frac{1}{4}$. For n=3, we have $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{5}$, and we know $\frac{1}{2} > \frac{1}{4} > \frac{1}{5}$. This supports the conjecture that $\frac{1}{n} > \frac{1}{n+1} > \frac{1}{n+2}$. From your work in Exercises 1 and 2, generalize how $\frac{1}{n}$ compares to $\frac{1}{n+m}$, where m and n are positive integers. 4. Since m is a positive integer being added to n, the denominator will increase, which will decrease the value of the rational expression overall. That is, $\frac{1}{n} > \frac{1}{n+m}$ for positive integers m and n. Will your conjecture change or stay the same if the numerator is 2 instead of 1? Make a conjecture about what 5. happens when the numerator is held constant, but the denominator increases for positive numbers. It will stay the same because this would be the same as multiplying the inequality by 2, and multiplication by a positive number does not change the direction of the inequality. If the numerator is held constant and the denominator increases, you are dividing by a larger number, so you get a smaller number overall.

Example (11 minutes)

• Suppose we want to compare the values of the rational expressions $\frac{x+1}{x}$ and $\frac{x+2}{x+1}$ for positive values of x. What are some ways to do this?

Ask students to suggest some methods of comparison. If needed, guide them to the ideas of using a numerical comparison through a table of values and a graphical comparison of the related rational functions $y = \frac{x+1}{x}$ and $y = \frac{x+2}{x+1}$ for x > 0.

• Let's start our comparison of $\frac{x+1}{x}$ and $\frac{x+2}{x+1}$ by looking at a table of values.









Example			
	x	$\frac{x+1}{x}$	$\frac{x+2}{x+1}$
	0.5	3	1.6667
	1	2	1.5000
	1.5	1.6667	1.4000
	2	1.5	1.3333
	5	1.2	1.1667
	10	1.1	1.0909
	100	1.01	1.0099

Have students complete the table below using their calculators and rounding to four decimal places.

Discussion

- From the table of values, it appears that $\frac{x+1}{x} > \frac{x+2}{x+1}$ for positive values of x. However, we have only checked 7 values of x, so we cannot yet say that this is the case for every positive value of x. How else can we compare the values of these two expressions?
 - Students should suggest graphing the functions $y = \frac{x+1}{x}$ and $y = \frac{x+2}{x+1}$.

Have students graph the two functions $y = \frac{x+1}{x}$ and $y = \frac{x+2}{x+1}$ on their calculators, and ask them to share their observations. Does the graph verify the conclusions we drew from the table above?

It seems from both the table of data and from the graph that $\frac{x+1}{x} > \frac{x+2}{x+1}$ for positive values of x, but we have not shown it conclusively. How can we use algebra to determine if this inequality is always true?

• We want to compare
$$\frac{x+1}{x}$$
 and $\frac{x+2}{x+1}$.

Ask students what they need to do before comparing fractions such as $\frac{3}{8}$

and $\frac{2}{7}$. Wait for someone to suggest that they need to find a common denominator.

Let's take a step back and see how we would compare the two fractions $\frac{3}{8}$ and $\frac{2}{7}$. First, we find the common denominator.





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x+1

x+2



Wait for a student to volunteer that the common denominator is 56.

Next, we rewrite each fraction as an equivalent fraction with denominator 56:

$$\frac{3}{8} = \frac{21}{56}$$
 and $\frac{2}{7} = \frac{16}{56}$.

- Since 21 > 16, and 56 is a positive number, we know that $\frac{21}{56} > \frac{16}{56}$; thus, we know that $\frac{3}{8} > \frac{2}{7}$.
- The process for comparing rational expressions is the same as the process for comparing fractions. As is
 always the case with inequalities, we need to be careful about changing the inequality if we multiply or divide
 by a negative number.
- What is the common denominator of the two expressions $\frac{x+1}{x}$ and $\frac{x+2}{x+1}$?

$$\ \ \, x(x+1)$$

• First, multiply the numerator and denominator of the first expression by (x + 1):

$$\frac{x+1}{x} = \frac{(x+1)(x+1)}{x(x+1)}$$
$$\frac{x+1}{x} = \frac{x^2+2x+1}{x(x+1)}.$$

Next, multiply the numerator and denominator of the second expression by x:

$$\frac{x+2}{x+1} = \frac{x(x+2)}{x(x+1)}$$
$$\frac{x+2}{x+1} = \frac{x^2+2x}{x(x+1)}.$$

Clearly, we have

$$x^2 + 2x + 1 > x^2 + 2x$$
,

and since x is always positive, we know that the denominator x(x + 1) is always positive. Thus, we see that

$$\frac{x^2 + 2x + 1}{x(x+1)} > \frac{x^2 + 2x}{x(x+1)},$$

so we have established that $\frac{x+1}{x} > \frac{x+2}{x+1}$ for all positive values of x.

 For rational expressions, numerical and visual comparisons can provide evidence that one expression is larger than another for specified values of the variable. However, finding common denominators and doing the algebra to show that one is larger than the other is the conclusive way to show that the values of one rational expression are consistently larger than the values of another.





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Lesson 23

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ALGEBRA II

Closing (6 minutes)

Ask students to do a side-by-side comparison of the different methods for comparing rational numbers to the extended method used for comparing rational expressions.

Rational Numbers	Rational Expressions
 <u>Visually</u>: Use area models or number lines to represent fractions and compare their relative sizes. Difficult with large numbers 	 <u>Visually</u>: Use a graphing utility to graph functions representing each expression and compare their relative heights. Easy with technology but inconclusive
 <u>Numerically</u>: Perform the division to find a decimal approximation to compare the sizes. <u>Algebraically</u>: Find equivalent fractions with common denominators and compare their numerators. 	 <u>Numerically</u>: Compare several values of the functions to see their relative sizes. Straightforward but tells us even less than graphing
	 <u>Algebraically</u>: Find equivalent fractions with common denominators and compare their numerators. Best way, but special care needs to be taken with values that may be negative.

How do you compare two rational expressions of the form $\frac{P}{Q}$?

Before comparing the expressions, find equivalent rational expressions with the same denominator. Then we can compare the numerators for values of the variable that do not cause the positive/negative signs to switch. Numerical and graphical analysis may be used to help understand the relative sizes of the expressions.

Lesson Summary

To compare two rational expressions, find equivalent rational expression with the same denominator. Then we can compare the numerators for values of the variable that do not cause the rational expression to change from positive to negative or vice versa.

We may also use numerical and graphical analysis to help understand the relative sizes of expressions.

Exit Ticket (7 minutes)



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Name

Date

Lesson 23: Comparing Rational Expressions

Exit Ticket

Use the specified methods to compare the following rational expressions: $\frac{x+1}{x^2}$ and $\frac{1}{x}$.

1. Fill out the table of values.

x	$\frac{x+1}{x^2}$	$\frac{1}{x}$
1		
10		
25		
50		
100		
500		

2. Graph $y = \frac{x+1}{x^2}$ and $y = \frac{1}{x}$ for positive values of x.

3. Find the common denominator, and compare numerators for positive values of *x*.







Exit Ticket Sample Solutions









Lesson 23 M1

ALGEBRA II

3.	Find the common denominator, and compare numerators for positive values of x .
	The common denominator of x and x^2 is x^2 .
	$\frac{1}{x} = \frac{1}{x} \cdot \frac{x}{x} = \frac{x}{x^2}$ $\frac{x+1}{x^2} = \frac{x+1}{x^2}$
	$x^2 = x^2$ For any value of x, x^2 is positive. Since
	x+1>x,
	we then have,
	$\frac{x+1}{x^2} > \frac{x}{x^2}$ $\frac{x+1}{x^2} > \frac{1}{x}.$

Problem Set Sample Solutions

1.	For parts (a)–(d), rewrite each rational expression as an equivalent rational expression so that all expressions have common denominator.		
	a.	$\frac{3}{5}, \frac{9}{10}, \frac{7}{15}, \frac{7}{21}$	
		$\frac{18}{30}, \frac{27}{30}, \frac{14}{30}, \frac{10}{30}$	
	b.	$\frac{m}{sd}, \frac{s}{dm}, \frac{d}{ms}$	
		$\frac{m^2}{msd}, \frac{s^2}{sdm}, \frac{d^2}{dms}$	
	c.	$\frac{1}{(2-x)^2}$, $\frac{3}{(2x-5)(2-x)}$	
		$\frac{(2x-5)}{(2-x)^2(2x-5)}, \frac{-3(2-x)}{(2x-5)(2-x)^2}$	
	d.	$\frac{3}{x^2-x}, \frac{5}{x}, \frac{2x+2}{2x^2-2}$	
		$\frac{3(x+1)}{x(x-1)(x+1)}, \frac{5(x-1)(x+1)}{x(x-1)(x+1)}, \frac{x(x+1)}{x(x-1)(x+1)}$	
2.	If x is	f x is a positive number, for which values of x is $x < \frac{1}{x}$?	
	Befor	Before we can compare two rational expressions, we need to express them as equivalent expressions with a	
	comn	rommon denominator. Since $x \neq 0$, we have $x = \frac{x^2}{x}$. Then $x < \frac{1}{x}$ exactly when $\frac{x^2}{x} < \frac{1}{x}$, which happens when	
	$x^2 <$	$x^2 < 1$. The only positive real number values of x that satisfy $x^2 < 1$ are $0 < x < 1$.	







3. Can we determine if
$$\frac{y}{y-1} \ge \frac{y+1}{y}$$
 for all values $y > 1$? Provide evidence to support your answer.
Before we can compare two rational expressions, we need to express them as equivalent expressions with a common denominator. Since $y > 1$, neither denominator is ever zero. Then $\frac{y}{y-1} = \frac{x^2}{y(y-1)}$ and $\frac{y+1}{y} = \frac{(y+1)(y-1)}{y(y-1)} = \frac{y^2-1}{y(y-1)}$. Since $y^2 > y^2 - 1$ for all values of y , we know that $\frac{y^2}{y(y-1)} > \frac{y^2-1}{y(y-1)}$. Then we can conclude that $\frac{y}{y-1} > \frac{y+1}{y}$ for all values $y > 1$.
4. For positive x , determine when the following rational expressions have negative denominators.
a. $\frac{3}{5}$
Never; 5 is never less than 0.
b. $\frac{x}{5-2x}$
 $5 - 2x < 0$ when $5 < 2x$, which is equivalent to $\frac{5}{2} < x$.
c. $\frac{x+3}{x^2+4x+8}$
For positive x , $x^2 + 4x + 8$ is never negative. One way to see this is that $x^2 + 4x + 8 = (x + 2)^2 + 4$, which is the sum of two positive numbers.
d. $\frac{3x^2}{(x-5)(x+3)(2x+3)}$
For positive x , $x + 3$ and $2x + 3$ are always positive. The number $x - 5$ is negative when $x < 5$, so the denominator is negative when $x < 5$.
5. Consider the rational expressions $\frac{x}{x-2}$ and $\frac{x}{x-4}$.
a. Evaluate each expression for $x = 6$.
If $x = 6$, then $\frac{x}{x-2} = 1$. 5 and $\frac{x}{x-4} = 3$.
b. Evaluate each expression for $x = 3$.
If $x = 3$, then $\frac{x}{x-2} = 3$ and $\frac{x}{x-4} = -3$.
c. Can you conclude that $\frac{x}{x-2} < \frac{x}{x-4}$ for all positive values of x ? Explain how you know.
No, because $\frac{x}{x-2} > \frac{x}{x-4}$ when $x = 3$, it is not true that $\frac{x}{x-2} < \frac{x}{x-4}$ for every positive value of x .







MP.3

MP.2



ALGEBRA II

Extension: Raphael claims that the calculation below shows that $\frac{x}{x-2} < \frac{x}{x-4}$ for all values of x, where d. $x \neq 2$ and $x \neq 4$. Where is the error in the calculation? Starting with the rational expressions $\frac{x}{x-2}$ and $\frac{x}{x-4}$, we need to first find equivalent rational expressions with a common denominator. The common denominator we will use is (x - 4)(x - 2). We then have $\frac{x}{x-2} = \frac{x(x-4)}{(x-4)(x-2)}$ $\frac{x}{x-4} = \frac{x(x-2)}{(x-4)(x-2)}$ Since $x^2 - 4x < x^2 - 2x$ for x > 0, we can divide each expression by (x - 4)(x - 2). We then have $\frac{x(x-4)}{(x-4)(x-2)} < \frac{x(x-2)}{(x-4)(x-2)'}$ and we can conclude that $\frac{x}{x-2} < \frac{x}{x-4}$ for all positive values of x. The error in logic in this calculation is that the denominator (x - 4)(x - 2) is not always a positive number for all positive values of x. In fact, if 2 < x < 4, then (x - 4)(x - 2) < 0. Thus, even though $x^2 - 4x < x^2 - 2x$ when x > 0, the inequality $\frac{x^2 - 4x}{(x-4)(x-2)} < \frac{x^2 - 2x}{(x-4)(x-2)}$ is not valid for every positive value of x. Consider the populations of two cities within the same state where the large city's population is P, and the small 6. city's population is Q. For each of the following pairs, state which of the expressions has a larger value. Explain your reasoning in the context of the populations. P + Q and Pa. The value of P + Q is larger than P. The expression P + Q represents the total population of the two cities, and P represents the population of the larger city. Since these quantities are populations of cities, we can assume they are greater than zero. b. $\frac{P}{P+Q}$ and $\frac{Q}{P+Q}$ The value of $\frac{P}{P+Q}$ is larger. As stated in part (a), P+Q represents the total population of the two cities. Hence, $\frac{P}{P+0}$ and $\frac{Q}{P+0}$ represent each city's respective fraction of the total population. Since P > Q, $\frac{P}{P+Q} > \frac{Q}{P+Q}$ 20 and P + 0c. The value of P + Q is larger than the value of 2Q. The population of the smaller of the two cities is represented by Q, so 2Q represents a population twice the size of the smaller city, but P > Q so P + Q > Q + Q and thus P + Q > 2Q. d. $\frac{P}{Q}$ and $\frac{Q}{P}$ The value of $\frac{P}{Q}$ is larger. These expressions represent the ratio between the populations of the cities. For instance, the larger city is $\frac{P}{Q}$ times larger than the smaller city. Since P > Q, $\frac{P}{Q} > 1 > \frac{Q}{P}$, we can say that there are $\frac{P}{Q}$ people in the larger city for every one person in the smaller city.







e. $\frac{P}{P+O}$ and $\frac{1}{2}$ The value of $\frac{P}{P+Q}$ is larger. Since P is the population of the larger city, the first city represents more than half of the total. f. $\frac{P+Q}{P}$ and P-QThe value of P-Q is larger. The expression P-Q represents the difference in population between the two cities. The expression $\frac{P+Q}{P}$ can represent the ratio of how much larger the total is compared to the population of the larger city, but we know that P represents more than half of the total; therefore, $\frac{P+Q}{P}$ cannot be larger than 2. Without the context, we could not say that P - Q is larger than 2, but in the context of the problem, since P is the population of a large city, and Q is the population of a small city, P - Q > 2. Thus, $P - Q > \frac{P + Q}{P}$. g. $\frac{P+Q}{2}$ and $\frac{P+Q}{Q}$ The value of $\frac{P+Q}{2}$ is larger. The sum divided by the number of cities represents the average population of the two cities and will be significantly higher than the ratio represented by $\frac{P+Q}{Q}$. Alternatively, Q is much larger than 2, so $\frac{P+Q}{2} < \frac{P+Q}{Q}$. h. $\frac{1}{p}$ and $\frac{1}{0}$ The value of $\frac{1}{0}$ is larger. The expression $\frac{1}{0}$ represents the proportion of the population of the second city a single citizen represents. Similarly for $\frac{1}{p}$, since the second city has a smaller population, each individual represents a larger proportion of the whole than in the first city.

EUREKA MATH

MP.2









Lesson 24: Multiplying and Dividing Rational Expressions

Student Outcomes

Students multiply and divide rational expressions and simplify using equivalent expressions.

Lesson Notes

This lesson quickly reviews the process of multiplying and dividing rational numbers using techniques students already know and translates that process to multiplying and dividing rational expressions (MP.7). This enables students to develop techniques to solve rational equations in Lesson 26 (**A-APR.D.6**). This lesson also begins developing facility with simplifying complex rational expressions, which is important for later work in trigonometry. Teachers may consider treating the multiplication and division portions of this lesson as two separate lessons.

Classwork

Opening Exercise (5 minutes)

Distribute notecard-sized slips of paper to students, and ask them to shade the paper to represent the result of $\frac{2}{3} \cdot \frac{4}{5}$. Circulate around the classroom to assess student proficiency. If many students are still struggling to remember the area model after the scaffolding, present the problem to them as shown. Otherwise, allow them time to do the multiplication on their own or with their neighbor and then progress to the question of the general rule.

• First, we represent $\frac{4}{5}$ by dividing our region into five vertical strips of equal area and shading 4 of the 5 parts.



• Now we need to find $\frac{2}{3}$ of the shaded area. So we divide the area horizontally



into three parts of equal area and then shade two of those parts.

Scaffolding:

If students do not remember the area model for multiplication of fractions, have them discuss it with their neighbor. If necessary, use an example like $\frac{1}{2} \cdot \frac{1}{2}$ to see if they can scale this to the problem presented.

If students are comfortable with multiplying rational numbers, omit the area model, and ask them to determine the following products.

$$\begin{array}{c} \frac{2}{3} \cdot \frac{3}{8} = \frac{1}{4} \\ \frac{1}{3} \cdot \frac{5}{6} = \frac{5}{24} \\ \frac{4}{7} \cdot \frac{8}{9} = \frac{32}{63} \end{array}$$



Multiplying and Dividing Rational Expressions




Thus, $\frac{2}{3} \cdot \frac{4}{5}$ is represented by the region that is shaded twice. Since 8 out of 15 subrectangles are shaded twice, we have $\frac{2}{3} \cdot \frac{4}{5} = \frac{8}{15}$. With this in mind, can we create a general rule about multiplying rational numbers?

Allow students to come up with this rule based on the example and prior experience. Have them discuss their thoughts with their neighbor and write the rule.

If a, b, c, and d are integers with $c \neq 0$ and $d \neq 0$, then $\frac{a}{c} \cdot \frac{b}{d} = \frac{ab}{cd}.$

The rule summarized above is also valid for real numbers.

Discussion (2 minutes)

 To multiply rational expressions, we follow the same procedure we use when multiplying rational numbers: we multiply together the numerators and multiply together the denominators. We finish by reducing the product to lowest terms.

```
If a, b, c, and d are rational expressions with b \neq 0, d \neq 0, then

\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.
```

Lead students through Examples 1 and 2, and ask for their input at each step.

Example 1 (4 minutes)

Give students time to work on this problem and discuss their answers with a neighbor before proceeding to class discussion.

Example 1

Make a conjecture about the product $\frac{x^3}{4y} \cdot \frac{y^2}{x}$. What will it be? Explain your conjecture, and give evidence that it is correct.

We begin by multiplying the numerators and denominators.

$$\frac{x^3}{4y} \cdot \frac{y^2}{x} = \frac{x^3 y^2}{4yx}$$

Scaffolding:

 To assist students in making the connection between rational numbers and rational expressions, show a sideby-side comparison of a numerical example from a previous lesson like the one shown.

Lesson 24

$$\frac{2}{3} \cdot \frac{4}{5} = \frac{8}{15} \qquad \frac{x^3}{4y} \cdot \frac{y^2}{x} = ?$$

 If students are struggling with this example, include some others, such as

$$\frac{x^2}{3} \cdot \frac{6}{x} \quad \text{OR} \quad \frac{y}{x^2} \cdot \frac{y^4}{x}.$$

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I: Multiplying and Dividing Rational Expressions

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Identify the greatest common factor (GCF) of the numerator and denominator. The GCF of x^3y^2 and 4xy is xy.

$$\frac{x^3}{4y} \cdot \frac{y^2}{x} = \frac{(xy)x^2y}{4(xy)}$$

Finally, we divide the common factor xy from the numerator and denominator to find the reduced form of the product:

$$\frac{x^3}{4y} \cdot \frac{y^2}{x} = \frac{x^2 y}{4}.$$

Note that the phrases "cancel xy" or "cancel the common factor" are intentionally avoided in this lesson. The goal is to highlight that it is division that allows these expressions to be simplified. Ambiguous words like "cancel" can lead students to simplify $\frac{\sin(x)}{x}$ to sin—they "canceled" the x!

It is important to understand why the numerator and denominator may be divided by x. The rule $\frac{na}{nb} = \frac{a}{b}$ works for rational expressions as well. Performing a simplification such as $\frac{x}{x^3y} = \frac{1}{x^2y}$ requires doing the following steps: $\frac{x}{x^{3}y} = \frac{x \cdot 1}{x \cdot x^{2}y} = \frac{x}{x} \cdot \frac{1}{x^{2}y} = 1 \cdot \frac{1}{x^{2}y} = \frac{1}{x^{2}y}.$

Example 2 (3 minutes)

Example 2

Before walking students through the steps of this example, ask them to try to find the product using the ideas of the previous example.

Find the following product: $\left(\frac{3x-6}{2x+6}\right) \cdot \left(\frac{5x+15}{4x+8}\right)$.

First, factor the numerator and denominator of each rational expression.

Identify any common factors in the numerator and denominator.

$$\left(\frac{3x-6}{2x+6}\right) \cdot \left(\frac{5x+15}{4x+8}\right) = \left(\frac{3(x-2)}{2(x+3)}\right) \cdot \left(\frac{5(x+3)}{4(x+2)}\right)$$
$$= \frac{15(x-2)(x+3)}{8(x+3)(x+2)}$$

MP.

The GCF of the numerator and denominator is x + 3.

Then, divide the common factor (x + 3) from the numerator and denominator, and obtain the reduced form of the product.

$$\left(\frac{3x-6}{2x+6}\right) \cdot \left(\frac{5x+15}{4x+8}\right) = \frac{15(x-2)}{8(x+2)}$$



Lesson 24:

Multiplying and Dividing Rational Expressions





Lesson 24

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ALGEBRA II





Exercises 1–3 (5 minutes)

Students can work in pairs on the following three exercises. Circulate around the class to informally assess their understanding. For Exercise 1, listen for key points such as "factoring the numerator and denominator can help" and "multiplying rational expressions is similar to multiplying rational numbers."



Discussion (5 minutes)

Recall that division of numbers is equivalent to multiplication of the numerator by the reciprocal of the denominator. That is, for any two numbers a and b, where $b \neq 0$, we have

$$\frac{a}{b} = a \cdot \frac{1}{b},$$

where the number $\frac{1}{b}$ is the multiplicative inverse of *b*. But, what if *b* is itself a fraction?

How do we evaluate a quotient such as $\frac{3}{5} \div \frac{4}{7}$?

• How do we evaluate $\frac{3}{5} \div \frac{4}{7}$?

Have students work in pairs to answer this and then discuss.

Scaffolding:

Students may be better able to generalize the procedure for dividing rational numbers by repeatedly dividing several

examples, such as
$$\frac{1}{2} \div \frac{3}{4}$$
,
 $\frac{2}{3} \div \frac{7}{10}$, and $\frac{1}{5} \div \frac{2}{9}$. After
dividing several of these
examples, ask students to
generalize the process (MP.8).









By our rule above, $\frac{3}{5} \div \frac{4}{7} = \frac{3}{5} \cdot \frac{1}{4/7}$. But, what is the value of $\frac{1}{4/7}$? Let x represent $\frac{1}{4/7}$, which is the multiplicative inverse of $\frac{4}{7}$. Then we have $x \cdot \frac{4}{7} = 1$

4x = 7

$$x = \frac{7}{4}.$$

Since we have shown that $\frac{1}{4/7} = \frac{7}{4}$, we can continue our calculation of $\frac{3}{5} \div \frac{4}{7}$ as follows:
$$\frac{3}{5} \div \frac{4}{7} = \frac{3}{5} \cdot \frac{1}{4/7}$$
$$= \frac{3}{5} \cdot \frac{7}{4}$$
$$= \frac{21}{20}.$$

This same process applies to dividing rational expressions, although we might need to perform the additional step of reducing the resulting rational expression to lowest terms. Ask students to generate the rule for division of rational numbers.

If a, b, c, and d are integers with $b \neq 0, c \neq 0$, and $d \neq 0$, then $\frac{a}{c} \div \frac{b}{d} = \frac{a}{c} \cdot \frac{d}{b}.$

The result summarized in the box above is also valid for real numbers.

Now that we know how to divide rational numbers, how do we extend this to divide rational expressions?

 Dividing rational expressions follows the same procedure as dividing rational numbers: we multiply the first term by the reciprocal of the second. We finish by reducing the product to lowest terms.

If *a*, *b*, *c*, and *d* are rational expressions with $b \neq 0$, $c \neq 0$, and $d \neq 0$, then $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$.

Scaffolding:

A side-by-side comparison may help as before.

$$\frac{3}{5} \div \frac{4}{7} = \frac{21}{20} \qquad \frac{x^3}{4y} \div \frac{y^2}{x} = ?$$

For struggling students, give

$$\frac{x^2y}{4} \div \frac{xy^2}{8} = \frac{2x}{y}$$

$$\frac{3y^2}{z-1} \div \frac{12y^5}{(z-1)^2} = \frac{(z-1)}{4y^3}.$$

For advanced students, give

•
$$\frac{x-3}{x^2+x-2} \div \frac{x^2-x-6}{x-1} = \frac{1}{(x+2)^2}$$

• $\frac{x^2-2x-24}{x^2-4} \div \frac{x^2+3x-4}{x^2+x-2} = \frac{x-6}{x-2}.$



4: Multiplying and Dividing Rational Expressions







Example 3 (3 minutes)

As in Example 2, ask students to apply their knowledge of rational-number division to rational expressions by working on their own or with a partner. Circulate to assist and assess understanding. Once students have made attempts to divide, use the scaffolded questions to develop the concept as necessary.

Example 3

Find the quotient and reduce to lowest terms: $\frac{x^2-4}{3x} \div \frac{x-2}{2x}$.

- First, we change the division of $\frac{x^2-4}{3x}$ by $\frac{x-2}{2x}$ into multiplication of $\frac{x^2-4}{3x}$ by the multiplicative inverse of $\frac{x-2}{2x}$. $\frac{x^2-4}{3x} \div \frac{x-2}{2x} = \frac{x^2-4}{3x} \cdot \frac{2x}{x-2}$
- Then, we perform multiplication as in the previous examples and exercises. That is, we factor the numerator and denominator and divide any common factors present in both the numerator and denominator.

$$\frac{x^2 - 4}{3x} \div \frac{x - 2}{2x} = \frac{(x - 2)(x + 2)}{3x} \cdot \frac{2x}{x - 2}$$
$$= \frac{2(x + 2)}{3}$$

Exercise 4 (3 minutes)

Allow students to work in pairs or small groups to evaluate the following quotient.

Exercises 4–5 4. Find the quotient and reduce to lowest terms: $\frac{x^2 - 5x + 6}{x + 4} \div \frac{x^2 - 9}{x^2 + 5x + 4}$ $\frac{x^2 - 5x + 6}{x + 4} \div \frac{x^2 - 9}{x^2 + 5x + 4} = \frac{x^2 - 5x + 6}{x + 4} \cdot \frac{x^2 + 5x + 4}{x^2 - 9}$ $= \frac{(x - 3)(x - 2)}{x + 4} \cdot \frac{(x + 4)(x + 1)}{(x - 3)(x + 3)}$ $= \frac{(x - 2)(x + 1)}{(x + 3)}$

Discussion (4 minutes)

What do we do when the numerator and denominator of a fraction are themselves fractions? We call a fraction that contains fractions a *complex fraction*. Remind students that the fraction bar represents division, so a complex fraction represents division between rational expressions.



4: Multiplying and Dividing Rational Expressions







Allow students the opportunity to simplify the following complex fraction.

$$\frac{\frac{12}{49}}{\frac{27}{28}}$$

Allow students to struggle with the problem before discussing solution methods.

$$\frac{\frac{12}{49}}{\frac{27}{28}} = \frac{12}{49} \div \frac{27}{28}$$
$$= \frac{12}{49} \div \frac{28}{27}$$
$$= \frac{3 \cdot 4}{7 \cdot 7} \cdot \frac{4 \cdot 7}{3^3}$$
$$= \frac{4^2}{7 \cdot 3^2}$$
$$= \frac{16}{63}$$

Notice that in simplifying the complex fraction above, we are merely performing division of rational numbers, and we already know how to do that. Since we already know how to divide rational expressions, we can also simplify rational expressions whose numerators and denominators are rational expressions.

x + 2

x + 5 $=\frac{x}{(x-3)^2}$

6 - *x* –

 $\overline{x^2-2x-3} \stackrel{\div}{=} \overline{x^2+6x+5}$

x + 2 $x^2 + 6x + 5$ $\overline{x^2-2x-3}$ $\overline{x^2-x-6}$ *x* + 2

 $\overline{(x-3)(x+1)}$ (x-3)(x+2)

 $x^2 - x - 6$

(x+5)(x+1)

Exercise 5 (4 minutes)

5.

Allow students to work in pairs or small groups to simplify the following rational expression.

x+2

 $\frac{1}{2x-3} =$

x-6 $x^2 + 6x + 5$

Simplify the rational expression.

Scaffolding: For struggling students, give a simpler example, such as

$$\frac{\left(\frac{2x}{3y}\right)}{\left(\frac{6x}{4y^2}\right)} = \frac{2x}{3y} \div \frac{6x}{4y^2}$$
$$= \frac{2x}{3y} \cdot \frac{4y^2}{6x}$$
$$= \frac{4}{9}y$$



Multiplying and Dividing Rational Expressions







Closing (3 minutes)

Ask students to summarize the important parts of the lesson in writing, to a partner, or as a class. Use this as an opportunity to informally assess understanding of the lesson. In particular, ask students to articulate the processes for multiplying and dividing rational expressions and simplifying complex rational expressions either verbally or symbolically.

Lesson Summary

In this lesson, we extended multiplication and division of rational numbers to multiplication and division of rational expressions.

- To multiply two rational expressions, multiply the numerators together and multiply the denominators together, and then reduce to lowest terms.
- To divide one rational expression by another, multiply the first by the multiplicative inverse of the second, and reduce to lowest terms.
- To simplify a complex fraction, apply the process for dividing one rational expression by another.

Exit Ticket (4 minutes)











Name

Date _____

Lesson 24: Multiplying and Dividing Rational Expressions

Exit Ticket

Perform the indicated operations, and reduce to lowest terms.

1. $\frac{x-2}{x^2+x-2} \cdot \frac{x^2-3x+2}{x+2}$





4: Multiplying and Dividing Rational Expressions







Exit Ticket Sample Solutions



Problem Set Sample Solutions

1.	Perfor	m the following operations:				
	a.	Multiply $rac{1}{3}(x-2)$ by 9.	b.	Divide $rac{1}{4}(x-8)$ by $rac{1}{12}$.	c.	Multiply $\frac{1}{4}\left(\frac{1}{3}x+2\right)$ by 12.
		3x - 6		3x - 24		<i>x</i> + 6
	d.	Divide $\frac{1}{3}\left(\frac{2}{5}x-\frac{1}{5}\right)$ by $\frac{1}{15}$.	e.	Multiply $\frac{2}{3}\left(2x+\frac{2}{3}\right)$ by $\frac{9}{4}$.	f.	Multiply $0.03(4 - x)$ by 100.
		2x - 1		3x + 1		12 - 3x
2.	Write	each rational expression as an $(a^3b^2 c) = a$	equiva	lent rational expression in lowest a^2+6a+9 $3a-9$	terms	5. 6 <i>x</i> 4 <i>x</i>
	a.	$\left(\frac{1}{c^2d^2}\cdot\frac{1}{ab}\right)\div\frac{1}{c^2d^3}$	t	a^2-9 $a+3$		c. $\frac{1}{4x-16} \div \frac{1}{x^2-16}$
		abcd		3		$\frac{3(x+4)}{8}$
	d.	$\frac{3x^2 - 6x}{3x + 1} \cdot \frac{x + 3x^2}{x^2 - 4x + 4}$	e	$\frac{2x^2 - 10x + 12}{x^2 - 4} \cdot \frac{2 + x}{3 - x}$		f. $\frac{a-2b}{a+2b} \div (4b^2-a^2)$
		$\frac{3x^2}{x-2}$		-2		$-\frac{1}{(a+2b)^2}$







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Lesson 24:

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Determine which of the following numbers is larger without using a calculator, $\frac{15^{16}}{16^{15}}$ or $\frac{20^{24}}{24^{20}}$. (Hint: We can 5. compare two positive quantities a and b by computing the quotient $\frac{a}{b}$. If $\frac{a}{b} > 1$, then a > b. Likewise, if $0 < \frac{a}{b} < 1$, then a < b.) $\frac{15^{16}}{16^{15}}\div\frac{20^{24}}{24^{20}}=\frac{15^{16}}{16^{15}}\cdot\frac{24^{20}}{20^{24}}$ $=\frac{3^{16}\cdot 5^{16}}{(2^4)^{15}}\cdot \frac{(2^3)^{20}\cdot 3^{20}}{(2^2)^{24}\cdot 5^{24}}$ $=\frac{2^{60}\cdot 3^{36}\cdot 5^{16}}{2^{108}\cdot 5^{24}}$ $=\frac{3^{36}}{2^{48}\cdot 5^8}$ $=\frac{9^{18}}{2^{40}\cdot 10^8}$ $=\left(\frac{9}{10}\right)^8\cdot\left(\frac{9^{10}}{2^{40}}\right)$ $=\left(\frac{9}{10}\right)^8 \cdot \left(\frac{9}{16}\right)^{10}$ Since $\frac{9}{16} < 1$, and $\frac{9}{10} < 1$, we know that $\left(\frac{9}{10}\right)^8 \cdot \left(\frac{9}{16}\right)^{10} < 1$. Thus, $\frac{15^{16}}{16^{15}} \div \frac{20^{24}}{24^{20}} < 1$, and we know that $\frac{15^{16}}{16^{15}} < \frac{20^{24}}{24^{20}}.$ Extension: One of two numbers can be represented by the rational expression $\frac{x-2}{x}$, where $x \neq 0$ and $x \neq 2$. 6. Find a representation of the second number if the product of the two numbers is 1. a. Let the second number be y. Then $\left(\frac{x-2}{x}\right) \cdot y = 1$, so we have $y=1\div\left(\frac{x-2}{x}\right)$ $=1\cdot\left(\frac{x}{x-2}\right)$ $=\frac{x}{x-2}$. b. Find a representation of the second number if the product of the two numbers is 0. Let the second number be z. Then $\left(\frac{x-2}{x}\right) \cdot z = 0$, so we have $z = \mathbf{0} \div \left(\frac{x-2}{x}\right)$ $= \mathbf{0} \cdot \left(\frac{x}{x-2}\right)$ = 0.

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Lesson 24:

4: Multiplying and Dividing Rational Expressions









Lesson 25: Adding and Subtracting Rational Expressions

Student Outcomes

Students perform addition and subtraction of rational expressions.

Lesson Notes

This lesson reviews addition and subtraction of fractions using the familiar number line technique that students have seen in earlier grades. This leads to an algebraic explanation of how to add and subtract fractions and an opportunity to practice MP.7. The lesson then moves to the process for adding and subtracting rational expressions by converting to equivalent rational expressions with a common denominator. As in the past three lessons, parallels are drawn between arithmetic of rational numbers and arithmetic of rational expressions.

Classwork

The four basic arithmetic operations are addition, subtraction, multiplication, and division. The previous lesson showed how to multiply and divide rational expressions. This lesson tackles the remaining operations of addition and subtraction of rational expressions, which are skills needed to address **A-APR.C.6**. As discussed in the previous lesson, rational expressions are worked with in the same way as rational numbers expressed as fractions. First, the lesson reviews the theory behind addition and subtraction of rational numbers.

Exercise 1 (8 minutes)

First, remind students how to add fractions with the same denominator. Allow them to work through the following sum individually. The solution should be presented to the class either by the teacher or by a student because the process of adding fractions will be extended to the new process of adding rational expressions.

Exercises 1–4

1. Calculate the following sum: $\frac{3}{10} + \frac{6}{10}$. One approach to this calculation is to factor out $\frac{1}{10}$ from each term. $\frac{3}{10} + \frac{6}{10} = 3 \cdot \frac{1}{10} + 6 \cdot \frac{1}{10}$ $= (3+6) \cdot \frac{1}{10}$ $= \frac{9}{10}$ Scaffolding:

If students need practice adding and subtracting fractions with a common denominator, have them compute the following.

 $\frac{2}{5} + \frac{1}{5}$ $\frac{5}{7} - \frac{3}{7}$ $\frac{17}{24} - \frac{12}{24}$

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Ask students for help in stating the rule for adding and subtracting rational numbers with the same denominator.

If a, b, and c are integers with $b \neq 0$, then $\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$ and $\frac{a}{b} - \frac{c}{b} = \frac{a-c}{b}$.

The result in the box above is also valid for real numbers a, b, and c.

- But what if the fractions have different denominators? Let's examine a technique to add the fractions $\frac{2}{r}$ and $\frac{1}{2}$.
- Recall that when we first learned to add fractions, we represented them on a number line. Let's first look at ²/₂.



If we try placing these two segments next to each other, the exact location of the endpoint is difficult to identify.



- The units on the two original graphs do not match. We need to identify a common unit in order to identify the endpoint of the combined segments. We need to identify a number into which both denominators divide without a remainder and write each fraction as an equivalent fraction with that number as the denominator; such a number is known as a *common denominator*.
- Since 15 is a common denominator of $\frac{2}{5}$ and $\frac{1}{3}$, we divide the interval [0, 1] into 15 parts of equal length. Now when we look at the segments of length $\frac{2}{5}$ and $\frac{1}{3}$ placed next to each other on the number line, we can see that the combined segment has length $\frac{11}{15}$.

• How can we do this without using the number line every time? The fraction $\frac{2}{5}$ is equivalent to $\frac{6}{15}$, and the fraction $\frac{1}{3}$ is equivalent to $\frac{5}{15}$. We then have

$$\frac{\frac{2}{5}}{\frac{1}{5}} + \frac{1}{3} = \frac{6}{15} + \frac{5}{15}$$
$$= \frac{11}{15}.$$

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Thus, when adding rational numbers, we have to find a common multiple for the two denominators and write each rational number as an equivalent rational number with the new common denominator. Then we can add the numerators together.

Have students discuss how to rewrite the original fraction as an equivalent fraction with the chosen common denominator. Discuss how the identity property of multiplication allows one to multiply the top and the bottom by the same number so that the product of the original denominator and the number gives the chosen common denominator.

Generalizing, let's add together two rational numbers, $\frac{a}{b}$ and $\frac{c}{d}$. The first step is to rewrite both fractions as equivalent fractions with the same denominator. A simple common denominator that could be used is the product of the original two denominators:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd}$$

Once we have a common denominator, we can add the two expressions together, using our previous rule for adding two expressions with the same denominator:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}.$$

We could use the same approach to develop a process for subtracting rational numbers:

$$\frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}$$

Now that we know to find a common denominator before adding or subtracting, we can state the general rule for adding and subtracting rational numbers. Notice that one common denominator that always works is the product of the two original denominators.

If a, b, c, and d are integers with $b \neq 0$ and $d \neq 0$, then $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$ and $\frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}$.

As with the other rules developed in this and the previous lesson, the rule summarized in the box above is also valid for real numbers.

Exercises 2–4 (5 minutes)

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Ask students to work in groups to write what they have learned in their notebooks or journals. Check in to assess their understanding. Then, have students work in pairs to quickly work through the following review exercises. Allow them to think about how to approach Exercise 4, which involves adding three rational expressions. There are multiple ways to approach this problem. They could generalize the process for two rational expressions, rearrange terms using the commutative property to combine the terms with the same denominator, and then add using the above process, or they could group the addends using the associative property and perform addition twice.

 $\frac{3}{20} - \frac{4}{15} = \frac{9}{60} - \frac{16}{60} = -$

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3.
$$\frac{\pi}{4} + \frac{\sqrt{2}}{5}$$

 $\frac{\pi}{4} + \frac{\sqrt{2}}{5} = \frac{5\pi}{20} + \frac{4\sqrt{2}}{20} = \frac{5\pi + 4\sqrt{2}}{20}$
4. $\frac{a}{m} + \frac{b}{2m} - \frac{c}{m}$
 $\frac{a}{m} + \frac{b}{2m} - \frac{c}{m} = \frac{2a}{2m} + \frac{b}{2m} - \frac{2c}{2m} = \frac{2a + b - 2c}{2m}$

Discussion (2 minutes)

- Before we can add rational numbers or rational expressions, we need to convert to equivalent rational expressions with the same denominators. Finding such a denominator involves finding a common multiple of the original denominators. For example, 60 is a common multiple of 20 and 15. There are other common multiples, such as 120, 180, and 300, but smaller numbers are easier to work with.
- To add and subtract rational expressions, we follow the same procedure as when adding and subtracting rational numbers. First, we find a denominator that is a common multiple of the other denominators, and then we rewrite each expression as an equivalent rational expression with this new common denominator. We then apply the rule for adding or subtracting with the same denominator.

If a, b, and c are rational expressions with $b \neq 0$, then $\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$ and $\frac{a}{b} - \frac{c}{b} = \frac{a-c}{b}$.

Example 1 (10 minutes)

Work through these examples as a class, getting input from students at each step.

```
Example 1
Perform the indicated operations below and simplify.
                 \frac{a+b}{4}+\frac{2a-b}{5}
          a.
                     A common multiple of 4 and 5 is 20, so we can write each expression as an equivalent rational expression
                    with denominator 20. We have \frac{a+b}{4} = \frac{5a+5b}{20} and \frac{2a-b}{5} = \frac{8a-4b}{20}, so that \frac{a+b}{4} + \frac{2a-b}{5} = \frac{5a+5b}{20} + \frac{8a-4b}{20} = \frac{13a+b}{20}.
                    \frac{4}{3x} - \frac{3}{5x^2}
          b.
                    A common multiple of 3x and 5x^2 is 15x^2, so we can write each expression as an equivalent rational expression with denominator 15x^2. We have \frac{4}{3x} - \frac{3}{5x^2} = \frac{20x}{15x^2} - \frac{9}{15x^2} = \frac{20x-9}{15x^2}.
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Adding and Subtracting Rational Expressions





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c.
$$\frac{3}{2x^2+2x} + \frac{5}{x^2-3x-4}$$

Since $2x^2 + 2x = 2x(x+1)$ and $x^2 - 3x - 4 = (x-4)(x+1)$, a common multiple of $2x^2 + 2x$ and $x^2 - 3x - 4$ is $2x(x+1)(x-4)$. Then we have $\frac{3}{2x^2+2x} + \frac{5}{x^2-3x-4} = \frac{3(x-4)}{2x(x+1)(x-4)} + \frac{5\cdot 2x}{2x(x+1)(x-4)} = \frac{13x-12}{2x(x+1)(x-4)}$.

Exercises 5-8 (8 minutes)

Have students work on these exercises in pairs or small groups.

Exercises 5-8 Perform the indicated operations for each problem below. $5. \quad \frac{5}{x-2} + \frac{3x}{4x-8}$ A common multiple is 4(x-2). $\frac{5}{x-2} + \frac{3x}{4x-8} = \frac{20}{4(x-2)} + \frac{3x}{4(x-2)} = \frac{3x+20}{4(x-2)}$ $6. \quad \frac{7m}{m-3} + \frac{5m}{3-m}$ *Notice that* (3 - m) = -(m - 3)*.* A common multiple is (m - 3). $\frac{7m}{m-3} + \frac{5m}{3-m} = \frac{7m}{m-3} + \frac{-5m}{m-3} = \frac{7m}{m-3} - \frac{5m}{m-3} = \frac{2m}{m-3}$ 7. $\frac{b^2}{b^2-2bc+c^2}-\frac{b}{b-c}$ A common multiple is (b - c)(b - c). $\frac{b^2}{b^2 - 2bc + c^2} - \frac{b}{b - c} = \frac{b^2}{(b - c)(b - c)} - \frac{b^2 - bc}{(b - c)(b - c)} = \frac{bc}{(b - c)^2}$ $8. \quad \frac{x}{x^2-1} - \frac{2x}{x^2+x-2}$ A common multiple is (x - 1)(x + 1)(x + 2). $=\frac{1}{(x-1)(x+1)(x+2)}$

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Example 2 (5 minutes)

Example 2

Complex fractions were introduced in the previous lesson with multiplication and division of rational expressions, but these examples require performing addition and subtraction operations prior to doing the division. Remind students that when rewriting a complex fraction as division of rational expressions, they should add parentheses to the expressions both in the numerator and denominator. Then they should work inside the parentheses first following the standard order of operations.

Simplify the following expression.

 $\frac{\frac{b^2+b-1}{2b-1}-1}{4-\frac{8}{(b+1)}}$

First, we can rewrite the complex fraction as a division problem, remembering to add parentheses.

$$\frac{\frac{b^2+b-1}{2b-1}-1}{4-\frac{8}{(b+1)}} = \left(\frac{b^2+b-1}{2b-1}-1\right) \div \left(4-\frac{8}{(b+1)}\right)$$

Remember that to divide rational expressions, we multiply by the reciprocal of the quotient. However, we first need to write each expression as a rational expression in lowest terms. For this, we need to find common denominators.

$$\frac{b^2 + b - 1}{2b - 1} - 1 = \frac{b^2 + b - 1}{2b - 1} - \frac{2b - 1}{2b - 1}$$
$$= \frac{b^2 - b}{2b - 1}$$

$$4 - \frac{8}{(b+1)} = \frac{4(b+1)}{b+1} - \frac{8}{(b+1)}$$
$$= \frac{4b-4}{(b+1)}$$
$$= \frac{4(b-1)}{b+1}$$

Now, we can substitute these equivalent expressions into our calculation above and continue to perform the division as we did in Lesson 24.

$$\frac{\frac{b^2+b-1}{2b-1}-1}{4-\frac{8}{(b+1)}} = \left(\frac{b^2+b-1}{2b-1}-1\right) \div \left(4-\frac{8}{(b+1)}\right)$$
$$= \left(\frac{b^2-b}{2b-1}\right) \div \left(\frac{4(b-1)}{b+1}\right)$$
$$= \left(\frac{b(b-1)}{2b-1}\right) \cdot \left(\frac{(b+1)}{4(b-1)}\right)$$
$$= \frac{b(b+1)}{4(2b-1)}$$

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MP.7





Closing (2 minutes)

Ask students to summarize the important parts of the lesson in writing, to a partner, or as a class. Use this opportunity to informally assess their understanding of the lesson. In particular, ask students to verbally or symbolically articulate the processes for adding and subtracting rational expressions.

Lesson Summary

In this lesson, we extended addition and subtraction of rational numbers to addition and subtraction of rational expressions. The process for adding or subtracting rational expressions can be summarized as follows:

- Find a common multiple of the denominators to use as a common denominator.
- Find equivalent rational expressions for each expression using the common denominator.
- Add or subtract the numerators as indicated and simplify if needed.

Exit Ticket (5 minutes)



5: Adding and Subtracting Rational Expressions









Name

Date _____

Lesson 25: Adding and Subtracting Rational Expressions

Exit Ticket

Perform the indicated operation.

1. $\frac{3}{a+2} + \frac{4}{a-5}$





5: Adding and Subtracting Rational Expressions







Exit Ticket Sample Solutions

Per	form the indicated operation.	
1.	$\frac{3}{a+2} + \frac{4}{a-5}$	
		$\frac{3}{a+2} + \frac{4}{a-5} = \frac{3a-15}{(a+2)(a-5)} + \frac{4a+8}{(a+2)(a-5)}$
		$=rac{7a-7}{(a+2)(a-5)}$
	4r 5	
2.	$\frac{1}{r+3} - \frac{1}{r}$	
		$\frac{4r}{r+3} - \frac{5}{r} = \frac{4r^2}{r(r+3)} - \frac{5r+15}{r(r+3)}$
		$=\frac{4r^2-5r-15}{r(r+3)}$

Problem Set Sample Solutions





5: Adding and Subtracting Rational Expressions







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2. Write as a single rational expression. a. $\frac{1}{x} - \frac{1}{x-1}$ c. $\frac{a-b}{a^2} + \frac{1}{a}$ $b. \qquad \frac{3x}{2y} - \frac{5x}{6y} + \frac{x}{3y}$ $\frac{x}{y}$ $\frac{2a-b}{a^2}$ $-\frac{1}{x(x-1)}$ $\mathsf{d.} \quad \frac{1}{p-2} - \frac{1}{p+2}$ $e. \quad \frac{1}{p-2} + \frac{1}{2-p}$ f. $\frac{1}{b+1} - \frac{b}{1+b}$ $\frac{1-b}{b+1}$ $\frac{4}{(p-2)(p+2)}$ g. $1-\frac{1}{1+p}$ i. $\frac{r}{s-r} + \frac{s}{r+s}$ h. $\frac{p+q}{p-q}-2$ $\frac{p}{1+p}$ $\frac{r^2+s^2}{(s-r)(r+s)}$ $\frac{3q-p}{p-q}$ j. $\frac{3}{x-4} + \frac{2}{4-x}$ $k. \qquad \frac{3n}{n-2} + \frac{3}{2-n}$ $I. \qquad \frac{8x}{3y-2x} + \frac{12y}{2x-3y}$ $\frac{3n-3}{n-2}$ $\frac{1}{x-4}$ m. $\frac{1}{\frac{2m-4n}{m}} - \frac{1}{\frac{2m+4n}{m}} -$ n. $\frac{1}{\frac{(2a-b)(a-c)}{(b-c)(b-2a)}} + \frac{1}{\frac{1}{(b-c)(b-2a)}}$ o. $\frac{b^2+1}{b^2-4} + \frac{1}{b+2} + \frac{1}{b-2}$ $\frac{1}{(b-c)(b-2a)}$ $\frac{b-a}{(a-c)(b-c)(2a-b)} \frac{b^2+2b+1}{(b-2)(b+2)}$ $-\frac{1}{m+2n}$ 3. Write each rational expression as an equivalent rational expression in lowest terms. b. $\frac{\frac{5x}{2}+1}{\frac{5x}{4}-\frac{1}{5x}}$ c. $\frac{1 + \frac{4x+3}{x^2+1}}{1 - \frac{x+7}{x^2+1}}$ a. $\frac{\frac{1}{a} - \frac{1}{2a}}{\frac{4}{2a}}$ **10***x* $\frac{x+2}{x-3}$ 8 5x - 2Extension: 4. Suppose that $x \neq 0$ and $y \neq 0$. We know from our work in this section that $\frac{1}{x} \cdot \frac{1}{y}$ is equivalent to $\frac{1}{xy}$. Is it also true that $\frac{1}{x} + \frac{1}{y}$ is equivalent to $\frac{1}{x+y}$? Provide evidence to support your answer. No, the rational expressions $\frac{1}{x} + \frac{1}{y}$ and $\frac{1}{x+y}$ are not equivalent. Consider x = 2 and y = 1. Then $\frac{1}{x+y} = \frac{1}{2+1} = \frac{1}{3}$, but $\frac{1}{x} + \frac{1}{y} = \frac{1}{2} + 1 = \frac{3}{2}$. Since $\frac{1}{3} \neq \frac{3}{2}$, the expressions $\frac{1}{x} + \frac{1}{y}$ and $\frac{1}{x+y}$ are not equivalent.

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Lesson 25:

Adding and Subtracting Rational Expressions





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5. Suppose that $x = \frac{2t}{1+t^2}$ and $y = \frac{1-t^2}{1+t^2}$. Show that the value of $x^2 + y^2$ does not depend on the value of t. $x^{2} + y^{2} = \left(\frac{2t}{1+t^{2}}\right)^{2} + \left(\frac{1-t^{2}}{1+t^{2}}\right)^{2}$ $= \frac{4t^2}{(1+t^2)^2} + \frac{(1-t^2)^2}{(1+t^2)^2}$ $=\frac{4t^2+(1-2t^2+t)}{(1+t^2)^2}$ $=\frac{1+2t^2+t^4}{1+2t^2+t^4}$ Since $x^2 + y^2 = 1$, the value of $x^2 + y^2$ does not depend on the value of t. 6. Show that for any real numbers a and b, and any integers x and y so that $x \neq 0, y \neq 0, x \neq y$, and $x \neq -y$, $\left(\frac{y}{x}-\frac{x}{y}\right)\left(\frac{ax+by}{x+y}-\frac{ax-by}{x-y}\right)=2(a-b).$ $\left(\frac{y}{x}-\frac{x}{y}\right)\left(\frac{ax+by}{x+y}-\frac{ax-by}{x-y}\right) = \left(\frac{y^2}{xy}-\frac{x^2}{xy}\right)\left(\frac{(ax+by)(x-y)}{(x+y)(x-y)}-\frac{(ax-by)(x+y)}{(x-y)(x+y)}\right)$ $= \left(\frac{y^2 - x^2}{xy}\right) \left(\frac{ax^2 - axy + bxy - by^2}{x^2 - y^2} - \frac{(ax^2 + axy - bxy - by^2)}{x^2 - y^2}\right)$ $= -\left(\frac{x^2 - y^2}{xy}\right) \left(\frac{-2axy + 2bxy}{x^2 - y^2}\right)$ $=-\left(\frac{1}{xy}\right)\left(\frac{-2xy(a-b)}{1}\right)$ = 2(a-b)Suppose that n is a positive integer. 7. Rewrite the product in the form $\frac{P}{Q}$ for polynomials P and Q: $\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n+1}\right)$. a. $\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n+1}\right) = \left(\frac{n+1}{n}\right)\left(\frac{n+2}{n+1}\right) = \left(\frac{n+2}{n}\right)$ Rewrite the product in the form $\frac{P}{Q}$ for polynomials P and Q: $\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n+1}\right)\left(1+\frac{1}{n+2}\right)$. b. $\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n+1}\right)\left(1+\frac{1}{n+2}\right) = \left(\frac{n+1}{n}\right)\left(\frac{n+2}{n+1}\right)\left(\frac{n+3}{n+2}\right) = \left(\frac{n+3}{n}\right)$ Rewrite the product in the form $\frac{P}{O}$ for polynomials P and Q: $\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n+1}\right)\left(1+\frac{1}{n+2}\right)\left(1+\frac{1}{n+2}\right)$ c. $\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n+1}\right)\left(1+\frac{1}{n+2}\right)\left(1+\frac{1}{n+3}\right) = \left(\frac{n+1}{n}\right)\left(\frac{n+2}{n+1}\right)\left(\frac{n+3}{n+2}\right)\left(\frac{n+4}{n+3}\right) = \left(\frac{n+4}{n+3}\right)\left(\frac{n+3}{n+3}\right)\left(\frac{n+3}{n+3}\right)\left(\frac{n+3}{n+3}\right)$ d. If this pattern continues, what is the product of *n* of these factors? If we have n of these factors, then the product will be $\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n+1}\right)\cdots\left(1+\frac{1}{n+(n-1)}\right)=\frac{n+n}{n}=2.$



MP.7

Adding and Subtracting Rational Expressions









Student Outcomes

Students solve rational equations, monitoring for the creation of extraneous solutions.

Lesson Notes

In the preceding lessons, students learned to add, subtract, multiply, and divide rational expressions so that in this lesson they can solve equations involving rational expressions (A-REI.A.2). The skills developed in this lesson are required to solve equations of the form f(x) = c for a rational function f and constant c in Lesson 27 and later in Module 3 (F-BF.B.4a).

There is more than one approach to solving a rational equation, and this section explores two such methods. The first method is to multiply both sides by the common denominator to clear fractions. The second method is to find equivalent forms of all expressions with a common denominator, set the numerators equal to each other, and solve the resulting equation. Either approach requires keeping an eye out for extraneous solutions; in other words, values that appear to be a solution to the equation but cause division by zero and are, thus, not valid. Throughout the work with rational expressions, students analyze the structure of the expressions in order to decide on their next algebraic steps (MP.7). Encourage students to check their answers by substituting their solutions back into each side of the equation separately.

Classwork

Exercises 1–2 (8 minutes)

Scaffolding: Let students solve this any way they can, and then discuss their answers. Focus on adding Struggling students may the fractions on the left and equating numerators or multiplying both sides by a common benefit from first solving the multiple. Indicate a practical technique of finding a common denominator. These first equation $\frac{x}{5} - \frac{2}{5} = \frac{1}{5}$. two exercises highlight MP.7, as students must recognize the given expressions to be of the form $\frac{a}{b} + \frac{c}{b}$ or $\frac{a}{b} + \frac{c}{d}$; by expressing the equations in the simplified form $\frac{A}{B} = \frac{C}{B}$, they More advanced students may try to solve realize that we must have A = C $\frac{x-1}{x+2} = \frac{3}{4}.$ Exercises 1-2 Solve the following equations for x, and give evidence that your solutions are correct. 1. $\frac{x}{2} + \frac{1}{2} = \frac{5}{6}$ Combining the expressions on the left, we have $\frac{3x}{6} + \frac{2}{6} = \frac{5}{6}$, so $\frac{3x+2}{6} = \frac{5}{6}$; therefore, 3x + 2 = 5. Then, x = 1. Or, using another approach: $6 \cdot \left(\frac{x}{2} + \frac{1}{3}\right) = 6 \cdot \left(\frac{5}{6}\right)$, so 3x + 2 = 5; then, x = 1. The solution to this equation is 1. To verify, we see that when x = 1, we have $\frac{x}{2} + \frac{1}{3} = \frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}$, so 1 is a valid solution.







2.
$$\frac{2x}{9} + \frac{5}{9} = \frac{8}{9}$$

Since the expressions already have a common denominator, we see that $\frac{2x}{9} + \frac{5}{9} = \frac{2x+5}{9}$, so we need to solve $\frac{2x+5}{9} = \frac{8}{9}$. It then follows that the numerators are equal, so $2x + 5 = 8$. Solving for x gives $x = \frac{3}{2}$. To verify, we see that when $x = \frac{3}{2}$, we have $\frac{2x}{9} + \frac{5}{9} = \frac{2(\frac{3}{2})}{9} + \frac{5}{9} = \frac{3}{9} + \frac{5}{9} = \frac{8}{9}$; thus, $\frac{3}{2}$ is a valid solution.

Remind students that two rational expressions with the same denominator are equal if the numerators are equal.

Discussion (2 minutes)

Now that students know how to add, subtract, multiply, and divide rational expressions, it is time to use some of those basic operations to solve equations involving rational expressions. An equation involving rational expressions is called a rational equation. Keeping the previous exercise in mind, this section looks at two different approaches to solving rational equations.

Example (6 minutes)

Ask students to try to solve this problem on their own. Have them discuss and explain their methods in groups or with neighbors. The teacher should circulate and lead a discussion of both methods once students have had a chance to try solving on their own.

Example

Solve the following equation: $\frac{x+3}{12} = \frac{5}{6}$. Equating Numerators Method: Obtain expressions on both sides with the same denominator and equate numerators. $\frac{x+3}{12}=\frac{5}{6}\cdot\frac{2}{2}$

 $\frac{x+3}{12} = \frac{10}{12}$

Thus, x + 3 = 10, and x = 7; therefore, 7 is the solution to our original equation.

<u>Clearing Fractions Method</u>: Multiply both sides by a common multiple of the denominators to clear the fractions, and then solve the resulting equation.

$$12 \cdot \left(\frac{x+3}{12}\right) = 12 \cdot \left(\frac{5}{6}\right)$$
$$x+3 = 10$$

We can see, once again, that the solution is 7.





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Discussion (3 minutes)

Ask students to discuss both methods used in the previous example. Which method do they prefer, and why? Does one method seem to be more efficient than the other? Have a few groups report their opinions to the class. At no time should students be required to use a particular method; just be sure they understand both approaches, and allow them to use whichever method seems more natural.

Exercise 3 (6 minutes)

Remind students that when saying a *solution* to an equation, this refers to a value of the variable, usually x, which results in a true number sentence. In Lesson 22, students learned that there are some values of the variable that are not allowed in order to avoid division by zero. Before students start working on the following exercise, ask them to identify the values of x that must be excluded. Wait for students to respond that $x \neq 0$ and $x \neq 2$ before having them work with a partner on the following exercise.

Exercises 3–7

3. Solve the following equation: $\frac{3}{x} = \frac{8}{x-2}$.

<u>Method 1</u>: Convert both expressions to equivalent expressions with a common denominator. The common denominator is x(x-2), so we use the identity property of multiplication to multiply the left side by $\frac{x-2}{x-2}$ and the right side by $\frac{x}{x}$. This does not change the value of the expression on either side of the equation.

$$\frac{\left(\frac{x-2}{x-2}\right)\cdot\left(\frac{3}{x}\right)}{\frac{3x-6}{x(x-2)}} = \frac{8x}{x(x-2)}$$

Since the denominators are equal, we can see that the numerators must be equal; thus, 3x - 6 = 8x. Solving for x gives a solution of $-\frac{6}{5}$. At the outset of this example, we noted that x cannot take on the value of 0 or 2, but there is nothing preventing x from taking on the value $-\frac{6}{5}$. Thus, we have found a solution. We can check our work. Substituting $-\frac{6}{5}$ into $\frac{3}{x}$ gives us $\frac{3}{(-6/5)} = -\frac{5}{2}$, and substituting $-\frac{6}{5}$ into $\frac{8}{x-2}$ gives us $\frac{8}{(-6/5)-2} = -\frac{5}{2}$. Thus, when $x = -\frac{6}{5}$, we have $\frac{3}{x} = \frac{8}{x-2}$; therefore, $-\frac{6}{5}$ is indeed a solution.

<u>Method 2</u>: Multiply both sides of the equation by the common denominator x(x - 2), and solve the resulting equation.

$$x(x-2)\left(\frac{3}{x}\right) = x(x-2)\left(\frac{8}{x-2}\right)$$
$$3(x-2) = 8x$$
$$3x-6 = 8x$$

From this point, we follow the same steps as we did in Method 1, and we get the same solution: $-\frac{6}{5}$



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Exercise 4 (6 minutes)

Have students continue to work with partners to solve the following equation. Walk around the room and observe student progress; if necessary, offer the following hints and reminders:

- Reminder: Ask students to identify excluded values of a. Suggest that they factor the denominator $a^2 4$. They should discover that $a \neq 2$ and $a \neq -2$ must be specified.
- Hint 1: Ask students to identify a common denominator of the three expressions in the equation. They should respond with (a 2)(a + 2), or equivalently, $a^2 4$.
- Hint 2: What do we need to do with this common denominator? They should determine that they need to find equivalent rational expressions for each of the terms with denominator (a 2)(a + 2).



Scaffolding:

Let students know that the word *extraneous* has meaning outside of the mathematics classroom; ask them to guess its definition, and then provide the actual definition.

EXTRANEOUS: Irrelevant or unrelated to the subject being dealt with.

Students may benefit from choral repetition, as well as a visual representation of this word.

Introduce the term *extraneous solution*. An invalid solution that may arise when we manipulate a rational expression is called an extraneous solution. An extraneous solution is a value that satisfies a transformed equation but does not satisfy the original equation.

Exercises 5–7 (8 minutes)

Give students a few minutes to discuss extraneous solutions with a partner. When do they occur, and how do you know when you have one? Extraneous solutions occur when one of the solutions found does not make a true number sentence when substituted into the original equation. The only way to know there is one is to note the values of the variable that will cause a part of the equation to be undefined. This lesson is concerned with division by zero; later lessons exclude values of the variable that would cause the square root of a negative number. Make sure that all students have an understanding of extraneous solutions before proceeding. Then, have them work in pairs on the following exercises.









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5. Solve the following equation. Remember to check for extraneous solutions. $\frac{4}{3x} + \frac{5}{4} = \frac{3}{x}$ First, note that we must have $x \neq 0$. Equating numerators: $\frac{16}{12x} + \frac{15x}{12x} = \frac{36}{12x}$ Then, we have 16 + 15x = 36, and the solution is $x = \frac{4}{3}$. Clearing fractions: $12x\left(\frac{4}{3x}+\frac{5}{4}\right)=12x\left(\frac{3}{x}\right)$ Then, we have 16 + 15x = 36, and the solution is $x = \frac{4}{3}$ The solution $\frac{4}{3}$ is valid since the only excluded value is 0. 6. Solve the following equation. Remember to check for extraneous solutions. $\frac{7}{b+3} + \frac{5}{b-3} = \frac{10b-2}{b^2-9}$ *First, note that we must have* $x \neq 3$ *and* $x \neq -3$ *.* Equating numerators: $\frac{7(b-3)}{(b-3)(b+3)} + \frac{5(b+3)}{(b-3)(b+3)} = \frac{10b-2}{(b-3)(b+3)}$ Matching numerators, we have 7b - 21 + 5b + 15 = 10b - 2, which leads to 2b = 4; therefore, b = 2. Clearing fractions: $(b-3)(b+3)\left(\frac{7}{b+3}+\frac{5}{b-3}\right) = (b-3)(b+3)\left(\frac{10b-2}{b^2-9}\right)$ We have 7(b-3) + 5(b+3) = 10b - 2, which leads to 2b = 4; therefore, b = 2. The solution 2 is valid since the only excluded values are 3 and -3. 7. Solve the following equation. Remember to check for extraneous solutions. $\frac{1}{x-6} + \frac{x}{x-2} = \frac{4}{x^2 - 8x + 12}$ First, note that we must have $x \neq 6$ and $x \neq 2$. Equating numerators: $\frac{x-2}{(x-6)(x-2)} + \frac{x^2 - 6x}{(x-6)(x-2)} = \frac{4}{(x-6)(x-2)}$ $x^2 - 5x - 2 = 4$ $x^2 - 5x - 6 = 0$ (x-6)(x+1) = 0The solutions are 6 and -1. **Clearing fractions:** $\left(\frac{1}{x-6} + \frac{x}{x-2}\right)(x-6)(x-2) = \left(\frac{4}{(x-6)(x-2)}\right)(x-6)(x-2)$ (x-2) + x(x-6) = 4 $x^2 - 6x + x - 2 = 4$ $x^2 - 5x - 6 = 0$ (x-6)(x+1) = 0The solutions are 6 and -1. Because x is not allowed to be 6 in order to avoid division by zero, the solution 6 is extraneous; thus, -1 is the only solution to the given rational equation.



Lesson 26: Solving Rational Equations



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Closing (2 minutes)

Ask students to summarize the important parts of the lesson in writing, to a partner, or as a class. Use this as an opportunity to informally assess understanding of the lesson. In particular, ask students to explain how to identify extraneous solutions and why they arise when solving rational equations.

Lesson Summary

In this lesson, we applied what we have learned in the past two lessons about addition, subtraction, multiplication, and division of rational expressions to solve rational equations. An extraneous solution is a solution to a transformed equation that is not a solution to the original equation. For rational functions, extraneous solutions come from the excluded values of the variable.

Rational equations can be solved one of two ways:

- 1. Write each side of the equation as an equivalent rational expression with the same denominator and equate the numerators. Solve the resulting polynomial equation, and check for extraneous solutions.
- 2. Multiply both sides of the equation by an expression that is the common denominator of all terms in the equation. Solve the resulting polynomial equation, and check for extraneous solutions.

Exit Ticket (4 minutes)











Name

Date _____

Lesson 26: Solving Rational Equations

Exit Ticket

Find all solutions to the following equation. If there are any extraneous solutions, identify them and explain why they are extraneous.

$$\frac{7}{b+3} + \frac{5}{b-3} = \frac{10b}{b^2 - 9}$$









Exit Ticket Sample Solutions

Find all solutions to the following equation. If there are any extraneous solutions, identify them and explain why they are extraneous.

$$\frac{7}{b+3} + \frac{5}{b-3} = \frac{10b}{b^2 - 9}$$

First, note that we must have $x \neq 3$ *and* $x \neq -3$ *.*

Using the equating numerators method: $\frac{7(b-3)}{(b-3)(b+3)} + \frac{5(b+3)}{(b-3)(b+3)} = \frac{10b}{(b-3)(b+3)}$ Matching numerators, we have 7b - 21 + 5b + 15 = 10b, which leads to 12b - 6 = 10b; therefore, b = 3. However, since the excluded values are 3 and -3, the solution 3 is an extraneous solution, and there is no solution to $\frac{7}{b+3} + \frac{5}{b-3} = \frac{10b}{b^2-9}$.

Problem Set Sample Solutions

1.	Solve the following equations, and check for extraneous solutions.					
	a.	$\frac{x-8}{x-4}=2$	b.	$\frac{4x-8}{x-2}=4$	c.	$\frac{x-4}{x-3}=1$
		0		All real numbers except 2		No solution
	d.	$\frac{4x-8}{x-2}=3$	e.	$\frac{1}{2a}-\frac{2}{2a-3}=0$	f.	$\frac{3}{2x+1} = \frac{5}{4x+3}$
		No solution		$-\frac{3}{2}$		-2
	g.	$\frac{4}{x-5} - \frac{2}{5+x} = \frac{2}{x}$	h.	$\frac{y+2}{3y-2} + \frac{y}{y-1} = \frac{2}{3}$	i.	$\frac{3}{x+1} - \frac{2}{1-x} = 1$
		$-\frac{5}{3}$		$\frac{5}{6}, -2$		0, 5
	j.	$\frac{4}{x-1} + \frac{3}{x} - 3 = 0$	k.	$\frac{x+1}{x+3} - \frac{x-5}{x+2} = \frac{17}{6}$	I.	$\frac{x+7}{4} - \frac{x+1}{2} = \frac{5-x}{3x-14}$
		$\frac{1}{3}$, 3		$0, -\frac{55}{17}$		5, 6
	m.	$\frac{b^2 - b - 6}{b^2} - \frac{2b + 12}{b} = \frac{b - 39}{2b}$	n.	$\frac{1}{p(p-4)}+1=\frac{p-6}{p}$	о.	$\frac{1}{h+3} = \frac{h+4}{h-2} + \frac{6}{h-2}$
		$3,\frac{4}{3}$		$\frac{23}{6}$		-8,-4
	p.	$\frac{m+5}{m^2+m} = \frac{1}{m^2+m} - \frac{m-6}{m+1}$				
		4, 1				













b.



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If x represents the first integer, then x + 1 represents the next integer. Suppose $\frac{1}{x} + \frac{1}{x+1} = \frac{3}{4}$. Then, $\frac{1}{x} + \frac{1}{x+1} = \frac{3}{4}$ $\frac{4(x+1)+4x}{4x(x+1)} = \frac{3x(x+1)}{4x(x+1)}$ $8x + 4 = 3x^2 + 3x$ $3x^2 - 5x - 4 = 0.$ The solutions to this quadratic equation are $\frac{5+\sqrt{73}}{6}$ and $\frac{5-\sqrt{73}}{6}$, so there are no integers that solve this equation. Thus, there are no pairs of consecutive integers whose reciprocals sum to $\frac{3}{4}$ Does there exist a pair of consecutive even integers whose reciprocals sum to $\frac{3}{4}$? Explain how you know. c. If x represents the first integer, then x + 2 represents the next even integer. Suppose $\frac{1}{x} + \frac{1}{x+2} = \frac{3}{4}$. Then, $\frac{1}{x} + \frac{1}{x+2} = \frac{3}{4}$ $\frac{4(x+2)+4x}{4x(x+2)} = \frac{3x(x+2)}{4x(x+2)}$ $8x + 8 = 3x^2 + 6x$ $3x^2 - 2x - 8 = 0.$ The solutions to this quadratic equation are $-\frac{4}{3}$ and 2; therefore, the only even integer x that solves the equation is 2. Then, 2 and 4 are consecutive even integers whose reciprocals sum to $\frac{3}{4}$. Does there exist a pair of consecutive *even* integers whose reciprocals sum to $\frac{5}{c}$? Explain how you know. d. If x represents the first integer, then x + 2 represents the next even integer. Suppose $\frac{1}{x} + \frac{1}{x+2} = \frac{5}{6}$. Then, $\frac{1}{x} + \frac{1}{x+2} = \frac{5}{6}$ $\frac{6(x+2)+6x}{6x(x+2)} = \frac{5x(x+2)}{6x(x+2)}$ $12x + 12 = 5x^2 + 10x$ $5x^2 - 2x - 12 = 0.$ The solutions to this quadratic equation are $\frac{1+\sqrt{61}}{5}$ and $\frac{1-\sqrt{61}}{5}$, so there are no integers that solve this equation. Thus, there are no pairs of consecutive even integers whose reciprocals sum to $\frac{5}{6}$

Does there exist a pair of consecutive integers whose reciprocals sum to $\frac{3}{4}$? Explain how you know.

EUREKA MATH

Lesson 26: Solving Rational Equations





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Lesson 27: Word Problems Leading to Rational Equations

Student Outcomes

Students solve word problems using models that involve rational expressions.

Lesson Notes

In the preceding lessons, students learned to add, subtract, multiply, and divide rational expressions and solve rational equations in order to develop the tools needed for solving application problems involving rational equations in this lesson (A-REI.A.2). Students develop their problem-solving and modeling abilities by carefully reading the problem description and converting information into equations (MP.1), thus creating a mathematical model of the problem (MP.4).

Classwork

Exercise 1 (13 minutes)

This lesson turns to some applied problems that can be modeled with rational equations, strengthening students' problem-solving and modeling experience in alignment with standards MP.1 and MP.4. These equations can be solved using the skills developed in previous lessons. Have students work in small groups to answer this set of four questions. At the end of the work time, ask different groups to present their solutions to the class. Suggest to students that they: (a) read the problem aloud, (b) paraphrase and summarize the problem in their own words, (c) find an equation that models the situation, and (d) say how it represents the quantities involved. Check to make sure that students understand the problem before they begin trying to solve it.

In Exercise 1, consider encouraging students to assign the variable m to the unknown quantity, and ask if they can arrive at an equation that relates m to the known quantities.



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Students may benefit from

having the problem read aloud

and summarized. They should

problem in their own words to

= 0.75, and ask

students how this models the

Students who may be working above grade level could be

challenged to write their own word problems that result in

rational equations.

be encouraged to restate the

If students are struggling,

present the equation

Scaffolding:

a partner.

12 + m

20 + m

situation.

ALGEBRA II

Exercise 1

- Anne and Maria play tennis almost every weekend. So far, Anne has won $12 \mbox{ out of } 20$ 1. matches.
 - а. How many matches will Anne have to win in a row to improve her winning percentage to 75%?

Suppose that Anne has already won 12 of 20 matches, and let m represent the number of additional matches she must win to raise her winning percentage to 75%. After playing and winning all of those additional m matches, she has won 12 + mmatches out of a total of 20 + m matches played. Her winning percentage is then 12 + m

and we want to find the value of m that solves the equation 20+m'

$$\frac{12+m}{20+m} = 0.75$$

Multiply both sides by 20 + m.

$$12 + m = 0.75(20 + m)$$
$$12 + m = 15 + 0.75m$$

Solve for m:

$$0.25m = 3$$

 $m = 12$

So, Anne would need to win 12 matches in a row in order to improve her winning percentage to 75%.

How many matches will Anne have to win in a row to improve her winning percentage to 90%? b.

This situation is similar to that for part (a), except that we want a winning percentage of 0.90, instead of 0.75. Again, we let m represent the number of matches Anne must win consecutively to bring her winning percentage up to 90%.

$$\frac{12+m}{20+m} = 0.90$$

Solve for m:

12 + m = 0.90(20 + m)12 + m = 18 + 0.90m0.10m = 6m = 60

In order for Anne to bring her winning percentage up to 90%, she would need to win the next 60 consecutive matches.

Can Anne reach a winning percentage of 100%? c.

> Allow students to come to the conclusion that Anne will never reach a winning percentage of 100% because she has already lost 8 matches.



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After Anne has reached a winning percentage of 90% by winning consecutive matches as in part (b), how d. many matches can she now lose in a row to have a winning percentage of 50%? Recall from part (b) that she had won 72 matches out of 80 to reach a winning percentage of 90%. We will now assume that she loses the next k matches in a row. Then, she will have won 72 matches out of 80 + kmatches, and we want to know the value of k that makes this a 50% win rate. $\frac{72}{80+k} = 0.50$ Solving the equation: 72 = 0.50(80 + k)72 = 40 + 0.50k32 = 0.50k64 = kThus, after reaching a 90% winning percentage in 80 matches, Anne can lose 64 matches in a row to drop to a 50% winning percentage.

Example (5 minutes)

Work this problem at the front of the room, but allow the class to provide input and steer the discussion. Depending on how students did with the first exercise, the teacher may lead a discussion of this problem as a class, ask students to work in groups, or ask students to work independently while targeting instruction with a small group that struggled on the first exercise.

Example

Working together, it takes Sam, Jenna, and Francisco two hours to paint one room. When Sam works alone, he can paint one room in 6 hours. When Jenna works alone, she can paint one room in 4 hours. Determine how long it would take Francisco to paint one room on his own. Consider how much can be accomplished in one hour. Sam, Jenna, and Francisco together can paint half a room in one hour. If Sam can paint one room in 6 hours on his own, then in one hour he can paint $\frac{1}{6}$ of the room. Similarly, Jenna can paint of the room in one hour. We do not yet know how much Francisco can paint in one hour, so we will say he can paint $\frac{1}{6}$ of the room. So, in one hour, Sam has painted $\frac{1}{6}$ of the room, Jenna has painted $\frac{1}{4}$ of the room, and all three together can paint $\frac{1}{2}$ the room, leading to the following equation for how much can be painted in one hour:

$$\frac{1}{6} + \frac{1}{4} + \frac{1}{f} = \frac{1}{2}.$$

A common multiple of the denominators is 12f. Multiplying both sides by 12f gives us:

$$\frac{12f}{6} + \frac{12f}{4} + \frac{12f}{f} = \frac{12f}{2}$$
$$2f + 3f + 12 = 6f$$

which leads us to the value of f:

f = 12.

So, Francisco can paint the room in 12 hours on his own.



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MP.4



Exercise 2 (5 minutes)

MP.1 & MP.4 Remind students that distance equals rate times time ($d = r \cdot t$) before having them work on this exercise in pairs or small groups. Be sure to have groups share their results before continuing to the next exercise.

Exe	rcises 2–4			
2.	Melissa walks 3 miles to the house of a friend and returns home on a bike. She averages 4 miles per hour faster when cycling than when walking, and the total time for both trips is two hours. Find her walking speed.			
	Using the relationship $d = r \cdot t$, we have $t = \frac{d}{r}$. The time it takes for Melissa to walk to her friend's house is $\frac{3}{r}$, and			
	the time to cycle back is $\frac{3}{r+4}$. Thus, we can write an equation that describes the combined time for both trips:			
	$\frac{3}{r}+\frac{3}{r+4}=2.$			
	A common multiple of the denominators is $r(r+4)$, so we multiply both sides of the equation by $r(r+4)$.			
	3(r+4) + 3r = 2r(r+4)			
	$3r + 12 + 3r = 2r^2 + 8r$			
	$2r^2 + 2r - 12 = 0$			
	2(r-2)(r+3) = 0			
	Thus, $r = -3$ or $r = 2$. Since r represents Melissa's speed, it does not make sense for r to be negative. So, the only			

solution is 2, which means that Melissa's walking speed is 2 miles per hour.

Exercise 3 (10 minutes)

3.	You have ${f 10}$ liters of a juice blend that is ${f 60\%}$ juice.				
	a.	How many liters of pure juice need to be added in order to make a blend that is 75% juice?			
		We start off with 10 liters of a blend containing 60% juice. Then, this blend contains $0.60(10) = 6$ liters of juice in the 10 liters of mixture. If we add A liters of pure juice, then the concentration of juice in the blend is $6+A$			
		$\frac{10+A}{10+A}$. We want to know which value of A makes this blend 75% juice.			
		$\frac{6+A}{10+A} = 0.75$			
		6 + A = 0.75(10 + A)			
		6 + A = 7.5 + 0.75A			
		0.25A = 1.5			
		A = 6			
		Thus, if we add 6 liters of pure juice, we have 16 liters of a blend that contains 12 liters of juice, meaning that the concentration of juice in this blend is 75% .			



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M1

Lesson 27

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b. How many liters of pure juice need to be added in order to make a blend that is 90% juice? $\frac{6+A}{10+A}=0.90$ 6 + A = 0.9(10 + A)6 + A = 9 + 0.9A3 = 0.1A*A* = 30 Thus, if we add 30 liters of pure juice, we will have 40 liters of a blend that contains 36 liters of pure juice, meaning that the concentration of juice in this blend is 90%. Write a rational equation that relates the desired percentage p to the amount A of pure juice that needs to c. be added to make a blend that is p% juice, where 0 . What is a reasonable restriction on the setof possible values of p? Explain your answer. $\frac{6+A}{10+A}=\frac{p}{100}$ We need to have 60 for the problem to make sense. We already have <math>60% juice; the percentage cannot decrease by adding more juice, and we can never have a mixture that is more than 100% juice. d. Suppose that you have added 15 liters of juice to the original 10 liters. What is the percentage of juice in this blend? $\frac{p}{100} = \frac{6+15}{10+15} = 0.84$ So, the new blend contains 84% pure juice. Solve your equation in part (c) for the amount A. Are there any excluded values of the variable p? Does this e. make sense in the context of the problem? $\frac{6+A}{10+A} = \frac{p}{100}$ 100(6+A) = p(10+A)600 + 100A = 10p + pA100A - pA = 10p - 600A(100 - p) = 10p - 600 $A = \frac{10p - 600}{100 - p}$ We see from the equation for A that $p \neq 100$. This makes sense because we can never make a 100% juice solution since we started with a diluted amount of juice.

Exercise 4 (5 minutes)

MP.4

MP.2

Allow students to work together in pairs or small groups for this exercise. This exercise is a bit different from the previous example in that the amount of acid comes from a diluted solution and not a pure solution. Be sure that students set up the numerator correctly. (If there is not enough time to do the entire problem, have students set up the equations in class and finish solving them for homework.)



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Closing (2 minutes)

MP.4

MP.2

Ask students to summarize the important parts of the lesson in writing, to a partner, or as a class. Use this as an opportunity to informally assess understanding of the lesson.

Exit Ticket (5 minutes)









Name

Date _____

Lesson 27: Word Problems Leading to Rational Equations

Exit Ticket

Bob can paint a fence in 5 hours, and working with Jen, the two of them painted a fence in 2 hours. How long would it have taken Jen to paint the fence alone?



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Exit Ticket Sample Solutions

Bob can paint a fence in 5 hours, and working with Jen, the two of them painted a fence in 2 hours. How long would it have taken Jen to paint the fence alone?

Let x represent the time it would take Jen to paint the fence alone. Then, Bob can paint the entire fence in 5 hours; therefore, in one hour he can paint $\frac{1}{5}$ of the fence. Similarly, Jen can paint $\frac{1}{x}$ of the fence in one hour. We know that it took them two hours to complete the job, and together they can paint $\frac{1}{2}$ of the fence in one hour. We then have to solve the equation:

 $\frac{1}{5} + \frac{1}{x} = \frac{1}{2}$ $\frac{2x}{10x} + \frac{10}{10x} = \frac{5x}{10x}$ 2x + 10 = 5x $x = \frac{10}{3}.$

Thus, it would have taken Jen 3 hours and 20 minutes to paint the fence alone.

Problem Set Sample Solutions



EUREKA MATH 7: Word Problems Leading to Rational Equations







M1

Lesson 27

The difference in the average speed of two trains is 16 miles per hour. The slower train takes 2 hours longer to 4. travel 170 miles than the faster train takes to travel 150 miles. Find the speed of the slower train. $\frac{150}{t} - \frac{170}{t+2} = 16$ We find that t = 3, so it takes 3 hours for the faster train to travel 150 miles, and it takes 5 hours for the slower train to travel 170 miles. The average speed of the slower train is 34 miles per hour. 5. A school library spends \$80 a month on magazines. The average price for magazines bought in January was 70 cents more than the average price in December. Because of the price increase, the school library was forced to subscribe to 7 fewer magazines. How many magazines did the school library subscribe to in December? $\frac{80}{x+0.70} = \frac{80}{x} - 7$ The solution to this equation is 2.50, so the average price in December is \$2.50. Thus the school subscribed to 32magazines in December. An investor bought a number of shares of stock for \$1,600. After the price dropped by \$10 per share, the investor 6. sold all but 4 of her shares for \$1, 120. How many shares did she originally buy? $\frac{1600}{x} = \frac{1120}{x-4} + 10$ This equation has two solutions: 32 and 20. Thus, the investor bought either 32 or 20 shares of stock. Newton's law of universal gravitation, $F = \frac{Gm_1m_2}{r^2}$, measures the force of gravity between two masses m_1 and m_2 , 7. where r is the distance between the centers of the masses, and G is the universal gravitational constant. Solve this equation for G. $G = \frac{Fr^2}{m_1 m_2}$ Suppose that $=\frac{x+y}{1-xy}$. 8. Show that when $x = \frac{1}{a}$ and $y = \frac{2a-1}{a+2}$, the value of *t* does not depend on the value of *a*. a. When simplified, we find that t = 2; therefore, the value of t does not depend on the value of a. For which values of *a* do these relationships have no meaning? b. If a is 0, then x has no meaning. If a = -2, then y has no meaning. Consider the rational equation $\frac{1}{R} = \frac{1}{r} + \frac{1}{r}$. Find the value of *R* when $x = \frac{2}{5}$ and $y = \frac{3}{4}$. $\frac{1}{R} = \frac{1}{2/5} + \frac{1}{3/4}$ So $R = \frac{6}{23}$



Lesson 27:

Word Problems Leading to Rational Equations



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27: Word Problems Leading to Rational Equations





Extension: 11. Suppose that Huck Finn can paint a fence in 5 hours. If Tom Sawyer helps him paint the fence, they can do it in 3 hours. How long would it take for Tom to paint the fence by himself? Huck paints the fence in 5 hours, so his rate of fence painting is $\frac{1}{5}$ fence per hour. Let T denote the percentage of the fence Tom can paint in an hour. Then 1 fence = $\left(\left(\frac{1}{5}+T\right)$ fence per hour $\right) \cdot (3 \text{ hours}).$ $3 = \frac{1}{\frac{1}{\frac{1}{5} + T}} = \frac{1}{\frac{1}{\frac{1}{5} + \frac{5T}{5}}} = \frac{5}{1 + 5T}$ 3(1+5T) = 515T = 2 $T = \frac{2}{15}$ So, Tom can paint $\frac{2}{15}$ of the fence in an hour. Thus, Tom would take $\frac{15}{2} = 7.5$ hours to paint the fence by himself. 12. Huck Finn can paint a fence in 5 hours. After some practice, Tom Sawyer can now paint the fence in 6 hours. How long would it take Huck and Tom to paint the fence together? a. The amount of fence that Huck can paint per hour is $\frac{1}{5}$, and the amount that Tom can paint per hour is $\frac{1}{6}$. So, together they can paint $\frac{1}{5} + \frac{1}{6}$ of the fence per hour. Suppose the entire job of painting the fence takes h hours. Then, the amount of the fence that is painted is $h\left(\frac{1}{5}+\frac{1}{6}\right)$. Since the entire fence is painted, we need to solve the equation $h\left(\frac{1}{5} + \frac{1}{6}\right) = 1$. $h\left(\frac{1}{5}+\frac{1}{6}\right)=1$ $h = \frac{1}{\frac{1}{1} + \frac{1}{4}} = \frac{30}{6+5} = \frac{30}{11}$

So, together they can paint the fence in $\frac{30}{11}$ hours, which is 2 hours and 44 minutes.



27: Word Problems Leading to Rational Equations





EUREKA MATH

Lesson 27:

27: Word Problems Leading to Rational Equations



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Lesson 28: A Focus on Square Roots

Student Outcomes

- Students solve simple radical equations and understand the possibility of extraneous solutions. They
 understand that care must be taken with the role of square roots so as to avoid apparent paradoxes.
- Students explain and justify the steps taken in solving simple radical equations.

Lesson Notes

In the next two lessons, students work with radical expressions and equations. They extend their understanding of the idea that not all operations are invertible, which was explored in Algebra I and continued in the previous lessons on solving rational equations. Squaring both sides of an equation in some cases produces an extraneous solution. They also continue to work with rational expressions and equations as seen in the previous lessons, but those expressions now contain radicals. This lesson highlights standards **A-REI.A.1**, which calls for students to be able to explain each step required to solve an equation, and **A-REI.A.2**, which calls for students to solve a radical equation and show how extraneous solutions might arise. It also addresses the standard MP.3 by building and analyzing arguments used in solving radical equations. In Example 2, students consider the difference between working with an *expression*, whose value must be preserved, and working with an *equation*, whose sides can be changed in value as long as equality is preserved. This difference addresses the standard MP.7 because students are stepping back to get an overview of expressions and equations as objects subject to different rules.

Classwork

Opening (1 minute)

Recall that working with radical expressions can be tricky, especially when negative numbers are involved. When solving a radical equation, one must always check the answers found to verify that they are indeed valid solutions to the equation. In some cases, extraneous solutions appear and must be eliminated. Recall that an *extraneous solution* is one that satisfies a transformed equation but not the original one.

Exercises 1-4 (7 minutes)

Give students a few minutes to work through the first four exercises, and then discuss the results as a whole class. Circulate the room to assess students' understanding.

Scaffolding:

- If students are struggling, show a few simpler examples such as solving $\sqrt{x} = 5$ or $\sqrt{x} = -5$.
- Another option would be to provide the following alternative model for students to complete.

Fill in the blanks to fully show and explain the solution process.

$$\sqrt{x} - 6 = 4$$

$$\underline{\sqrt{x} = 10}$$
Added 6 to both sides to isolate the radical.
$$\sqrt{x^2} = 10^2$$

$$x = 100$$

$$\underline{Square \ both \ sides.}$$



Lesson 28: A Focus on Square Roots





Exercises 1–4

For Exercises 1–4, describe each step taken to solve the equation. Then, check the solution to see if it is valid. If it is not a valid solution, explain why.

1.	$\sqrt{x}-6=4$		2.	$\sqrt[3]{x}-6=4$	
	$\sqrt{x} = 10$	Add 6 to both sides.		$\sqrt[3]{x} = 10$	Add 6 to both sides.
	<i>x</i> = 100	Square both sides.		<i>x</i> = 1000	Cube both sides.
	<i>Check:</i> $\sqrt{100} - 6 =$	= 10 - 6 = 4		<i>Check:</i> $\sqrt[3]{1000} - 6$	= 10 - 6 = 4
	So 100 is a valid sol	lution.		So 1,000 is a valid s	olution.
3.	$\sqrt{x} + 6 = 4$		4.	$\sqrt[3]{x} + 6 = 4$	
	$\sqrt{x} = -2$			$\sqrt[3]{x} = -2$	
	x = 4			x = -8	
	Check: $\sqrt{4} + 6 = 2$ a valid solution.	$+ 6 = 8$, and $8 \neq 4$, so 4 is not		Check: $\sqrt[3]{-8} + 6 = \frac{1}{2}$ solution.	-2 + 6 = 4, so -8 is a valid

Discussion

MP.3

Consider each of the following questions, one at a time.

- What was the first step taken in solving the radical equations in Exercises 1 and 2?
 - D The radical was isolated.
- What was the second step taken?
 - Both sides were squared or cubed to eliminate the radical.
- What happened in Exercise 3?
 - The same steps were used to solve the equation as were used in Exercise 1, but this time the solution found did not work. There is no solution to the equation.
- Why did that happen?

This is one of the focal points of the lesson. Ask students to answer in writing or discuss with a partner before sharing their answers with the rest of the class. In the discussion, emphasize that 4 is an *extraneous solution*; it is the solution to x = 4 but not to the original equation.

- For 4 to be a solution, $\sqrt{4}$ would need to equal -2. Even though $(-2)^2 = 4$, we define $\sqrt{4} = 2$ so that $f(x) = \sqrt{x}$ takes on only one value for $x \ge 0$ and is thus a function. As a result, the square root of a positive number is only the positive value. Therefore, 4 is an extraneous solution.
- What other types of equations sometimes have extraneous solutions?
 - Rational equations can have extraneous solutions that create zero in the denominator.
- Why did the solution process work in Exercise 4?
 - The cube root of a negative number is negative, so a cube root equation does not have the same issues with the negative numbers as a square root does.





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Example 1 (5 minutes)

Work through the example to solidify the steps in solving a radical equation. Depending on how students did with the first four exercises, consider having them continue working with a partner. Or, work through this example with the whole class at once. Be sure students can explain and justify the steps they are taking.

Example 1

```
Solve the following radical equation. Be sure to check your solutions.
                                                             \sqrt{3x+5} - 2 = -1
                                                   Solution: \sqrt{3x+5} = 1
                                                                3x + 5 = 1
                                                                     3x = -4
                                                                      x = -\frac{4}{3}
Check: \sqrt{3\left(-\frac{4}{3}\right)+5} - 2 = \sqrt{-4+5} - 2 = \sqrt{1} - 2 = -1, so -\frac{4}{3} is a valid solution.
```

Discussion

- What was the first step you took?
 - I isolated the radical.
- Why did you do that first? .
 - Isolating the radical allows it to be eliminated by squaring or cubing both sides of the equation.
- What was the next step?
 - I squared both sides.
- Why did you do that?
 - The purpose was to eliminate the radical from the equation.
- Even though we are solving a new type of equation, does this feel like a familiar process?
 - Yes. When solving an equation, we work on undoing any operation by doing the inverse. To undo a square root, we use the inverse, so we square the expression.
- How do the steps we are following relate to your previous experiences with solving other types of equations?
 - We are still following the basic process to solve an equation, which is to undo any operation on the same side as the variable by using the inverse operation.
- Why is it important to check the solution?
 - Sometimes extraneous solutions appear because the square root of a positive number or zero is never negative.

Summarize (in writing or with a partner) what you have learned about solving radical equations. Be sure that you explain what to do when you get an extraneous solution.







Exercises 5–15 (15 minutes)

Allow students time to work the problems individually and then check with a partner. Circulate around the room. Make sure students are checking for extraneous solutions.

Exer	cises 5–15		
Solv	e each radical equation. Be sure to check your solutions.		
5.	$\sqrt{2x-3}=11$	6.	$\sqrt[3]{6-x} = -3$
	62		33
-	$\sqrt{x+5}$ 0 - 12	0	$\sqrt{4\pi - 7} = \sqrt{2\pi + 9}$
7.	$\sqrt{x} + 5 - 9 = -12$ No solution	0.	$\sqrt{4x} - 7 = \sqrt{5x} + 9$
9.	$-12\sqrt{x-6}=18$	10.	$3\sqrt[3]{x+2}=12$
	No solution		62
11	$\sqrt{r^2-5}=2$	12	$\sqrt{r^2 + 8r} = 3$
	33		-9. 1
	-, -		-, -

- Which exercises produced extraneous solutions?
 - Exercises 7 and 9
- Which exercises produced more than one solution? Why?
 - Exercises 11 and 12 because after eliminating the radical, the equation became a quadratic equation.
 Both solutions were valid when checked.
- Write an example of a radical equation that has an extraneous solution. Exchange with a partner and confirm that the example does in fact have an extraneous solution.

Compute each product, and combine like terms.					
13. $(\sqrt{x}+2)(\sqrt{x}-2)$	14. $(\sqrt{x}+4)(\sqrt{x}+4)$	$15. (\sqrt{x-5})(\sqrt{x-5})$			
x-4	$x + 8\sqrt{x} + 16$	<i>x</i> – 5			

In the next example and exercises, we are working with equations and expressions that contain quotients and radicals. The purpose of these problems is to continue to build fluency working with radicals, to build on the work done in the previous lessons on rational expressions and equations, and to highlight MP.7, which calls for students to recognize and make use of structure in an expression.







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Example 2 (5 minutes)

Work through the two examples as a class, making sure students understand the differences between working with an *expression* and working with an *equation*.

Example 2				
Rationalize the denominator in each expression. That is, rewrite the expression so that there is a rational expression in the denominator.				
a.	$\frac{x-9}{\sqrt{x-9}}$	b.	$\frac{x-9}{\sqrt{x+3}}$	
	$\frac{x-9}{\sqrt{x-9}} \cdot \frac{\sqrt{x-9}}{\sqrt{x-9}}$		$\frac{x-9}{\sqrt{x}+3} \cdot \frac{\sqrt{x}-3}{\sqrt{x}-3}$	
	$\frac{(x-9)\sqrt{x-9}}{x-9}$		$\frac{(x-9)(\sqrt{x}-3)}{x-9}$	
	$\sqrt{x-9}$		$\sqrt{x}-3$	

- What do the directions mean by "rationalize the denominator?"
 - Remove the radical from the denominator so that the denominator is a rational expression.
- How can we accomplish this goal in part (a)?
 - Multiply the numerator and denominator by $\sqrt{x-9}$.
- Why not just square the expression?
 - ^D We are working with an expression, not an equation. You cannot square the expression because you would be changing its value. You can multiply the numerator and denominator by $\sqrt{x-9}$ because that is equivalent to multiplying by 1. It does not change the value of the expression.
- Can we take the same approach in part (b)?
 - No, multiplying by $\sqrt{x} + 3$ would not remove the radical from the denominator.
- Based on Exercise 13, what number should we multiply the numerator and the denominator by in part (b) in order to make the denominator rational?
 - $\sqrt{x}-3$
- In these examples, what was accomplished by rationalizing the denominator?
 - ^D It allowed us to create an equivalent expression that is simpler.
- Why would that be advantageous?
 - It would be easier to work with if we were evaluating it for a particular value of x.

Exercises 16-18 (5 minutes)

Allow students time to work on the three problems and then debrief. Students may have taken different approaches on Exercise 17, such as squaring both sides first or rationalizing the denominator. Share a few different approaches and compare.



MP.7







Closing (2 minutes)

Ask students to respond to these questions in writing or with a partner. Use this as an opportunity to informally assess students' understanding.

- Explain to your neighbor how to solve a radical equation. What steps do you take and why?
 - Isolate the radical, and then eliminate it by raising both sides to an exponent. The radical is isolated so that both sides can be squared or cubed as a means of eliminating the radical.
- How is solving a radical equation similar to solving other types of equations we have solved?
 - ^D We are isolating the variable by undoing any operation on the same side.
- Why is it important to check the solutions?
 - Remember that the square root of a number takes on only the positive value. When solving a radical equation involving a square root, squaring both sides of the equation in the process of solving may make the negative "disappear" and may create an extraneous solution.

Exit Ticket (5 minutes)











Name ____

Date _____

Lesson 28: A Focus on Square Roots

Exit Ticket

Consider the radical equation $3\sqrt{6-x} + 4 = -8$.

1. Solve the equation. Next to each step, write a description of what is being done.

2. Check the solution.

3. Explain why the calculation in Problem 1 does not produce a solution to the equation.







Exit Ticket Sample Solutions

Consider the radical equation $3\sqrt{6-x} + 4 = -8$. Solve the equation. Next to each step, write a description of what is being done. 1. $3\sqrt{6-x} = -12$ Subtract 4 from both sides. Divide both sides by 3 in order to isolate the radical. $\sqrt{6-x} = -4$ 6 - x = 16Square both sides to eliminate the radical. Subtract 6 from both sides and divide by -1. x = -102. Check the solution. $3\sqrt{6-(-10)} + 4 = 3\sqrt{16} + 4 = 3(4) + 4 = 16$, and $\neq -8$, so -10 is not a valid solution. 3. Explain why the calculation in Problem 1 does not produce a solution to the equation. Because the square root of a positive number is positive, $3\sqrt{6-x}$ will be positive. A positive number added to 4 cannot be -8.

Problem Set Sample Solutions







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Is the statement $\sqrt[3]{x^3} = x$ true for all *x*-values? Explain. b. Yes, this statement is true for all x-values. For example, if x = 2, then $\sqrt[3]{2^3} = 2$. If x = -2, then $\sqrt[3]{(-2)^3} = -2$. Since the cube root of a positive number is positive, and the cube root of a negative number is negative, this statement is true for any value of x. Rationalize the denominator in each expression. $4. \quad \frac{2}{\sqrt{x-12}}$ $5. \quad \frac{1}{\sqrt{x+3}-\sqrt{x}}$ 3. $\frac{4-x}{2+\sqrt{x}}$ $\frac{2\sqrt{x-12}}{x-12}$ $\frac{\sqrt{x+3} + \sqrt{x}}{3}$ $2-\sqrt{x}$ Solve each equation, and check the solutions. 6. $\sqrt{x+6} = 3$ 7. $2\sqrt{x+3} = 6$ x = 3x = 68. $\sqrt{x+3} + 6 = 3$ 9. $\sqrt{x+3} - 6 = 3$ No solution *x* = 78 10. $16 = 8 + \sqrt{x}$ 11. $\sqrt{3x-5} = 7$ *x* = 64 *x* = 18 12. $\sqrt{2x-3} = \sqrt{10-x}$ 13. $3\sqrt{x+2} + \sqrt{x-4} = 0$ $x = \frac{13}{3}$ No solution 15. $\frac{12}{\sqrt{x+9}} = 3$ 14. $\frac{\sqrt{x+9}}{4} = 3$ *x* = 135 *x* = 7 17. $\sqrt{x^2 - 6x} = 4$ 16. $\sqrt{x^2 + 9} = 5$ x = 8 or x = -2x = 4 or x = -418. $\frac{5}{\sqrt{x-2}} = 5$ 19. $\frac{5}{\sqrt{x}-2} = 5$ x = 3*x* = 9 20. $\sqrt[3]{5x-3} + 8 = 6$ 21. $\sqrt[3]{9-x} = 6$ x = -1x = -207





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therefore, the sum cannot equal 0.



22. Consider the inequality √x² + 4x > 0. Determine whether each x-value is a solution to the inequality.
a. x = -10 b. x = -4 c. x = 10 d. x = 4
Yes No Yes Yes
23. Show that a-b/√a-√b = √a + √b for all values of a and b such that a > 0 and b > 0 and a ≠ b.
If we multiply the numerator and denominator of a-b/√a-√b by √a + √b to rationalize the denominator, then we have a-b/√a-√b = a-b/√a-√b · √a+√b = (a-b)(√a+√b)/(a-b) = √a + √b.
24. Without actually solving the equation, explain why the equation √x + 1 + 2 = 0 has no solution. The value of √x + 1 must be positive, which is then added to 2. The sum of two positive numbers is positive;













Student Outcomes

Students develop facility in solving radical equations.

Lesson Notes

In the previous lesson, students were introduced to the notion of solving radical equations and checking for extraneous solutions (A-REI.A.2). Students continue this work by looking at radical equations that contain variables on both sides. The main point to stress to students is that radical equations become polynomial equations through exponentiation. So we really have not left the notion of polynomials that have been studied throughout this module. This lesson also provides opportunities to emphasize MP.7 (look for and make use of structure).

Classwork

Discussion (5 minutes)

Before beginning the lesson, remind students of past experiences by providing the following scenario, which illustrates a case when an operation performed to both sides of an equation has changed the set of solutions.

Carlos and Andrea were solving the equation $x^2 + 2x = 0$. Andrea says that there are two solutions, 0 and -2. Carlos says the only solution is -2 because he divided both sides by x and got x + 2 = 0. Who is correct and why?

- Do both 0 and −2 satisfy the original equation?
 - Yes. If we replace x with either 0 or -2, the answer is 0.
- What happened when Carlos divided both sides of the equation by *x*?
 - He changed the solutions from 0 and -2 to simply -2. He lost one solution to the equation.
- What does this say about the solution of equations after we have performed algebraic operations on both sides?
 - Performing algebraic steps may alter the set of solutions to the original equation.

Now, Carlos and Andrea are solving the equation $\sqrt{x} = -3$. Andrea says the solution is 9 because she squared both sides and got x = 9. Carlos says there is no solution. Who is correct? Why?

- Was Andrea correct to square both sides?
 - Yes. To eliminate a radical from an equation, we raise both sides to an exponent.
- Is she correct that the solution is 9?
 - No. Carlos is correct. If we let x = 9, then we get $\sqrt{9} = 3$, and $3 \neq -3$, so 9 is not a solution.

Scaffolding

- Use several examples to illustrate that if a > 0, then an equation of the form $\sqrt{x} = -a$ will not have a solution (e.g., $\sqrt{x} = -4$, $\sqrt{x} = -5$).
- Extension: Write an equation that has an extraneous solution of x = 50.



MP.3





- What is the danger in squaring both sides of an equation?
 - It sometimes produces an equation whose solution set is not equivalent to that of the original equation. If both sides of $\sqrt{x} = -3$ are squared, the equation x = 9 is produced, but 9 is not a solution to the original equation. The original equation has no solution.
- Because of this danger, what is the final essential step of solving a radical equation?
 - Checking the solution or solutions to ensure that an extraneous solution was not produced by the step of squaring both sides.
- How could we have predicted that the equation would have no solution?
 - The square root of a number is never equal to a negative value, so there is no x-value so that $\sqrt{x} = -3$.

Example 1 (5 minutes)

MP.1

While this problem is difficult, students should attempt to solve it on their own first, by applying their understandings of radicals. Students should be asked to verify the solution they come up with and describe their solution method. Discuss Example 1 as a class once they have worked on it individually.

Example 1

```
Solve the equation 6 = x + \sqrt{x}.
                                                             6-x=\sqrt{x}
                                                          (6-x)^2 = \sqrt{x}^2
                                                   36 - 12x + x^2 = x
                                                    x^2 - 13x + 36 = 0
                                                    (x-9)(x-4) = 0
The solutions are 9 and 4.
Check x = 9:
                                                               Check x = 4:
                                                                                4 + \sqrt{4} = 4 + 2 = 6
                  9 + \sqrt{9} = 9 + 3 = 12
                  6 \neq 12
So, 9 is an extraneous solution.
The only valid solution is 4.
```

- How does this equation differ from the ones from yesterday's lesson?
 - *There are two x*'s; *one inside and one outside of the radical.*
- Explain how you were able to determine the solution to the equation above.
 - Isolate the radical and square both sides. Solve the resulting equation.
- Did that change the way in which the equation was solved?
 - Not really; we still eliminated the radical by squaring both sides.
- What type of equation were we left with after squaring both sides?
 - A quadratic polynomial equation
- Why did 9 fail to work as a solution?
 - The square root of 9 takes only the positive value of 3.







Exercises 1-4 (13 minutes)

Allow students time to work the problems independently and then pair up to compare solutions. Use this time to informally assess student understanding by examining their work. Display student responses, making sure that students checked for extraneous solutions.

Exe Solv	Exercises 1–4 Solve.					
1.	$3x = 1 + 2\sqrt{x}$ The only solution is 1. Note that $\frac{1}{9}$ is an extraneous solution.	2.	$3 = 4\sqrt{x} - x$ The two solutions are 9 and 1.			
3.	$\sqrt{x+5} = x-1$ The only solution is 4. Note that -1 is an extraneous solution.	4.	$\sqrt{3x+7} + 2\sqrt{x-8} = 0$ There are no solutions.			

- When solving Exercise 1, what solutions did you find? What happened when you checked these solutions?
 - The solutions found were $\frac{1}{9}$ and 1. Only 1 satisfies the original equation, so $\frac{1}{9}$ is an extraneous solution.
- Did Exercise 2 have any extraneous solutions?
 - No. Both solutions satisfied the original equation.
- Looking at Exercise 4, could we have predicted that there would be no solution?
 - Yes. The only way the two square roots could add to zero would be if both of them produced a zero, meaning that 3x + 7 = 0 and x - 8 = 0. Since x cannot be both $-\frac{7}{3}$ and 8, both radicals cannot be simultaneously zero. Thus, at least one of the square roots will be positive, and they cannot sum to zero.

Example 2 (5 minutes)

Example 2

Solve the equation $\sqrt{x} + \sqrt{x+3} = 3$.

 $\sqrt{x+3} = 3 - \sqrt{x}$

 $\left(\sqrt{x+3}\right)^2 = (3-\sqrt{x})^2$

 $1 = \sqrt{x}$ 1 = x

 $x+3=9-6\sqrt{x}+x$

MP.7

MP.7

What do we do when there is no way to isolate the radical? What is going to be the easiest way to square both sides? Give students time to work on Example 2 independently. Point out that even though we had to square both sides twice, we were still able to rewrite the equation as a polynomial.

Scaffolding:

What if we had squared both sides of the equation as it was presented? Have early finishers work out the solution this way and share with the class.



Lesson 29: Solving Radical Equations





Check:

 $\sqrt{1} + \sqrt{1+3} = 1 + 2 = 3$

So the solution is 1.



Exercises 5–6 (7 minutes)

Allow students time to work the problems independently and then pair up to compare solutions. Circulate to assess understanding. Consider targeted instruction with a small group of students while others work independently. Display student responses, making sure that students check for extraneous solutions.

Exe	rcises 5–6		
Solv	ve the following equations.		
5.	$\sqrt{x-3} + \sqrt{x+5} = 4$	6.	$3 + \sqrt{x} = \sqrt{x + 81}$
	4		144

Closing (5 minutes)

Ask students to respond to these prompts in writing or with a partner. Use these responses to informally assess their understanding of the lesson.

- How did these equations differ from the equations seen in the previous lesson?
 - ^a Most of them contained variables on both sides of the equation or a variable outside of the radical.
- How were they similar to the equations from the previous lesson?
 - They were solved using the same process of squaring both sides. Even though they were more complicated, the equations could still be rewritten as a polynomial equation and solved using the same process seen throughout this module.
- Give an example where $a^n = b^n$ but $a \neq b$.
 - □ We know that $(-3)^2 = 3^2$ but $-3 \neq 3$.

Lesson Summary

If a = b and n is an integer, then $a^n = b^n$. However, the converse is not necessarily true. The statement $a^n = b^n$ does not imply that a = b. Therefore, it is necessary to check for extraneous solutions when both sides of an equation are raised to an exponent.

Exit Ticket (5 minutes)







Name ____

Date _____

Lesson 29: Solving Radical Equations

Exit Ticket

1. Solve $\sqrt{2x + 15} = x + 6$. Verify the solution(s).

2. Explain why it is necessary to check the solutions to a radical equation.







Exit Ticket Sample Solutions

```
Solve \sqrt{2x+15} = x+6. Verify the solution(s).
1.
                                               2x + 15 = x^2 + 12x + 36
                                                      0 = x^2 + 10x + 21
                                                      0 = (x + 3)(x + 7)
     The solutions are -3 and -7.
     Check x = -3:
                                                                 Check x = -7:
                                                                                 \sqrt{2(-7)+15}=\sqrt{1}=1
                     \sqrt{2(-3)+15} = \sqrt{9} = 3
                           -3 + 6 = 3
                                                                                 -7 + 6 = -1
     So, -3 is a valid solution.
                                                                 Since -1 \neq 1, we see that -1 is an extraneous solution.
     Therefore, the only solution to the original equation is -3.
2. Explain why it is necessary to check the solutions to a radical equation.
     Raising both sides of an equation to a power can produce an equation whose solution set is not equivalent to that of
     the original equation. In the problem above, x = -7 does not satisfy the equation.
```

Problem Set Sample Solutions

Solve.					
1.	$\sqrt{2x-5} - \sqrt{x+6} = 0$	2.	$\sqrt{2x-5} + \sqrt{x+6} = 0$		
	11		No solution		
3.	$\sqrt{x-5} - \sqrt{x+6} = 2$	4.	$\sqrt{2x-5} - \sqrt{x+6} = 2$		
	NO SOLUTION		43		
5.	$\sqrt{x+4} = 3 - \sqrt{x}$	6.	$\sqrt{x+4} = 3 + \sqrt{x}$		
	$\frac{23}{36}$		No solution		
7.	$\sqrt{x+3} = \sqrt{5x+6} - 3$	8.	$\sqrt{2x+1} = x-1$		
	6		4		
9.	$\sqrt{x+12} + \sqrt{x} = 6$	10.	$2\sqrt{x} = 1 - \sqrt{4x - 1}$		
	4		$\frac{1}{4}$		

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Student Outcomes

Students solve linear systems in three variables algebraically.

Lesson Notes

Students solved systems of linear equations in two variables using substitution and elimination in Grade 8 and then encountered the topic again in Algebra I when solving systems of linear equalities and inequalities. This lesson begins with a quick review of the method of elimination to solve a linear system in two variables along with an application problem. We then solve a system of equations in three variables using algebraic techniques.

Classwork

Opening (2 minutes)

This lesson transitions from solving 2-by-2 systems of linear equations as in Algebra I to solving systems of equations involving linear and nonlinear equations in two variables in the next two lessons. These nonlinear systems are solved algebraically using substitution or by graphing each equation and finding points of intersection, if any. This lesson helps remind students how to solve linear systems of equations and introduces them to 3-by-3 systems of linear equations (analyzed later using matrices in Precalculus and Advanced Topics).

Exercises 1–3 (8 minutes)



After this review of using elimination to solve a system, guide students through the setup of the following problem, and then let them solve using the techniques reviewed in Exercises 1 and 2.







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Solve this problem using a system of two equations in two variables.

Solution: *Milliliters of* **20**% *solution: x* ml Milliliters of 45% solution: y ml Write one equation to represent the total amounts of each solution needed: x + y = 120.Since 30% of 120 ml is 36, we can write one equation to model the acidic portion: 0.20x + 0.45y = 36.Writing these two equations as a system: x + y = 1200.20x + 0.45y = 36To solve, multiply both sides of the top equation by either 0.20 to eliminate x or 0.45 to eliminate y. The following steps will eliminate x: 0.20(x + y) = 0.20(120)0.20x + 0.45y = 40which gives 0.20x + 0.20y = 240.20x + 0.45y = 36.Replacing the top equation with the difference between the bottom equation and top equation results in a new system with the same solutions: 0.25y = 120.20x + 0.45y = 36.The top equation can quickly be solved for y, y = 48,and substituting y = 48 back into the original first equation allows us to find x: x + 48 = 120x = 72.

Thus, we need $48\ ml$ of the 45% solution and $72\ ml$ of the 20% solution.





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Lesson 30

M1

ALGEBRA II





Discussion (5 minutes)

In the previous exercises we solved systems of two linear equations in two variables using the method of elimination. However, what if we have three variables? For example, what are the solutions to the following system of equations?

MP.1

$$2x + 3y - z = 5$$
$$4x - y - z = -1$$

Allow students time to work together and struggle with this system and realize that they cannot find a unique solution. Include the following third equation, and ask students if they can solve it now.

$$x + 4y + z = 12$$

Give students an opportunity to consider solutions or other ideas on how to begin the process of solving this system. After considering their suggestions and providing feedback, guide them through the process in the example below.

Example (9 minutes)

Example Determine the values for x, y, and z in the following system: 2x + 3y - z = 5 (1) 4x - y - z = -1 (2) x + 4y + z = 12 (3)

Suggest numbering the equations as shown above to help organize the process.

• Eliminate *z* from equations (1) and (2) by subtraction. Replace equation (1) with the result.

$$2x + 3y - z = 5$$
$$-(4x - y - z) = -(-1)$$
$$-2x + 4y = 6$$

Our goal is to find two equations in two unknowns. Thus, we will also eliminate z from equations (2) and (3) by adding as follows. Replace equation (3) with the result.

$$4x - y - z = -1$$
$$x + 4y + z = 12$$
$$5x + 3y = 11$$

• Our new system of three equations in three variables has two equations with only two variables in them:

$$-2x + 4y = 6$$
$$4x - y - z = -1$$
$$5x + 3y = 11.$$







Scaffolding:

Ask students if they can eliminate two of the variables from either equation. (They cannot.) Have a discussion around what that means graphically. (The graph of the solution set is a line, not a point, so there is no single solution to the system.) These two equations now give us a system of two equations in two variables, which we reviewed how to solve in Exercises 1–2.

$$-2x + 4y = 6$$
$$5x + 3y = 11$$

At this point, let students solve this individually or with partners, or guide them through the process if necessary.

• To get matching coefficients, we need to multiply both equations by a constant:

$$5(-2x + 4y) = 5(6) \rightarrow -10x + 20y = 30$$

2(5x + 3y) = 2(11) $\rightarrow 10x + 6y = 22.$

• Replacing the top equation with the sum of the top and bottom equations together gives the following:

$$26y = 52$$
$$10x + 6y = 22.$$

• The new top equation can be solved for *y*:

y = 2.

• Replace y = 2 in one of the equations to find x:

$$5x + 3(2) = 11$$

 $5x + 6 = 11$
 $5x = 5$
 $x = 1$.

• Replace x = 1 and y = 2 in any of the original equations to find z:

$$2(1) + 3(2) - z = 5$$

2 + 6 - z = 5
8 - z = 5
z = 3.

• The solution, x = 1, y = 2, and z = 3, can be written compactly as an ordered triple of numbers (1, 2, 3).

Consider pointing out to students that the point (1, 2, 3) can be thought of as a point in a three-dimensional coordinate plane, and that it is, like a two-by-two system of equations, the intersection point in three-space of the three planes given by the graphs of each equation. These concepts are not the point of this lesson, so addressing them is optional.

Point out that a linear system involving three variables requires three equations in order for the solution to possibly be a single point.

The following problems provide examples of situations that require solving systems of equations in three variables.







Lesson 30

ALGEBRA II



Exercise 4 (8 minutes)

Exercises 4–5 Given the system below, determine the values of r, s, and u that satisfy all three equations. r + 2s - u = 8 s + u = 4 r - s - u = 2Adding the second and third equations together produces the equation r = 6. Substituting this into the first equation and adding it to the second gives 6 + 3s = 12, so that s = 2. Replacing s with 2 in the second equation gives u = 2. The solution to this system of equations is (6, 2, 2).

Exercise 5 (6 minutes)

Find the equation of the form $y = ax^2 + bx + c$ whose graph passes through the points (1, 6), (3, 20), and (-2, 15). We find a = 2, b = -1, c = 5; therefore, the quadratic equation is $y = 2x^2 - x + 5$.

Students may need help getting started on Exercise 5. A graph of the points may help.



• Since we know three ordered pairs, we can create three equations.

$$6 = a + b + c$$
$$20 = 9a + 3b + c$$
$$15 = 4a - 2b + c$$

.

Ask students to explain where the three equations came from. Then have them use the technique from Example 1 to solve this system.

Have students use a graphing utility to plot $y = 2x^2 - x + 5$ along with the original three points to confirm their answer.





MP.7

Lesson 30: Linear Systems in Three Variables





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Closing (2 minutes)

- We've seen that in order to find a single solution to a system of equations in two variables, we need to have two equations, and in order to find a single solution to a system of equations in three variables, we need to have three equations. How many equations do you expect we will need to find a single solution to a system of equations in four variables? What about five variables?
 - It seems that we will need four equations in four variables to find a single solution, and that we will need five equations in five variables to find a single solution.

Exit Ticket (5 minutes)



: Linear Systems in Three Variables







Name

Date _____

Lesson 30: Linear Systems in Three Variables

Exit Ticket

For the following system, determine the values of p, q, and r that satisfy all three equations:

2p + q - r = 8q + r = 4p - q = 2.







Exit Ticket Sample Solutions

For the following system, determine the values of p, q, and r that satisfy all three equations: 2p + q - r = 8 q + r = 4 p - q = 2. p = 4, q = 2, r = 2, or equivalently (4, 2, 2)

Problem Set Sample Solutions

Sol	Solve the following systems of equations.					
1.	x + y = 3	2.	r = 2(s-t)			
	y + z = 6		2t = 3(s - r)			
	x + z = 5		r+t=2s-3			
	x = 1, y = 2, z = 4 or (1, 2, 4)		r = 2, s = 4, t = 3, or (2, 4, 3)			
3.	2a+4b+c=5	4.	2x + y - z = -5			
	a-4b=-6		4x - 2y + z = 10			
	2b + c = 7		2x + 3y + 2z = 3			
	a = -2, b = 1, c = 5 or (-2, 1, 5)		$x = \frac{1}{2}, y = -2, z = 4 \text{ or } \left(\frac{1}{2}, -2, 4\right)$			
5.	r+3s+t=3	6.	x - y = 1			
	2r - 3s + 2t = 3		2y + z = -4			
	-r+3s-3t=1		x - 2z = -6			
	$r = 3, s = \frac{1}{3}, t = -1 \text{ or } \left(3, \frac{1}{3}, -1\right)$		x = -2, y = -3, z = 2 or (-2, -3, 2)			
7.	x = 3(y - z)	8.	p+q+3r=4			
	y = 5(z - x)		2q + 3r = 7			
	x + y = z + 4		p-q-r=-2			
	x = 3, y = 5, z = 4 or (3, 5, 4)		p = 2, q = 5, r = -1 or (2, 5, -1)			
9.	$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 5$	10.	$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 6$			
	$\frac{1}{-} + \frac{1}{-} = 2$		$\frac{1}{1} + \frac{1}{2} = 5$			
	x y 1 1		<i>b c</i> 1 1			
	$\frac{1}{x} - \frac{1}{z} = -2$		$\frac{1}{a} - \frac{1}{b} = -1$			
	$x = 1, y = 1, z = \frac{1}{3} or(1, 1, \frac{1}{3})$		$a = 1, b = \frac{1}{2}, c = \frac{1}{3} \text{ or } \left(1, \frac{1}{2}, \frac{1}{3}\right)$			







Lesson 30

11. Find the equation of the form $y = ax^2 + bx + c$ whose graph passes through the points (1, -1), (3, 23), and (-1, 7). $y = 4x^2 - 4x - 1$ 12. Show that for any number t, the values x = t + 2, y = 1 - t, and z = t + 1 are solutions to the system of equations below. x + y = 3y + z = 2(In this situation, we say that t parameterizes the solution set of the system.) x + y = (t + 2) + (1 - t) = 3y + z = (1 - t) + (t + 1) = 213. Some rational expressions can be written as the sum of two or more rational expressions whose denominators are the factors of its denominator (called a partial fraction decomposition). Find the partial fraction decomposition for $\frac{1}{n(n+1)}$ by finding the value of A that makes the equation below true for all n except 0 and -1. $\frac{1}{n(n+1)} = \frac{A}{n} - \frac{1}{n+1}$ Adding $\frac{1}{n+1}$ to both sides of the equations, we have $\frac{A}{n} = \frac{1}{n(n+1)} + \frac{1}{(n+1)}$ $=\frac{1}{n(n+1)}+\frac{n}{n(n+1)}$ $=\frac{(n+1)}{n(n+1)}$ $=\frac{1}{n}$ so A = 1 and thus $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$. 14. A chemist needs to make 40 ml of a 15% acid solution. He has a 5% acid solution and a 30% acid solution on hand. If he uses the 5% and 30% solutions to create the 15% solution, how many ml of each does he need? He needs 24 ml of the 5% solution and 16 ml of the 30% solution. 15. An airplane makes a 400-mile trip against a head wind in 4 hours. The return trip takes 2.5 hours, the wind now being a tail wind. If the plane maintains a constant speed with respect to still air, and the speed of the wind is also constant and does not vary, find the still-air speed of the plane and the speed of the wind. The speed of the plane in still wind is 130 mph, and the speed of the wind is 30 mph. 16. A restaurant owner estimates that she needs the same number of pennies as nickels and the same number of dimes as pennies and nickels together. How should she divide \$26 between pennies, nickels, and dimes? She will need 200 dimes (\$20 worth), 100 nickels (\$5 worth), and 100 pennies (\$1 worth) for a total of \$26.



Linear Systems in Three Variables







Lesson 31: Systems of Equations

Student Outcomes

- Students solve systems of linear equations in two variables and systems of a linear and a quadratic equation in two variables.
- Students understand that the points at which the two graphs of the equations intersect correspond to the solutions of the system.

Lesson Notes

Students review the solution of systems of linear equations, move on to systems of equations that represent a line and a circle and systems that represent a line and a parabola, and make conjectures as to how many points of intersection there can be in a given system of equations. They sketch graphs of a circle and a line to visualize the solution to a system of equations, solve the system algebraically, and note the correspondence between the solution and the intersection. Then they do the same for graphs of a parabola and a line.

The principal standards addressed in this lesson are **A-REI.C.6** (solve systems of linear equations exactly and approximately, e.g., with graphs, focusing on pairs of linear equations in two variables) and **A-REI.C.7** (solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically). The standards MP.5 (use appropriate tools strategically) and MP.8 (look for and express regularity in repeated reasoning) are also addressed.

Materials

Graph paper, straightedge, compass, and a tool for displaying graphs (e.g., projector, interactive white board, white board, chalk board, or squared poster paper)

Classwork

Exploratory Challenge 1 (8 minutes)

In this exercise, students review ideas about systems of linear equations from Module 4 in Grade 8 (A-REI.C.6). Consider distributing graph paper for students to use throughout this lesson. Begin by posing the following problem for students to work on individually:

Exploratory	Challenge 1	
Exploratory	enancinge 1	

a. Sketch the lines given by x + y = 6 and -3x + y = 2 on the same set of axes to solve the system graphically. Then solve the system of equations algebraically to verify your graphical solution.

Scaffolding:

Circulate to identify students who might be asked to display their sketches and solutions.







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Once students have made a sketch, ask one of them to use the display tool and draw the two graphs for the rest of the class to see. While the student is doing that, ask the other students how many points are shared (one) and what the coordinates of that point are.



Point out that in this case, there is one solution. Now change the problem as follows. Then discuss the question as a class, and ask one or two students to show their sketches using the display tool.



Point out that in this case, there is no solution. Now change the problem again as follows, and again discuss the question as a class. Then ask one or two students to show their sketches using the display tool.





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Point out that in this third case, there are infinitely many solutions. Discuss the following problem as a class.



Exploratory Challenge 2 (12 minutes)

In this exercise, students move on to a system of a linear and a quadratic equation (A-REI.C.6). Begin by asking students to work in pairs to sketch graphs and develop conjectures about the following item:

Exploratory Challenge 2

a. Suppose that instead of equations for a pair of lines, you were given an equation for a circle and an equation for a line. What possibilities are there for the two figures to intersect? Sketch a graph for each possibility.

Scaffolding:

- Circulate to assist pairs of students who might be having trouble coming up with all three possibilities.
- For students who are ready, ask them to write equations for the graphs they have sketched.

Once students have made their sketches, ask one pair to use the display tool and draw the graphs for the rest of the class to see.









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Next, students should continue to work in pairs to sketch graphs and develop conjectures about the following item (A-REI.C.6):

b. Graph the parabola with equation $y = x^2$. What possibilities are there for a line to intersect the parabola? Sketch each possibility.

Scaffolding: Circulate again to assist pairs of students who might be having trouble coming up with all three possibilities.

Once students have made their sketches, ask one pair to use the display tool and draw the graphs for the rest of the class to see.



Next, ask students to work on the following problem individually (A-REI.C.7):





Lesson 31: Systems of Equations



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Once students have made a sketch, ask one of them to use the display tool to draw the two graphs for the rest of the class to see. While the student is doing that, ask the other students how many points are shared (two) and what the coordinates of those points are.

Students should see that they can substitute the value for y in the second equation into the first equation. In other words, they need to solve the following quadratic equation (A-REI.B.4).

d. Substitute y = 2x + 2 into the equation x² + y² = 1, and solve the resulting equation for x. Factoring or using the quadratic formula, students should find that the solutions to x² + (2x + 2)² = 1 are -1 and -3/5.
e. What does your answer to part (d) tell you about the intersections of the circle and the line from part (c)? There are two intersections of the line and the circle. When x = -1, then y = 0, as the sketch shows, so (-1,0) is a solution. When x = -3/5, then y = 2(-3/5) + 2 = 4/5, so (-3/5, 4/5) is another solution.

Note that the problem above does not explicitly tell students to look for intersection points. Thus, the exercise assesses not only whether they can solve the system but also whether they understand that the intersection points of the graphs correspond to solutions of the system.

Students should understand that to solve the system of equations, they look for points that lie on the line and the circle. The points that lie on the circle are precisely those that satisfy $x^2 + y^2 = 1$, and the points that lie on the line are those that satisfy y = 2x + 2. So points on both are in the intersection.

Exercise 1 (8 minutes)

Pose the following three-part problem for students to work on individually, and then discuss as a class.





Lesson 31: Systems of Equations



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Exercises 2-6 (8 minutes)

Students need graph paper for this portion of the lesson. Complete Exercise 2 in groups so students can check answers with each other. Then they can do Exercises 3–6 individually or in groups as they choose. Assist with the exercises if students have trouble understanding what it means to "verify your results both algebraically and graphically."

Scaffolding (for advanced learners):

Create two different systems of one linear equation and one quadratic equation that have one solution at (0, 2).

2. By solving the equations as a system, find the points common to the line with equation x - y = 6 and the circle with equation $x^2 + y^2 = 26$. Graph the line and the circle to show those points. (5, -1) and (1, -5). See picture to the right.



Lesson 31: Systems of Equations









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Closing (4 minutes)

MP.1

Ask students to respond to these questions with a partner or in writing. Share their responses as a class.

- How does graphing a line and a quadratic curve help you solve a system consisting of a linear and a quadratic equation?
- What are the possibilities for the intersection of a line and a quadratic curve, and how are they related to the number of solutions of a system of linear and quadratic equations?

Present and discuss the Lesson Summary.

Be sure to note that in the case of the circle, the reverse process of solving the equation for the circle first—for either xor y—and then substituting in the linear equation would have yielded an equation with a complicated radical expression and might have led students to miss part of the solution by considering only the positive square root.

Lesson Summary

Here are some steps to consider when solving systems of equations that represent a line and a quadratic curve.

- Solve the linear equation for y in terms of x. This is equivalent to rewriting the equation in 1. slope-intercept form. Note that working with the quadratic equation first would likely be more difficult and might cause the loss of a solution.
- Replace y in the quadratic equation with the expression involving x from the slope-intercept form of the 2. linear equation. That will yield an equation in one variable.
- Solve the quadratic equation for x. 3.
- 4. Substitute x into the linear equation to find the corresponding value of y.
- 5. Sketch a graph of the system to check your solution.

Exit Ticket (5 minutes)



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Scaffolding:

Perhaps create a chart with the summary that can serve as a reminder to students.



Name

Date

Lesson 31: Systems of Equations

Exit Ticket

Make and explain a prediction about the nature of the solution to the following system of equations, and then solve the system.

$$x^2 + y^2 = 25$$
$$4x + 3y = 0$$

Illustrate with a graph. Verify your solution, and compare it with your initial prediction.





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Exit Ticket Sample Solutions



Problem Set Sample Solutions

Problem 4 yields a system with no real solution, and the graph shows that the circle and line have no point of intersection in the coordinate plane. In Problems 5 and 6, the curve is a parabola. In Problem 5, the line intersects the parabola in two points, whereas in Problem 6, the line is tangent to the parabola, and there is only one point of intersection. Note that there would also have been only one point of intersection if the line had been the line of symmetry of the parabola.

1. Where do the lines given by y = x + b and y = 2x + 1 intersect? Since we do not know the value of b, we cannot solve this problem by graphing, and we will have to approach it algebraically. Eliminating y gives the equation x + b = 2x + 1 x = b - 1. Since x = b - 1, we have y = x + b = (b - 1) + b = 2b - 1. Thus, the lines intersect at the point (b - 1, 2b - 1).





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Lesson 31: Systems of Equations



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Lesson 31: Systems of Equations













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C Lesson 32: Graphing Systems of Equations

Student Outcomes

- Students develop facility with graphical interpretations of systems of equations and the meaning of their solutions on those graphs. For example, they can use the distance formula to find the distance between the centers of two circles and thereby determine whether the circles intersect in 0, 1, or 2 points.
- By completing the squares, students can convert the equation of a circle in general form to the center-radius form and, thus, find the radius and center. They can also convert the center-radius form to the general form by removing parentheses and combining like terms.
- Students understand how to solve and graph a system consisting of two quadratic equations in two variables.

Lesson Notes

This lesson is an extension that goes beyond what is required in the standards. In particular, the standard **A-REI.C.7** (solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically) does not extend to a system of two quadratic equations, which is a natural culmination of the types of systems formed by linear and quadratic equations. The lesson also addresses standard MP.8 (look for and express regularity in repeated reasoning).

The lesson begins with a brief review of the distance formula, and its connection both to the Pythagorean theorem and to the center-radius equation of a circle. The distance formula is used extensively in the next few lessons, so be sure to review it with students. Students also briefly review how to solve and graph a system of a linear equation and an equation of a circle. They then move to the main focus of the lesson, which is graphing and solving systems of pairs of quadratic equations whose graphs include parabolas as well as circles

Materials

This lesson requires use of graphing calculators or computer software, such as the Wolfram Alpha engine, the GeoGebra package, or the Geometer's Sketchpad software for graphing geometric figures, plus a tool for displaying graphs, such as a projector, smart board, white board, chalk board, or squared poster paper.

Classwork

Opening (1 minute)

Begin with questions that should remind students of the distance formula and how it is connected to the Pythagorean theorem.

- Suppose you have a point *A* with coordinates (1, 3). Find the distance *AB* if *B* has coordinates:
 - 1. (4,2)2. (-3,1)3. (x,y)Answer: $AB = \sqrt{10}$ Answer: $AB = 2\sqrt{5}$ Answer: $AB = \sqrt{(x-1)^2 + (y-3)^2}$









If students cannot recall the distance formula (in the coordinate plane), they may need to be reminded of it.

THE DISTANCE FORMULA: Given two points (x_1, y_1) and (x_2, y_2) , the distance d between these points is given by the formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

If *A* and *B* are points with coordinates (x_1, y_1) and (x_2, y_2) , then the distance between them is the length *AB*. Draw horizontal and vertical lines through *A* and *B* to intersect in point *C* and form right triangle ΔABC . The length of the horizontal side is the difference in the *x*-coordinates $|x_2 - x_1|$, and the length of the vertical side is the difference in the *y*-coordinates $|y_2 - y_1|$. The Pythagorean theorem gives the length of the hypotenuse as $(AB)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$. Taking the square root gives the distance formula.



Opening Exercise (3 minutes)

Make sure the students all have access to, and familiarity with, some technology (calculator or computer software) for graphing lines and circles in the coordinate plane. Have them work individually on the following exercise.



Students should compare the graph they have drawn with that of a neighbor.





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Exercise 1 (5 minutes)

This exercise reviews the solution of a simple system consisting of a linear equation and the equation of a circle from the perspective of the defining property of a circle (**A-REI.C.7**).



Example 1 (5 minutes)

It is important to keep in mind that not all quadratic equations in two variables represent circles.











Exercise 2 (5 minutes)

Allow students time to think these questions over, draw some pictures, and discuss with a partner before discussing as a class.

Exercise 2 Consider a circle with radius 5 and another circle with radius 3. Let *d* represent the distance between the two centers. We want to know how many intersections there are of these two circles for different values of *d*. Draw figures for each case. a. What happens if *d* = 8? *If the distance is* 8, *then the circles touch at only one point. We say that the circles are <u>externally tangent</u>. b. What happens if <i>d* = 10? *If the distance is* 10, *the circles do not intersect, and neither circle is inside the other*. c. What happens if *d* = 1? *If the distance is* 1, *the circles do not intersect, but one circle lies inside the other*.









Example 2 (5 minutes)

Example 2

Find the distance between the centers of the two circles with equations below, and use that distance to determine in how many points these circles intersect.

$$x^2 + y^2 = 5$$

 $(x-2)^2 + (y-1)^2 = 3$

The first circle has center (0, 0), and the second circle has center (2, 1). Using the distance formula, the distance between the centers of these circles is $d = \sqrt{(2-0)^2 + (1-0)^2} = \sqrt{5}.$

Since the distance between the centers is between the sum and the difference of the two radii, that is, $\sqrt{5} - \sqrt{3} < \sqrt{5} < \sqrt{5} + \sqrt{3}$, we know that the circles must intersect in two distinct points.



- Find the coordinates of the intersection points of the circles.
 - Multiplying out the terms in the second equation gives

$$x^2 - 4x + 4 + y^2 - 2y + 1 = 3.$$

- We subtract the first equation: $x^2 + y^2 = 5$.
- The reason for subtracting is that we are removing repeated information in the two equations.
 - We get -4x 2y = -7, which is the equation of the line through the two intersection points of the circles.







- To find the intersection points, we find the intersection of the line -4x - 2y = -7 and the circle $x^2 + y^2 = 5$.
- . As with the other systems of quadratic curves and lines, we solve the linear equation for y and substitute it into the quadratic equation to find two solutions for *x*: $x = \frac{7}{5} - \frac{\sqrt{51}}{10}$, and $x = \frac{7}{5} + \frac{\sqrt{51}}{10}$.
- The corresponding y-values are

$$y = \frac{7}{10} + \frac{\sqrt{51}}{5}$$
, and $y = \frac{7}{10} - \frac{\sqrt{51}}{5}$.

The graph of the circles and the line through the intersection points is shown to the right.

Exercise 3 (4 minutes)

This exercise concerns a system of equations that represents circles that do not intersect.



Example 3 (10 minutes)

Work through this example with the whole class, showing students how to find the tangent to a circle at a point and one way to determine how many points of intersection there are for a line and a circle.









Lesson 32

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When b < -13 or b > 39, the line and circle do not intersect.









Closing (2 minutes)

Ask students to summarize how to convert back and forth between the center-radius equation of a circle and the general quadratic equation of a circle.

Ask students to speculate about what might occur with respect to intersections if one or two of the quadratic equations in the system are not circles.

Exit Ticket (5 minutes)









Name

Date _____

Lesson 32: Graphing Systems of Equations

Exit Ticket

1. Find the intersection of the two circles

and

$$x^2 + y^2 + 4x + 2y - 9 = 0.$$

 $x^2 + y^2 - 2x + 4y - 11 = 0$

2. The equations of the two circles in Question 1 can also be written as follows:

$$(x-1)^2 + (y+2)^2 = 16$$

and

 $(x+2)^2 + (y+1)^2 = 14.$

Graph the circles and the line joining their points of intersection.

3. Find the distance between the centers of the circles in Questions 1 and 2.



Graphing Systems of Equations

Lesson 32:



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Exit Ticket Sample Solutions



Problem Set Sample Solutions

In this Problem Set, after solving some problems dealing with the distance formula, students continue converting between forms of the equation of a circle and then move on to solving and graphing systems of quadratic equations, some of which represent circles and some of which do not.

1. Use the distance formula to find the distance between the points (-1, -13) and (3, -9). Using the formula with (-1, -13) and (3, -9), $d = \sqrt{(3 - (-1))^2 + ((-9) - (-13))^2}$ $d = \sqrt{4^2 + 4^2} = \sqrt{16 + 16} = \sqrt{16} \cdot \sqrt{2} = 4\sqrt{2}.$ Therefore, the distance is $4\sqrt{2}$.



Lesson 32: Graphing Systems of Equations



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Lesson 32





Lesson 32: Graphing Systems of Equations







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Lesson 32: Graphing Systems of Equations









Student Outcomes

Students model the locus of points at equal distance between a point (focus) and a line (directrix). They
construct a parabola and understand this geometric definition of the curve. They use algebraic techniques to
derive the analytic equation of the parabola.

Lesson Notes

A Newtonian reflector telescope uses a parabolic mirror to reflect light to the focus of the parabola, bringing the image of a distant object closer to the eye. This lesson uses the Newtonian telescope to motivate the discussion of parabolas. The precise definitions of a parabola and the axis of symmetry of a parabola are given here. Figure 1 to the right depicts this definition of a parabola. In this diagram, $FP_1 = P_1Q_1$, $FP_2 = P_2Q_2$, $FP_3 = P_3Q_3$ illustrate that for any point *P* on the parabola, the distance between *P* and *F* is equal to the distance between *P* and the line *L* along a segment perpendicular to *L*.



PARABOLA: (G-GPE.A.2) A parabola with directrix L and focus F is the set of all points in the plane that are equidistant from the point F and line L.

AXIS OF SYMMETRY OF A PARABOLA: (G-GPE.A.2) The *axis of symmetry of a parabola* given by a focus point and a directrix is the perpendicular line to the directrix that passes through the focus.

VERTEX OF A PARABOLA: (**G-GPE.A.2**) The *vertex of a parabola* is the point where the axis of symmetry intersects the parabola.

This lesson focuses on deriving the analytic equation for a parabola given the focus and directrix (**G.GPE.A.2**) and showing that it is a quadratic equation. In doing so, students are able to tie together many powerful ideas from geometry and algebra, including transformations, coordinate geometry, polynomial equations, functions, and the Pythagorean theorem.

Parabolas all have the reflective property illustrated in Figure 2. Rays entering the parabola parallel to the axis of symmetry will reflect off the parabola and pass through the focus point F. A Newtonian telescope uses this property of parabolas.

Parabolas have been studied by mathematicians since at least the 4th century B.C. James Gregory's *Optical Promata*, printed in 1663, contains the first known plans for a reflecting telescope using parabolic mirrors, though the idea itself was discussed earlier by many astronomers and mathematicians, including Galileo Galilei, as early as 1616. Isaac Newton, for whom the telescope is now named, needed such a telescope to prove his theory that white light is made up of a spectrum of colors. This theory explained why earlier telescopes that worked by refraction distorted the colors of objects in the sky. However, the technology did not exist at the time to accurately construct a parabolic mirror because of









difficulties accurately engineering the curve of the parabola. In 1668, he built a reflecting telescope using a spherical mirror instead of a parabolic mirror, which distorted images but made the construction of the telescope possible. Even with the image distortion caused by the spherical mirror, Newton was able to see the moons of Jupiter without color distortion. Around 1721, John Hadley constructed the first reflecting telescope that used a parabolic mirror.

A Newtonian telescope reflects light back into the tube and requires a second mirror to direct the reflected image to the eyepiece. In a modern Newtonian telescope, the primary mirror is a paraboloid—the surface obtained by rotating a parabola around its axis of symmetry—and a second flat mirror positioned near the focus reflects the image directly to the eyepiece mounted along the side of the tube. A quick image search of the Internet will show simple diagrams of these types of telescopes. This type of telescope remains a popular design today, and many amateur astronomers build their own Newtonian telescopes. The diagram shown in the student pages is adapted from this image, which can be accessed at http://en.wikipedia.org/wiki/File:Newton01.png#filelinks.



Classwork

Opening (3 minutes)

The Opening Exercise below gets students thinking about reflections on different-shaped lines and curves. According to physics, the measure of the angle of reflection of a ray of light is equal to the measure of the angle of incidence when it is bounced off a flat surface. For light reflecting on a curved surface, angles can be measured using the ray of light and the line tangent to the curve where the light ray touches the curve. As described below in the scaffolding box, it is important for students to understand that the shape of the mirror will result in different reflected images. After giving students a few minutes to work on this, ask for their ideas.

- How does each mirror reflect the light?
 - Mirror 1 bounces light straight back at you.
 - Mirror 2 bounces light at a 90° angle across to the other side of the mirror, then it bounces at another 90° angle away from the mirror.
 - Mirror 3 would bounce the light at different angles at different points because the surface is curved.



Lesson 33: The Definition of a Parabola



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Lesson 33

Some background information that can help the teacher process the Opening Exercise with students is summarized below.

- Semicircular mirrors do not send all rays of light to a single focus point; this fact can be seen by carefully
 drawing the path of three rays of light and noticing that they each intersect the other rays in different points
 after they reflect.
- From physics, the angle of reflection is congruent to the angle of incidence with a line tangent to the curve. On
 a curved mirror, the slope of the tangent line changes, so the rays of light reflect at different angles.
- Remember when working with students to focus on the big ideas: These mirrors will not reflect light back to a single point. Simple student diagrams are acceptable.

After debriefing the opening with the class, introduce the idea of a telescope that uses mirrors to reflect light. We want a curved surface that focuses the incoming light to a single point in order to see reflected images from outer space. The question below sets the stage for this lesson. The rest of the lesson defines a parabola as a curve that meets the requirements of the telescope design.

Is there a curved shape that accomplishes this goal?

Scaffolding:

Do not get sidetracked if students struggle to accurately draw the reflected light. Work through these as a class if necessary. Emphasize that the light is reflected differently as the mirror's curvature changes.

Opening Exercise (2 minutes)





MP.1 & MP.4

Lesson 33: The Definition of a Parabola





To transition from the Opening Exercise to the next discussion, tell students that telescopes work by reflecting light. To create an image without distortion using a telescope, the reflected light needs to focus on one point. Mirror 3 comes the closest to having this property but does not reflect the rays of light back to a single point. Model this by showing a sample of a student solution or by providing a teacher created sketch.

Discussion (15 minutes): Telescope Design

Lead a whole class discussion that ties together the definition of a parabola and its reflective property with Newton's telescope design requirements. A Newtonian telescope needs a mirror that focuses all the light on a single point to prevent a distorted image. A parabola by definition meets this requirement. During this discussion, share the definition of a parabola and how the focus and directrix give the graph its shape. If the distance between the focus and the directrix changes, the parabola's curvature changes. If a parabola is rotated 180° around the axis of symmetry, a curved surface called a *paraboloid* is produced; this is the shape of a parabolic mirror. A Newtonian telescope requires a fairly flat mirror in order to see images of objects that are astronomically far away, so the mirror in a Newtonian telescope is a parabola with a relatively large distance between its focus and directrix.







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In the diagram below, the dotted and solid lines show the incoming light. Model how to add these additional lines to the diagram. Make sure students annotate this on their student pages.



Next, discuss the definition of parabola that appears in the student pages and the reflective property of parabolic curves. Take time to explain what the term equidistant means and how distance is defined between a given point and a given line as the shortest distance, which is always the length of the segment that lies on a line perpendicular to the given line whose endpoints are the given point and the intersection point of the given line and the perpendicular line. Ask students to recall the definition of a circle from Lessons 30–31 and use this to explore the definition of a parabola. Before reading through the definition, give students a ruler and ask them to measure the segments $\overline{FP_1}, \overline{Q_1P_1}, \overline{FP_2}, \overline{Q_2P_2}$, etc., in Figure 2. Then have them locate a few more points on the curve and measure the distance from the curve to point *F* and from the curve to the horizontal line *L*.





Lesson 33: The Definition of a Parabola





Then, move on to Figures 4 and 5. Here we transition back to thinking about the telescope and show how a mirror in the shape of a parabola (as opposed to say a semi-circle or other curve) reflects light to the focus point. Talk about fun house mirrors (modeled by some smartphone apps) that distort images as an example of how other curved surfaces reflect light differently.

- If we want the light to be reflected to the focus at exactly the same time, then what must be true about the distances between the focus and any point on the mirror and the distances between the directrix and any point on the mirror?
 - Those distances must be equal.

Scaffolding:

To help students master the new vocabulary associated with parabolas, make a poster using the diagram shown below and label the parts. Adjust as needed for students, but make sure to include the focus point and directrix in the poster along with marked congruent segments illustrating the definition.



Figure 4 below shows several different line segments representing the reflected light with one endpoint on the curved mirror that is a parabola and the other endpoint at the focus. Anywhere the light hits this type of parabolic surface, it always reflects to the focus, F, at exactly the same time.

Figure 5 shows the same image with a directrix. Imagine for a minute that the mirror was not there. Then, the light would arrive at the directrix all at the same time. Since the distance from each point on the parabolic mirror to the directrix is the same as the distance from the point on the mirror to the focus, and the speed of light is constant, it takes the light the same amount of time to travel to the focus as it would have taken it to travel to the directrix. In the diagram, this means that $AF = AF_A$, $BF = BF_B$, and so on. Thus, the light rays arrive at the focus at the same time, and the image is not distorted.



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Lesson 33: The Definition of a Parabola



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To further illustrate the definition of a parabola, ask students to mark on Figure 5 how the lengths AF, BF, CF, DF, and EF are equal to the lengths AF_A , BF_B , CF_C , DF_D , and EF_E , respectively.

- How does this definition fit the requirements for a Newtonian telescope?
 - The definition states exactly what we need to make the incoming light hit the focus at the exact same time since the distance between any point on the curve to the directrix is equal to the distance between any point on the curve and the focus.
- A parabola looks like the graph of what type of function?
 - It looks like the graph of a quadratic function.

Transition to Example 1 by announcing that the prediction that an equation for a parabola would be a quadratic equation will be confirmed using a specific example.

Example (13 minutes): Finding an Analytic Equation for a Parabola

This example derives an equation for a parabola given the focus and directrix. Work through this example, and give students time to record the steps.. Refer students back to the way the distance formula was used in the definition of a circle, and explain that it can be used here as well to find an analytic equation for this type of curve.

Example: Finding an Analytic Equation for a Parabola

Given a focus and a directrix, create an equation for a parabola.

Scaffolding:

- Provide some additional practice with using distance formula to find the length of a line segment for struggling learners.
- Post the distance formula on the classroom wall.
- Draw a diagram with the points F(0,2), F'(4,0), and A(4,5) labeled. Have them find the lengths FA and F'A.







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Create an equation that relates the two lengths, and solve it for y. $P = \{(x, y) | \sqrt{(x-0)^2 + (y-2)^2} = y \}.$ Therefore, The two segments have equal lengths. $y = \sqrt{(x-0)^2 + (y-2)^2}$ The length of each segment $y^2 = x^2 + (y - 2)^2$ Square both sides of the equation. $y^2 = x^2 + y^2 - 4y + 4$ Expand the binomial. $4y = x^2 + 4$ Solve for y. $y = \frac{1}{4}x^2 + 1$ Replacing this equation in the definition of $P = \{(x, y) | (x, y) \text{ is equidistant from } F \text{ and the } x \text{-axis} \}$ gives the statement $P = \{(x, y) \mid y = \frac{1}{4}x^2 + 1\}.$ Thus, the parabola P is the graph of the equation $y = \frac{1}{4}x^2 + 1$. Verify that this equation appears to match the graph shown. Consider the point where the y-axis intersects the parabola; let this point have coordinates (0, b). From the graph, we see that 0 < b < 2. The distance from the focus (0,2) to (0,b) is 2 - b units, and the distance from the directrix to (0, b) is b units. Since (0, b) is on the parabola, we have 2 - b = b, so that b = 1. From this perspective, we see that the point (0,1) must be on the parabola. Does this point satisfy the equation we found? Let x=0. Then our equation gives $y = \frac{1}{4}x^2 + 1 = \frac{1}{4}(0)^2 + 1$, so (0, 1) satisfies the equation. This is the only point that we have determined to be on the parabola at this point, but it provides evidence that the equation matches the graph.

Use the questions below to work through Example 1. Have students mark the congruent segments on their diagram and record the derivation of the equation as it is worked out in front of the class. Remind students that when the distance formula is worked with, the Pythagorean theorem is being applied in the coordinate plane. This refers back to their work in both Grade 8 and high school Geometry.

- According to the definition of a parabola, which two line segments in the diagram must have equal measure? Mark them congruent on your diagram.
 - The length AF must be equal to AF'.
- How long is \overline{AF} ? How do you know?
 - It is y units long. The y-coordinate of point A is y.
- Recall the distance formula, and use it to create an expression equal to the length of AF.
 - The distance formula is $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$, where (x_1, y_1) and (x_2, y_2) are two points in the Cartesian plane.
- How can you tell if this equation represents a quadratic function?

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• The degree of x will be 2, and the degree of y will be 1, and each x will correspond to exactly one y.

A marked up diagram is shown to the right.







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Lesson 33

Exercises (4 minutes)



Closing (3 minutes)

In this lesson, limit the discussion to parabolas with a horizontal directrix. Later lessons show that all parabolas are similar and that the equations are quadratic regardless of the orientation of the parabola in the plane. Have students answer these questions individually in writing. Then discuss their responses as a whole class.

- What is a parabola?
 - A parabola is a geometric figure that represents the set of all points equidistant from a point called the focus and a line called the directrix.
- Why are parabolic mirrors used in telescope designs?
 - Parabolic mirrors are used in telescope designs because they focus reflected light to a single point.
- What type of analytic equation can be used to model parabolas?
 - A parabola whose directrix is a horizontal line can be represented by a quadratic equation in x, given by $y = ax^2 + bx + c$.





 Lesson Summary

 PARABOLA: A parabola with directrix line L and focus point F is the set of all points in the plane that are equidistant from the point F and line L.

 Axis or SYMMETRY: The axis of symmetry of a parabola given by a focus point and a directrix is the perpendicular line to the directrix that passes through the focus.

 VERTEX OF A PARABOLA: The vertex of a parabola is the point where the axis of symmetry intersects the parabola.

 In the Cartesian plane, the distance formula can help in deriving an analytic equation for a parabola.

 axis of symmetry

 focus

 vertex

 directrix

Exit Ticket (5 minutes)









Name_____

Date _____

Lesson 33: The Definition of a Parabola

Exit Ticket

1. Derive an analytic equation for a parabola whose focus is (0,4) and directrix is the *x*-axis. Explain how you got your answer.

2. Sketch the parabola from Question 1. Label the focus and directrix.











Exit Ticket Sample Solutions








Problem Set Sample Solutions

These questions are designed to reinforce the ideas presented in this lesson. The first few questions focus on applying the definition of a parabola to sketch parabolas. Then the questions scaffold to creating an analytic equation for a parabola given its focus and directrix. Finally, questions near the end of the Problem Set help students to recall transformations of graphs of functions to prepare them for work in future lessons on proving when parabolas are congruent and that all parabolas are similar.





Lesson 33: The Definition of a Parabola

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5. Find the distance from the point (-1, 3) to the line y = 5. The distance is 2 units. 6. Find the distance from the point (x, 4) to the line y = -1. The distance is 5 units. 7. Find the distance from the point (x, -3) to the line y = 2. The distance is 5 units. 8. Find the values of x for which the point (x, 4) is equidistant from (0, 1), and the line y = -1. If $\sqrt{(x-0)^2 + (4-1)^2} = 5$, then x = 4 or x = -4. Find the values of x for which the point (x, -3) is equidistant from (1, -2), and the line y = 2. 9. If $\sqrt{(x-1)^2 + (-3-(-2))^2} = 5$, then $x = 1 + 2\sqrt{6}$ or $x = 1 - 2\sqrt{6}$. 10. Consider the equation $y = x^2$. Find the coordinates of the three points on the graph of $y = x^2$ whose x-values are 1, 2, and 3. а. The coordinates are (1, 1), (2, 4), and (3, 9). Show that each of the three points in part (a) is equidistant from the point $(0, \frac{1}{4})$, and the line $y = -\frac{1}{4}$. b. For (1, 1), show that $\sqrt{(1-0)^2 + \left(1-\frac{1}{4}\right)^2} = \sqrt{1+\frac{9}{16}} = \sqrt{\frac{25}{16}} = \frac{5}{4}$ and $1-\left(-\frac{1}{4}\right)=\frac{5}{4}.$ For (2, 4), show that $\sqrt{(2-0)^2 + \left(4 - \frac{1}{4}\right)^2} = \sqrt{4 + \frac{225}{16}} = \sqrt{\frac{289}{16}} = \frac{17}{4}$ and $4 - \left(-\frac{1}{4}\right) = \frac{17}{4}.$ For (3, 9), show that $\sqrt{(3-0)^2 + \left(9 - \frac{1}{4}\right)^2} = \sqrt{9 + \left(\frac{35}{4}\right)^2} = \sqrt{\frac{1369}{16}} = \frac{37}{4}$ and $9 - \left(-\frac{1}{4}\right) = \frac{37}{4}.$







c.

 $y = x^{2}$.



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The distance from (x, y) to $\left(0, \frac{1}{4}\right)$ is $\sqrt{(x-0)^2 + \left(y - \frac{1}{4}\right)^2}$, and the distance from (x, y) to the line $y = -\frac{1}{4}$ is $y - \left(-\frac{1}{4}\right) = y + \frac{1}{4}$. Setting these distances equal gives $\sqrt{(x-0)^2 + \left(y - \frac{1}{4}\right)^2} = y + \frac{1}{4}$ $\sqrt{x^2 + \frac{y^2 - \frac{1}{2}y + \frac{1}{16}}} = y + \frac{1}{4}$ $x^{2} + y^{2} - \frac{1}{2}y + \frac{1}{16} = \left(y + \frac{1}{4}\right)^{2}$ $x^2 - \frac{1}{2}y + \frac{1}{16} = \frac{1}{2}y + \frac{1}{16}$ Thus, if a point (x, y) is the same distance from the point $\left(0, \frac{1}{4}\right)$, and the line $y = -\frac{1}{4'}$ then (x, y) lies on the parabola $y = x^2$. 11. Consider the equation $y = \frac{1}{2}x^2 - 2x$. Find the coordinates of the three points on the graph of $y = \frac{1}{2}x^2 - 2x$ whose x-values are -2, 0, and 4. a. The coordinates are (-2, 6), (0, 0), (4, 0). Show that each of the three points in part (a) is equidistant from the point $\left(2, -\frac{3}{2}\right)$ and the line $y = -\frac{5}{2}$. b. For (-2, 6), show that $\sqrt{(-2-2)^2 + \left(6 - \left(-\frac{3}{2}\right)\right)^2} = \sqrt{16 + \frac{225}{4}} = \sqrt{\frac{289}{4}} = \frac{17}{2}$ and $6 - \left(-\frac{5}{2}\right) = \frac{17}{2}.$ For (0, 0), show that $\int ((0-2)^2 + (0-(-\frac{3}{2}))^2 = \sqrt{4+\frac{9}{4}} = \sqrt{\frac{25}{4}} = \frac{5}{2}$ and $0-\left(-\frac{5}{2}\right)=\frac{5}{2}.$

Show that if the point with coordinates (x, y) is equidistant from the point $(0, \frac{1}{4})$, and the line $y = -\frac{1}{4}$, then

For (4, 0), show that

$$\sqrt{(4-2)^2 + \left(0 - \left(-\frac{3}{2}\right)\right)^2} = \sqrt{4 + \frac{9}{4}} = \sqrt{\frac{25}{4}} = \frac{5}{2}$$
$$0 - \left(-\frac{5}{2}\right) = \frac{5}{2}.$$

and



Lesson 33: The Definition of a Parabola





Show that if the point with coordinates (x, y) is equidistant from the point $\left(2, -\frac{3}{2}\right)$, and the line $y = -\frac{5}{2}$ c. then $y = \frac{1}{2}x^2 - 2x$. The distance from (x, y) to $\left(2, -\frac{3}{2}\right)$ is $\sqrt{(x-2)^2 + \left(y + \frac{3}{2}\right)^2}$, and the distance from (x, y) to the line $y = -\frac{5}{2}$ is $y - \left(-\frac{5}{2}\right) = y + \frac{5}{2}$. Setting these distances equal gives $\sqrt{(x-2)^2 + \left(y + \frac{3}{2}\right)^2} = y + \frac{5}{2}$ $\sqrt{x^2 - 4x + y^2 + 3y + \frac{25}{4}} = y + \frac{5}{2}$ $x^2 - 4x + y^2 + 3y + \frac{25}{4} = y^2 + 5y + \frac{25}{4}$ $x^2 - 4x = 2y$ $\frac{1}{2}(x^2 - 2x) = y.$ Thus, if a point (x, y) is the same distance from the point $\left(2, -\frac{3}{2}\right)$, and the line $y = -\frac{5}{2}$, then (x, y) lies on the parabola $y = \frac{1}{2}(x^2 - 2x).$ 12. Derive the analytic equation of a parabola with focus (1,3) and directrix y = 1. Use the diagram to help you work this problem. Label a point (x, y) anywhere on the parabola. а. Write an expression for the distance from the point (x, y) to the b. directrix. y – 1 Write an expression for the distance from the point (x, y) to the c. focus (1, 3). $\sqrt{(x-1)^2+(y-3)^2}$ Apply the definition of a parabola to create an equation in terms of x and y. Solve this equation for y. d. $y-1 = \sqrt{(x-1)^2 + (y-3)^2}$ $(y-1)^2 = (x-1)^2 + (y-3)^2$ $y^2 - 2y + 1 = (x - 1)^2 + y^2 - 6y + 9$ $4y = (x - 1)^2 + 8$ $y = \frac{1}{4}(x-1)^2 + 2$ $y = \frac{1}{4}x^2 - \frac{1}{2}x + \frac{9}{4}$

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Lesson 34: Are All Parabolas Congruent?

Student Outcomes

- Students learn the vertex form of the equation of a parabola and how it arises from the definition of a
 parabola.
- Students perform geometric operations, such as rotations, reflections, and translations, on arbitrary parabolas to discover standard representations for their congruence classes. In doing so, they learn that all parabolas with the same distance p between the focus and the directrix are congruent to the graph of $y = \frac{1}{2n}x^2$.

Lesson Notes

This lesson builds upon the previous lesson and applies transformations to show that all parabolas with the same distance between their focus and directrix are congruent. Recall that two figures in the plane are congruent if there exists a finite sequence of rigid motions that maps one onto the other, so it makes sense to discuss congruency of parabolas. The lesson closes with a theorem and proof detailing the answer to the question posed in the lesson title. By using transformations in this lesson to determine the conditions under which two parabolas are congruent, this lesson builds coherence with the work students did in Geometry. This lesson specifically asks students to consider how we can use transformations to prove two figures are congruent. Additionally, the lesson reinforces the connections between geometric transformations and transformations of the graphs of functions.

There are many opportunities to provide scaffolding in this lesson for students who are not ready to move quickly to abstract representation. Use technology, patty paper or transparencies, and simple hand-drawn graphs as appropriate to support student learning throughout this lesson. Use the anchor poster created in Lesson 33, and keep key vocabulary words and formulas (e.g., the distance formula) displayed for student reference. Consider breaking this lesson up into two days; on the first day, explore the definition of congruent parabolas, sketch parabolas given their focus and directrix, and explore the consequences of changing the distance between the focus and directrix, *p*. On the second day, derive the analytic equation for a parabola with a given focus and directrix and vertex at the origin, and prove the theorem on parabola congruence.

Classwork

Opening Exercise (7 minutes)

Allow students to discuss their approaches to this exercise with a partner or in small groups. Keep encouraging students to consider the definition of a parabola as they try to sketch the parabolas. Encourage students who draw a haphazard curve to consider how they could make sure their graph is the set of points equidistant from the focus and directrix. Throughout this lesson, provide students with access to graphing calculators or other computer graphing software that they can use to test and confirm conjectures. Model this further, and provide additional scaffolding by using an online applet located at the website http://www.intmath.com/plane-analytic-geometry/parabola-interactive.php. Within the applet, students can move either the focus or the directrix to change the value of *p*. They can also slide a point along the parabola noting the equal distances between that point and the focus and that point and the directrix. Depending on the level of students, the teacher can begin these exercises by moving directly to the applet or by having them start the exercises and use the applet later in this section.



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Debrief this exercise using the following questions, which can also be used as scaffolds if students are struggling to begin this problem. During the debrief, record student thinking on chart paper to be used for reference at the end of this lesson as a means to confirm or refute their conjectures.

At some point in this discussion, students should recognize that it would be nice if there was a name for the distance between the focus and directrix of the parabola. When appropriate, let them know that this distance is denoted by p in this lesson and lessons that follow. If they do not mention it here, it is brought into the discussion in Exercise 5.



MP.3



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- If students struggle with back to the anchor poster.
- equation for the parabola as was shown in Example 1

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- How can you use the definition of a parabola to quickly locate at least three points on the graph of the parabola with a focus (0,1) and directrix y = -1?
 - Along the *y*-axis, the distance between the focus and the directrix is 2, so one point, the vertex, will be halfway between them at (0,0). Since the distance between the focus and the directrix is p = 2 units, if we go 2 units to the left and right of the focus, we will be able to locate two more points on the parabola at (2,1) and (-2,1).
- How can you use the definition of a parabola to quickly locate at least three points on the graph of the parabola with a focus (0,2) and directrix y = -2?
 - The distance between the focus and the directrix is p = 4 units, so the vertex is halfway between the focus and directrix at (0,0). Moving right and left 4 units from the focus gives points (4,2) and (-4,2).
- Generalize this process of finding three points on a parabola with a given focus and directrix.
 - Find the vertex halfway between the focus and directrix, and let p be the distance between the focus and directrix. Then, sketch a line through the focus parallel to the directrix. Locate the two points p units along that line in either direction from the focus. The vertex and these two points are all on the parabola.
- How does this process help us determine if two parabolas are congruent?
 - Two parabolas will be congruent if the two points found through this process are the same distance away from the focus. That is, two parabolas will be congruent if they have the same value of p or the same distance between the focus and directrix.

Exercises 1–5 (5 minutes)

Students practice drawing parabolas given a focus and a directrix. These parabolas are NOT all oriented with a horizontal directrix. Let students struggle with how to construct a sketch of the parabola. Remind them of their work in the Opening Exercise. The teacher may choose to do one problem together to model the process. Draw a line perpendicular to the directrix through the focus. Locate the midpoint of the segment connecting the focus and directrix. Then, create a square on either side of this line with a side length equal to the distance between the focus and the directrix. One vertex of each square that is not on the directrix or axis of symmetry is another point on the parabola. If dynamic geometry software is available, students could also model this construction using technology.













Give students time to work through these exercises alone or in small groups. Then, have a few students present their approaches on the board. Be sure to emphasize that by applying the definition, a fairly accurate sketch of a parabola can be produced.

- What two geometric objects determine the set of points that forms a parabola?
 - A point called the focus and a line called the directrix

Direct students to compare and contrast Exercises 1–3 and discuss the implications within their small groups.



Direct students' attention to the diagram in Exercise 5. Have students respond individually and then discuss in small groups.





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M1

Lesson 34



Example 1 (12 minutes): Derive an Equation for a Parabola

In this example, lead students through the process of creating an equation that represents a parabola with a horizontal directrix, a vertex at the origin, and the distance between the focus and directrix p > 0. This process is similar to the work done in yesterday's lesson, except students are working with a general case instead of a specified value for p. Scaffolding may be necessary for students who are not ready to move to the abstract level. For those students, continued modeling with selected p values and use of the applet mentioned earlier will help bridge the gap between concrete and abstract. Also, recall that several of the exercises in the previous lesson worked through this process with specifc points. Remind students for whom this work is tedious that by deriving a general formula, the work going forward can be simplified, which is the heart of MP.7 and MP.8.

Scaffolding:

For more advanced students, derive the general vertex form of a parabola with the vertex at (h, k) and the horizontal directrix and distance pbetween the focus and directrix:

$$y = \pm \frac{1}{2p}(x-h)^2 + k.$$







- What are the coordinates of the vertex?
 - The coordinates of the vertex are (0,0).
- Find a formula for the distance between the focus and the point (x, y).

$$\Box \quad \sqrt{(x-0)^2 + \left(y - \frac{1}{2}p\right)^2}$$

• Find a formula for the distance between (*x*, *y*) and the directrix.

$$y = y + \frac{1}{2}p$$

 By the definition of a parabola, these distances are equal. Create an equation.

$$y + \frac{1}{2}p = \sqrt{(x-0)^2 + \left(y - \frac{1}{2}p\right)^2}$$

- Solve the equation for *y*.
 - Start by squaring both sides. When you expand the squared binomials, the y^2 and $\frac{1}{4}p^2$ terms drop out of the equation. You may need to provide some additional scaffolding for students who are still not fluent with expanding binomial expressions.

$$\left(y + \frac{1}{2}p\right)^{2} = x^{2} + \left(y - \frac{1}{2}p\right)^{2}$$
$$\left(y + \frac{1}{2}p\right)^{2} - \left(y - \frac{1}{2}p\right)^{2} = x^{2}$$
$$y^{2} + py + \frac{1}{4}p^{2} - y^{2} + py - \frac{1}{4}p = x^{2}$$
$$2py = x^{2}$$
$$y = \frac{1}{2p}x^{2}$$

• Therefore,
$$P = \{(x, y) | y = \frac{1}{2p}x^2\}$$

Scaffolding:

As an alternative to Example 1, use graphing software to explore the relationship between the distance between the focus and directrix, and the coefficient of x^2 . Organize the results in a table like the one shown below, and ask students to generalize the patterns they are seeing.

Equation	Focus	Directrix	Distance from focus to directrix
$y = \frac{1}{2}x^2$	$\left(0,\frac{1}{2}\right)$	$y = -\frac{1}{2}$	1
$y = \frac{1}{4}x^2$	(0,1)	y = -1	2
$y = \frac{1}{6}x^2$	$\left(0,\frac{3}{2}\right)$	$y = -\frac{3}{2}$	3
$y = \frac{1}{8}x^2$	(0,2)	y = -2	4
$y = \frac{1}{2p}x^2$	$\left(0,\frac{1}{2}p\right)$	$y = -\frac{1}{2}p$	p

 How does this result verify the conjecture that as p increases, the parabola gets flatter? (Note to teacher: Now is a good time to demonstrate with the applet

http://www.intmath.com/plane-analytic-geometry/parabola-interactive.php.)

• We can confirm the conjecture that the graph of a parabola vertically shrinks as p increases because the expression $\frac{1}{2n}$ will get smaller as p gets larger; thus, the parabola appears flatter.

Discussion (3 minutes)

MP.7

The goal of this brief discussion is to introduce the vertex form of the equation of a parabola without going through the entire derivation of the formula and to make the connection between what we are doing now with parabolas and what was done in Algebra I, Module 2, Topic B with quadratic functions. Completing the square on any quadratic function y = f(x) produces the equation of a parabola in vertex form, which could then be quickly graphed since the coordinates of the vertex and distance p from focus to directrix are known.





Discussion

We have shown that any parabola with a distance p > 0 between the focus $\left(0, \frac{1}{2}p\right)$ and directrix $y = -\frac{1}{2}p$ has a vertex at the origin and is represented by a quadratic equation of the form $y = \frac{1}{2n}x^2$.

Suppose that the vertex of a parabola with a horizontal directrix that opens upward is (h, k), and the distance from the focus to directrix is p > 0. Then, the focus has coordinates $\left(h, k + \frac{1}{2}p\right)$, and the directrix has equation $y = k - \frac{1}{2}p$. If we go through the above derivation with focus $\left(h, k + \frac{1}{2}p\right)$ and directrix $y = k - \frac{1}{2}p$, we should not be surprised to get a quadratic equation. In fact, if we complete the square on that equation, we can write it in the form $y = \frac{1}{2n}(x - h)^2 + k$.

In Algebra I, Module 4, Topic B, we saw that any quadratic function can be put into vertex form: $f(x) = a(x - h)^2 + k$. Now we see that any parabola that opens upward can be described by a quadratic function in vertex form, where $a = \frac{1}{2n}$.

If the parabola opens downward, then the equation is $y = -\frac{1}{2p}(x-h)^2 + k$, and the graph of any quadratic equation of this form is a parabola with vertex at (h, k), distance p between focus and directrix, and opening downward. Likewise, we can derive analogous equations for parabolas that open to the left and right. This discussion is summarized in the box below.

Vertex Form of a Parabola

Given a parabola P with vertex (h, k), horizontal directrix, and distance p > 0 between focus and directrix, the analytic equation that describes the parabola P is

- $y = \frac{1}{2p}(x-h)^2 + k$ if the parabola opens upward, and
- $y = -\frac{1}{2n}(x-h)^2 + k$ if the parabola opens downward.

Conversely, if p > 0, then

- The graph of the quadratic equation $y = \frac{1}{2p}(x-h)^2 + k$ is a parabola that opens upward with vertex at (h, k) and distance p from focus to directrix, and
- The graph of the quadratic equation $y = -\frac{1}{2p}(x-h)^2 + k$ is a parabola that opens downward with vertex at (h, k) and distance p from focus to directrix.

Given a parabola P with vertex (h, k), vertical directrix, and distance p > 0 between focus and directrix, the analytic equation that describes the parabola P is

- $x = \frac{1}{2p}(y-k)^2 + h$ if the parabola opens to the right, and
- $x = -\frac{1}{2n}(y-k)^2 + h$ if the parabola opens to the left.

Conversely, if p > 0, then

- The graph of the quadratic equation $x = \frac{1}{2p}(y-k)^2 + h$ is a parabola that opens to the right with vertex at (h, k) and distance p from focus to directrix, and
- The graph of the quadratic equation $x = -\frac{1}{2p}(y-k)^2 + h$ is a parabola that opens to the left with vertex at (h, k) and distance p from focus to directrix.







Example 2 (8 minutes)

The goal of this section is to present and prove the theorem that all parabolas that have the same distance from the focus to the directrix (that is, the same value of p) are congruent. Start by having students sketch the parabola defined by the focus and directrix on the diagram shown on the student pages.



A more concrete approach also appears below and could be used as an alternative approach to the formal proof or as a precursor to the formal proof. If this additional scaffolding is provided, this lesson may need to extend to an additional day to provide time to prove the theorem below rather than just demonstrate it by the examples provided in the scaffolds.

The following examples provide concrete evidence to support the proof provided on the next pages. Use this before or after the proof based on the needs of students. Print graphs of these parabolas on paper. Then, print the graph of $y = \frac{1}{2}x^2$ on a transparency. Use this to help students understand the rigid transformations that map the given parabola

onto $y = \frac{1}{2}x^2$.

• The graphs of $y = \frac{1}{2}x^2$ (in blue) and another parabola (in red) are shown in each coordinate plane. Describe a series of rigid transformations that map the red parabola onto the graph of $y = \frac{1}{2}x^2$.



^D There are many sequences of transformations that can take the red parabola to the blue one. For the graphs on the left, translate the red graph one unit vertically up and one unit to the left, then reflect the resulting graph about the *x*-axis. For the graphs on the right, rotate the graph 90° counter-clockwise about the point (-3,0), then translate the graph 3 units to the right.





- Suppose we changed the distance between the focus and the directrix to 2 units instead of 1, and then mapped the given parabolas onto the graph of $y = \frac{1}{4}x^2$. Would the resulting graphs be the same? Why?
 - The resulting graphs would be the same since the distance between the focus and the directrix for the parabola $y = \frac{1}{4}x^2$ is 2 units. The parabolas would map exactly onto the parabola whose vertex is (0,0).
- Under what conditions would two parabolas be congruent? How could you verify this using transformations that map one parabola onto the other?
 - If the distance between the focus and the directrix is the same, then the two parabolas will be congruent. You could describe a series of rotations, reflections, and/or translations that will map one parabola onto the other. These rigid transformations preserve the size and shape of the graph and show that the two figures are congruent.





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Lesson 34

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Depending on the level of students, begin with the scaffold examples using $y = \frac{1}{2}x^2$; the proof of this theorem can be presented directly, or students can try to work through the proof with some scaffolding and support.

Consider these discussion questions to help students get started with thinking about a proof if they work collaboratively in small groups to create a proof.

- How can we apply transformations to show that every parabola is congruent to $y = \frac{1}{2p}x^2$, where p is the distance between the focus and the directrix?
 - ^{\Box} The parabola's vertex is (0,0). We could translate any parabola so that its vertex is also (0,0).
 - Then we would need to rotate the parabolas so that the directrix is a horizontal line and the focus is a point along the *y*-axis.

The teacher may need to remind students that translations and rotations are rigid transformations and, therefore, guarantee that the parabolas determined by the focus and directrix as they are translated and rotated remain the same shape and size. For more information, see Module 1 from high school Geometry.

Exercises 6-9 (4 minutes): Reflecting on the Theorem

Have students respond individually and then share within their groups. Post a few responses on the board for a whole class debrief, and correct any misconceptions at that point.

Exercises 6–9: Reflecting on the Theorem				
6.	Restate the results of the theorem from Example 2 in your own words.			
	All parabolas that have the same distance between the focus point and the directrix are congruent.			
7.	Create the equation for a parabola that is congruent to $y=2x^2$. Explain how you determined your answer.			
	$y = 2x^2 + 1$. As long as the coefficient of the x^2 term is the same, the parabolas will be congruent.			
8.	Create an equation for a parabola that IS NOT congruent to $y=2x^2$. Explain how you determined your answer.			
	$y = x^2$. As long as the coefficient of the x^2 term is different, the parabolas will not be congruent.			
9.	Vrite the equation for two different parabolas that are congruent to the parabola with focus point $(0,3)$ and lirectrix line $y = -3$.			
	The distance between the focus and the directrix is 6 units. Therefore, any parabola with a coefficient of $\frac{1}{2p} = \frac{1}{2(6)} = \frac{1}{12}$ will be congruent to this parabola. Here are two options: $y = \frac{1}{12}x^2 + 1$ and $x = \frac{1}{12}y^2$.			







Closing (2 minutes)

Revisit the conjecture from the beginning of this lesson: Under what conditions are two parabolas congruent? Give students time to reflect on this question in writing before reviewing the points listed below. Summarize the following key points as this lesson is wrapped up. In the next lesson, students consider whether or not all parabolas are similar. This lesson has established that given a distance p between the directrix and focus, all parabolas with equal values of p

are congruent to the parabola that is the graph of the equation $y = \frac{1}{2p}x^2$.

- The points of a parabola are determined by the directrix and a focus.
- Every parabola is congruent to a parabola defined by a focus on the y-axis and a directrix that is parallel to the x-axis.
- All parabolas that have the same distance between the focus and the directrix are congruent.
- When the focus is at $(0, \frac{1}{2}p)$ and the directrix is given by the equation $y = -\frac{1}{2}p$, then the parabola is the graph of the equation $y = \frac{1}{2n}x^2$.
- When the vertex is at (h, k), and the distance from the focus to directrix is p > 0, then:
 - If it opens upward, the parabola is the graph of the equation $y = \frac{1}{2p}(x h)^2 + k$;
 - If it opens downward, the parabola is the graph of the equation $y = -\frac{1}{2p}(x-h)^2 + k$;
 - If it opens to the right, the parabola is the graph of the equation $x = \frac{1}{2p}(y-k)^2 + h$;
 - If it opens to the left, the parabola is the graph of the equation $x = -\frac{1}{2n}(y-k)^2 + h$.

Exit Ticket (4 minutes)









Name

Date _____

Lesson 34: Are All Parabolas Congruent?

Exit Ticket

Which parabolas shown below are congruent to the parabola that is the graph of the equation $y = \frac{1}{12}x^2$? Explain how you know.









Exit Ticket Sample Solutions



Problem Set Sample Solutions

Problems 1–9 in this Problem Set review how to create the analytic equation of a parabola. Students may use the process from the previous lesson or use the vertex form of the equation of a parabola included in this lesson. Starting with Problem 10, the focus of the Problem Set shifts to recognizing when parabolas are congruent.

1. Show that if the point with coordinates
$$(x, y)$$
 is equidistant from $(4, 3)$, and the line $y = 5$, then $y = -\frac{1}{4}x^2 + 2x$.
Students might start with the equation $\sqrt{(x-4)^2 + (y-3)^2} = 5 - y$ and solve for y as follows:
 $\sqrt{(x-4)^2 + y^2 - 6y + 9} = 5 - y$
 $(x-4)^2 + y^2 - 6y + 9 = 25 - 10y + y^2$
 $(x-4)^2 = -4y + 16$
 $4y = -(x-4)^2 + 16$
 $y = -\frac{1}{4}(x^2 - 8x + 16) + 4$
 $y = -\frac{1}{4}x^2 + 2x$.
Or, they might apply what we have learned about the vertex form of the equation of a parabola. Since the directrix is above the focus, we know the parabola opens downward, so p will be negative. Since the distance from the point (4, 3) to the line $y = 5$ is 2 units, we know that $p = -2$. The vertex is halfway between the focus and directrix, so the coordinates of the vertex are (4, 4). Then, the vertex form of the equation that represents the parabola is
 $y = -\frac{1}{4}(x-4)^2 + 4$
 $y = -\frac{1}{4}x^2 + 2x$.

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2. Show that if the point with coordinates (x, y) is equidistant from the point (2, 0) and the line y = -4, then $y = \frac{1}{8}(x-2)^2 - 2.$ Students might start with the equation $\sqrt{(x-2)^2 + (y-0)^2} = y + 4$, and then solve it for y, or they might apply the vertex form of the equation of a parabola. Since the vertex is above the directrix, we know that the parabola opens upward, and p > 0. Since the distance from the point (2, 0) to the line y = -4 is 4 units, we know that p = 4. The vertex is halfway between the focus and directrix, so the vertex is (2, -2). Thus, the equation that represents the parabola is $y = \frac{1}{9}(x-2)^2 - 2$. Find the equation of the set of points which are equidistant from (0,2) and the x-axis. Sketch this set of points. 3. The focus is (0, 2), and the directrix is the x-axis. Thus, the vertex is the point (0,1), which is halfway between the vertex and directrix. Since the parabola opens upward, p > 0, so p = 2. Then, the vertex form of the equation of the parabola is $y=\frac{1}{4}x^2+1.$ х Find the equation of the set of points which are equidistant from the origin and the line y = 6. Sketch this set of 4. points. $y=-\frac{1}{12}x^2+3$ 5. Find the equation of the set of points which are equidistant from (4, -2) and the line y = 4. Sketch this set of points. $y = -\frac{1}{12}(x-4)^2 + 1$ 10 х

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Lesson 34: Are All Parabolas Congruent?











Lesson 34: Are All Parabolas Congruent?

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MP.3



14. Jemma thinks that the parabola with equation $y = \frac{1}{3}x^2$ is NOT congruent to the parabola with equation $y = -\frac{1}{3}x^2 + 1$. Do you agree or disagree? Create a convincing argument to support your reasoning. Jemma is wrong. These two parabolas are congruent. If you translate the graph of $y = -\frac{1}{3}x^2 + 1$ down one unit and then reflect the resulting graph about the x-axis, the resulting parabola will have equation $y = \frac{1}{2}x^2$. Alternately, the focus and directrix of each parabola are the same distance apart, 1.5 units. 15. Let *P* be the parabola with focus (2, 6) and directrix y = -2. Write an equation whose graph is a parabola congruent to P with focus (0, 4). a. The equation $y = \frac{1}{16}x^2$ is one option. The directrix for this parabola is y = -4. Another possible solution would be the parabola with focus (0, 4) and directrix y = 12. The equation would be $y = -\frac{1}{16}x^2 + 8$. Write an equation whose graph is a parabola congruent to P with focus (0, 0). b. $y = \frac{1}{16}x^2 - 4$ c. Write an equation whose graph is a parabola congruent to P with the same directrix but different focus. The focus would be a reflection of the original focus across the directrix, or (2, -10). The equation would be $y = -\frac{1}{16}(x-2)^2 - 6.$ Write an equation whose graph is a parabola congruent to P with the same focus but with a vertical directrix. d. $x = \frac{1}{16}(y-6)^2 - 2$ or $x = -\frac{1}{16}(y-6)^2 + 8$ 16. Let *P* be the parabola with focus (0, 4) and directrix y = x. Sketch this parabola. а. The sketch is shown to the right. By how many degrees would you have to rotate P about the b. focus to make the directrix line horizontal? One possible answer is a clockwise rotation of 45°. Write an equation in the form $y = \frac{1}{2a}x^2$ whose graph is a c. parabola that is congruent to P. The distance between the focus and the directrix is $2\sqrt{2}$. The equation is $y = \frac{1}{4\sqrt{2}}x^2$. Write an equation whose graph is a parabola with a vertical directrix that is congruent to P. d. Since the exact focus and directrix are not specified, there are infinitely many possible parabolas. A vertical directrix does require that the y-term be squared. Thus, $x = \frac{1}{4\sqrt{2}}y^2$ satisfies the conditions specified in the problem.







e. Write an equation whose graph is P', the parabola congruent to P that results after P is rotated clockwise 45° about the focus.

The directrix will be $y = 4 - 2\sqrt{2}$. The equation is $y = \frac{1}{4\sqrt{2}}x^2 + 4 - \sqrt{2}$.

f. Write an equation whose graph is P'', the parabola congruent to P that results after the directrix of P is rotated 45° about the origin.

The focus will be $(2\sqrt{2}, 2\sqrt{2})$, and the directrix will be the *x*-axis. The equation is $y = \frac{1}{4\sqrt{2}} (x - 2\sqrt{2})^2 + \sqrt{2}$.

Extension:

MP.2

17. Consider the function $f(x) = \frac{2x^2 - 8x + 9}{-x^2 + 4x - 5}$, where x is a real number.

a. Use polynomial division to rewrite f in the form $f(x) = q + \frac{r}{-x^2+4x-5}$ for some real numbers q and r.

Using polynomial division, $f(x) = -2 + \frac{-1}{-x^2+4x-5}$.

b. Find the *x*-value where the maximum occurs for the function *f* without using graphing technology. Explain how you know.

We can rewrite f as $f(x) = -2 + \frac{1}{x^2 - 4x + 5}$. Since $x^2 - 4x + 5 = (x - 2)^2 + 1$, the graph of $y = x^2 - 4x + 5$ is a parabola with vertex (2, 1) that opens upward. Thus, the lowest point on the graph is (2, 1). The function f will take on its maximum value when $\frac{1}{x^2 - 4x + 5}$ is maximized; this happens when the value of $x^2 - 4x + 5$ is minimized. Since we have already seen that $x^2 - 4x + 5$ is minimized at x = 2, the expression $\frac{1}{x^2 - 4x + 5}$ takes on its maximum value when x = 2, and, thus, the original function f takes on its maximum value when x = 2.









Student Outcomes

• Students apply the geometric transformation of dilation to show that all parabolas are similar.

Lesson Notes

In the previous lesson, students used transformations to prove that all parabolas with the same distance between the focus and directrix are congruent. In the process, they made a connection between geometry, coordinate geometry, transformations, equations, and functions. In this lesson, students explore how dilation can be applied to prove that all parabolas are similar.

Students may express disagreement with or confusion about the claim that all parabolas are similar because the various graphical representations of parabolas they have seen do not appear to have the "same shape." Because a parabola is an open figure as opposed to a closed figure, like a triangle or quadrilateral, it is not easy to see similarity among parabolas. Students must understand that *similar* is strictly defined via similarity transformations; in other words, two parabolas are similar if there is a sequence of translations, rotations, reflections, and dilations that takes one parabola to the other. In the last lesson, students saw that every parabola is congruent to the graph of the equation

 $y = \frac{1}{2n}x^2$ for some p > 0; in this lesson, students need only consider dilations of parabolas in this form.

When students claim that two parabolas are not similar, they should be reminded that the parts of the parabolas they are looking at may well appear to be different in size or magnification, but the parabolas themselves are not different in shape. Remind students that similarity is established by dilation; in other words, by magnifying a figure in both the horizontal and vertical directions. By analogy, although circles with different radii have different curvature, every student should agree that any circle can be dilated to be the same size and shape as any other circle; thus, all circles are similar.

Quadratic curves such as parabolas belong to a family of curves known as *conic sections*. The technical term in mathematics for how much a conic section deviates from being circular is *eccentricity*, and two conic sections with the same eccentricity are similar. Circles have eccentricity 0, and parabolas have eccentricity 1. After this lesson, consider asking students to research and write a report on eccentricity.

Classwork

Provide graph paper for students as they work the first seven exercises. They first examine three congruent parabolas and then make a conjecture about whether or not all parabolas are similar. Finally, they explore this conjecture by graphing parabolas of the form $y = ax^2$ that have different *a*-values.

Scaffolding:

- Allow students access to graphing calculators or software to focus on conceptual understanding if they are having difficulty sketching the graphs.
- Consider providing students with transparencies with a variety of parabolas drawn on them (as in prior lesson), such as y = x², y = ¹/₂x², and y = ¹/₄x² to help them illustrate these principles.









Exercises 1–7 (4 minutes)





Lesson 35: Are All Parabolas Similar?





6. What would it mean for two parabolas to be similar? How could we use geometric transformation to determine if two parabolas are similar?

Two parabolas should be similar if we can transform one onto the other by a sequence of rotations, reflections, translations and dilations.

7. Use your work in Exercises 1–6 to make a conjecture: Are all parabolas similar? Explain your reasoning.

(Student answers will vary.) It seems that any pair of parabolas should be similar because we can line up the vertices through a sequence of rotations, reflections and translations, then we should be able to dilate the width of one parabola to match the other.

Discussion

After students have examined the fact that when the *a*-value in the equation of the parabola is changed, the resulting graph is basically the same shape, this point can be further emphasized by exploring the graph of $y = x^2$ on a graphing calculator or graphing program on the computer. MP.3 Use the same equation but different viewing windows so students can see that an image can be created of what appears to be a different parabola by transforming the dimensions of the viewing window. However, the images are just a dilation of the original that is created when the scale is changed. See the images to the right. Each figure is a graph of the equation $y = x^2$ with different scales on the horizontal and vertical axes.







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Exercise 8 (5 minutes)

In this exercise, students derive the analytic equation for a parabola given its graph, focus, and directrix. Students have worked briefly with parabolas with a vertical directrix in previous lessons, so this exercise is an opportunity for the teacher to assess whether or not students are able to transfer and extend their thinking to a slightly different situation.







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Discussion (8 minutes)

After giving students time to work through Exercises 1–8, ask the following questions to revisit concepts from Algebra I, Module 3.

- In the previous exercise, is P a function of x?
 - No, because the *x*-value 2 corresponds to two *y*-values.
- Is P a function of y?
 - Yes, if we take *y* to be in the domain and *x* to be in the range, then each *y*-value on *P* corresponds to exactly one *x*-value, which is the definition of a function.

These two questions remind students that just because we typically use the variable x to represent the domain element of an algebraic function, this does not mean that it must always represent the domain element.

Next, transition to summarizing what was learned in the last two lessons. We have defined a parabola and determined the conditions required for two parabolas to be congruent. Use the following questions to summarize these ideas.

- What have we learned about the definition of a parabola?
 - ^a The points on a parabola are equidistant from the directrix and the focus.
- What transformations can be applied to a parabola to create a parabola congruent to the original one?
 - If the directrix and the focus are transformed by a rigid motion (e.g., translation, rotation, or reflection), then the new parabola defined by the transformed directrix and focus will be congruent to the original.

Essentially, every parabola that has a distance of p units between its focus and directrix is congruent to a parabola with focus $\left(0, \frac{1}{2}p\right)$ and directrix $y = -\frac{1}{2}p$. What is the equation of this parabola?

$$P = \left\{ (x, y) \mid y = \frac{1}{2p} x^2 \right\}$$

Thus, all parabolas that have the same distance between the focus and the directrix are congruent.

The family of graphs given by the equation $y = \frac{1}{2p}x^2$ for p > 0 describes the set of non-congruent parabolas, one for each value of p.

Ask students to consider the question from the lesson title. Chart responses to revisit at the end of this lesson to confirm or refute their claims.

Discussion

Do you think that all parabolas are similar? Explain why you think so.

Yes, they all have the same basic shape.

What could we do to show that two parabolas are similar? How might you show this?

Since every parabola can be transformed into a congruent parabola by applying one or more rigid transformations, perhaps similar parabolas can be created by applying a dilation which is a non-rigid transformation.

To check to see if all parabolas are similar, it only needs to be shown that any parabola that is the graph of $y = \frac{1}{2p}x^2$ for p > 0 is similar to the graph of $y = x^2$. This is done through a dilation by some scale factor k > 0 at the origin (0,0).



MP.3





Note that a dilation of the graph of a function is the same as performing a horizontal scaling followed by a vertical scaling that students studied in Algebra I, Module 3.

Exercises 9–12 (8 minutes)

The following exercises review the function transformations studied in Algebra I that are required to define dilation at the origin. These exercises provide students with an opportunity to recall what they learned in a previous course so that they can apply it here. Students must read points on the graphs to determine that the vertical scaling is by a

factor of 2 for the graphs on the left and by a factor of $\frac{1}{2}$ for the graphs on the right. In Algebra I, Module 3, students saw that the graph of a function can be transformed with a non-rigid transformation in two ways: *vertical scaling* and *horizontal scaling*.

A vertical scaling of a graph by a scale factor k > 0 takes every point (x, y) on the graph of y = f(x) to (x, ky). The result of the transformation is given by the graph of y = kf(x).

A *horizontal scaling* of a graph by a scale factor k > 0 takes every point (x, y) on the graph of y = f(x) to (kx, y). The result of the transformation is given by the graph of $y = f\left(\frac{1}{k}x\right)$.

Scaffolding:

- Allow students access to graphing calculators or software to focus on conceptual understanding if they are having difficulty sketching the graphs.
- The graphs shown in Exercises 9 and 10 are $f(x) = x^2$, $g(x) = 2x^2$, and $h(x) = \frac{1}{2}x^2$. The graphs shown in Exercises 11 and 12 are $f(x) = x^2$, $g(x) = (\frac{1}{2}x)^2$, and $h(x) = (2x)^2$.



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When these exercises are debriefed, model marking up the diagrams to illustrate the vertical and horizontal scaling. A sample is provided below.

Marked up diagrams for vertical scaling in Exercises 9 and 10:





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Marked up diagrams for horizontal scaling in Exercises 11 and 12:

After working through Exercises 9–12, pose the following discussion question.

- If a dilation by scale factor k involves both horizontal and vertical scaling by a factor of k, how could we express the dilation of the graph of y = f(x)?
 - You could combine both types of scaling. Thus, $y = kf\left(\frac{1}{k}x\right)$.

Explain the definition of dilation at the origin as a combination of a horizontal and then vertical scaling by the same factor. Exercises 1–3 in the Problem Set will address this idea further.

Definition: A dilation at the origin D_k is a horizontal scaling by k > 0 followed by a vertical scaling by the same factor k. In other words, this dilation of the graph of y = f(x) is the graph of the equation $y = kf(\frac{1}{k}x)$.

It is important for students to clearly understand that this dilation of the graph of y = f(x) is the graph of the equation $y = kf(\frac{1}{k}x)$. Remind students of the following two facts that they studied in Geometry:

- 1. When one figure is a dilation of another figure, the two figures are similar.
- 2. A dilation at the origin is just a particular type of dilation transformation.

Thus, the graph of y = f(x) is similar to the graph of $y = kf\left(\frac{1}{k}x\right)$. Students may realize here that their thinking about "stretching" the graph creating a similar parabola is not quite enough to prove that all parabolas are similar because they must consider both a horizontal and vertical dilation in order to connect back to the geometric definition of similar figures.





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Example (5 minutes): Dilation at the Origin

This example helps students gain a level of comfort with the notation and mathematics before moving on to proving that all parabolas are similar.

Example: Dilation at the Origin Let $f(x) = x^2$ and let k = 2. Write a formula for the function g that results from dilating f at the origin by a factor of $\frac{1}{2}$. The new function will have equation $g(x) = 2f(\frac{1}{2}x)$. Since $f(x) = x^2$, the new function will have equation $g(x) = 2\left(\frac{1}{2}x\right)^2$. That is, $g(x) = \frac{1}{2}x^2$. What would the results be for k = 3, 4, or 5? What about $k = \frac{1}{2}$? For k = 3, $g(x) = \frac{1}{2}x^2$. For k = 4, $g(x) = \frac{1}{4}x^2$. For k = 5, $g(x) = \frac{1}{5}x^2$. For $k = \frac{1}{2}$, $g(x) = 2x^2$.

After working through this example, the following questions help prepare students for the upcoming proof using a general parabola from the earlier discussion.

- Based on this example, what can you conclude about these parabolas?
 - They are all similar to one another because they represent dilations of the graph at the origin of the original function.
- Based on this example, what can you conclude about these parabolas?
- Is this enough information to prove ALL parabolas are similar?
 - No, we have only proven that these specific parabolas are similar.
- How could we prove that all parabolas are similar?
 - We would have to use the patterns we observed here to make a generalization and algebraically show that it works in the same way.

Scaffolding:

Some students might find this derivation easier if the parabola $y = ax^2$ is used. Then, the proof would be as follows:

If $f(x) = ax^2$, then the graph of f is similar to the graph of the equation $y = k \left(a \left(\frac{1}{k} x \right) \right)^2$

Simplifying the right side gives

 $y = \frac{a}{k}x^2$.

This new parabola should be similar to

 $y = x^2$, which it will be if $\frac{a}{b} = 1$.

Therefore, let a = k. Thus, dilating the graph of $y = ax^2$ about the origin by a factor of *a*, students see that this parabola is similar to $y = x^2$.

To further support students, supply written reasons, such as those provided, as these steps are worked through on the board.







Discussion (8 minutes): Prove All Parabolas Are Similar

In this discussion, work through a dilation at the origin of a general parabola with equation $y = \frac{1}{2n}x^2$ to transform it to a basic parabola with equation $y = x^2$ by selecting the appropriate value of k. At that point, it can be argued that all parabolas are similar. Walk through the outline below slowly, and ask the class for input at each step, but expect that much of this discussion will be teacher-centered. For students not ready to show this result at an abstract level, have them work in small groups to show that a few parabolas, such as

$$y = \frac{1}{2}x^2$$
, $y = 4x^2$, and $y = \frac{1}{8}x^2$,

are similar to $y = x^2$ by finding an appropriate dilation about the origin. Then, generalize from these examples in the following discussion.

- Recall from Lesson 34 that any parabola is congruent to an "upright" parabola of the form $y = \frac{1}{2n}x^2$, where p is the distance between the vertex and directrix. That is, given any parabola we can rotate, reflect and translate it so that it has its vertex at the origin and axis of symmetry along the γ -axis. We now want to show that all parabolas of the form $y = \frac{1}{2p}x^2$ are similar to the parabola $y = x^2$. To do this, we apply a dilation at the origin to the parabola $y = \frac{1}{2n}x^2$. We just need to find the right value of k for the dilation.
- Recall that the graph of y = f(x) is similar to the graph of $y = kf\left(\frac{1}{k}x\right)$.

If $f(x) = \frac{1}{2p}x^2$, then the graph of f is similar to the graph of the equation $y = kf\left(\frac{1}{k}x\right) = k\left(\frac{1}{2p}\left(\frac{1}{k}x\right)^2\right)$, which simplifies to $y = \frac{1}{2nk}x^2$.

We want to find the value of k that dilates the graph of $f(x) = \frac{1}{2p}x^2$ into $y = x^2$. That is, we need to choose the dilation factor k so that $y = \frac{1}{2p}x^2$ becomes $y = x^2$; therefore, we want $\frac{1}{2nk} = 1$. Solving this equation for k gives $k = \frac{1}{2p}$.

Therefore, if we dilate the parabola $y = \frac{1}{2p}x^2$ about the origin by a factor of $\frac{1}{2p}$, we have

$$y = kf\left(\frac{1}{k}x\right)$$
$$= k\left(\frac{1}{2p}\left(\frac{1}{k}x\right)^{2}\right)$$
$$= \frac{1}{2pk}x^{2}$$
$$= x^{2}.$$

Thus, we have shown that the original parabola is similar to $y = x^2$.

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Lesson 35

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- In the previous lesson, we showed that any parabola is congruent to a parabola given by $y = \frac{1}{2p}x^2$ for some value of p. Now, we have shown that every parabola with equation of the form $y = \frac{1}{2p}x^2$ is similar to our basic parabola given by $y = x^2$. Then, any parabola in the plane is similar to the basic parabola given by $y = x^2$.
- Further, all parabolas are similar to each other because we have just shown that they are all similar to the same parabola.

Closing (3 minutes)

Revisit the title of this lesson by asking students to summarize what they learned about the reason why all parabolas are similar. Then, take time to bring closure to this cycle of three lessons. The work students have engaged in has drawn together three different domains: geometry, algebra, and functions. In working through these examples and exercises and engaging in the discussions presented here, students can gain an appreciation for how mathematics can model real-world scenarios. The past three lessons show the power of using algebra and functions to solve problems in geometry.

Combining the power of geometry, algebra, and functions is one of the most powerful techniques available to solve science, technology and engineering problems.



Exit Ticket (4 minutes)





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Lesson 35: Are All Parabolas Similar?

NYS COMMON CORE MATHEMATICS CURRICULUM

Exit Ticket

1. Describe the sequence of transformations that transform the parabola P_x into the similar parabola P_y .



2. Are the two parabolas defined below similar or congruent or both? Justify your reasoning.

Parabola 1: The parabola with a focus of (0,2) and a directrix line of y = -4Parabola 2: The parabola that is the graph of the equation $y = \frac{1}{6}x^2$





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Exit Ticket Sample Solutions



Problem Set Sample Solutions

MP.3





Lesson 35: Are All Parabolas Similar?



2. Let f(x) = -|x| + 1. The graph of f is shown below. On the same axes, graph the function g, where $g(x) = f(\frac{1}{3}x)$. Then, graph the function h, where h(x) = 3g(x). Based on your work in Problems 1 and 2, describe the resulting function when the original function is transformed 3. with a horizontal and then a vertical scaling by the same factor, k. The resulting function is scaled by a factor of k in both directions. It is a dilation about the origin of the original figure and is similar to it. 4. Let $f(x) = x^2$. What are the focus and directrix of the parabola that is the graph of the function $f(x) = x^2$? а. Since $\frac{1}{2n} = 1$, we know $p = \frac{1}{2}$, and that is the distance between the focus and the directrix. The point (0,0)is the vertex of the parabola and the midpoint of the segment connecting the focus and the directrix. Since the distance between the focus and vertex is $\frac{1}{2}p = \frac{1}{4'}$, which is the same as the distance between the vertex and directrix; therefore, the focus has coordinates $\left(0,\frac{1}{4}\right)$, and the directrix is $y=-\frac{1}{4}$ b. Describe the sequence of transformations that would take the graph of f to each parabola described below. Focus: $\left(0, -\frac{1}{4}\right)$, directrix: $y = \frac{1}{4}$ i. This parabola is a reflection of the graph of f across the x-axis. Focus: $\left(\frac{1}{4}, 0\right)$, directrix: $x = -\frac{1}{4}$ ii. This parabola is a 90° clockwise rotation of the graph of f. Focus: (0, 0), directrix: $y = -\frac{1}{2}$ iii. This parabola is a vertical translation of the graph of f down $\frac{1}{4}$ unit. Focus: $(0, \frac{1}{4})$, directrix: $y = -\frac{3}{4}$ iv. This parabola is a vertical scaling of the graph of f by a factor of $\frac{1}{2}$ and a vertical translation of the resulting image down $\frac{1}{4}$ unit.











Lesson 35: Are All Parabolas Similar?













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Lesson 36: Overcoming a Third Obstacle to Factoring— What If There Are No Real Number Solutions?

Student Outcomes

Students understand the possibility that there might be no real number solution to an equation or system of
equations. Students identify these situations and make the appropriate geometric connections.

Lesson Notes

Lessons 36–40 provide students with the necessary tools to find solutions to polynomial equations outside the realm of the real numbers. This lesson illustrates how to both analytically and graphically identify a system of equations that has no real number solution. In the next lesson, the imaginary unit *i* is introduced, and students begin to work with complex numbers through the familiar geometric context of rotation. Students realize that the set of complex numbers inherits the arithmetic and algebraic properties from the real numbers. The work with complex solutions to polynomial equations in these lessons culminates with the fundamental theorem of Algebra in Lesson 40, the final lesson in this module.

Classwork

Opening (1 minutes)

This lesson illustrates how to identify a system of equations that has no real number solution, both graphically and analytically. In this lesson, students explore systems of equations that have no real number solutions.

Opening Exercise 1 (5 minutes)

Instruct students to complete the following exercise individually and then to pair up with a partner after a few minutes to compare their answers. Allow students to search for solutions analytically or graphically as they choose. After a few minutes, ask students to share their answers and solution methods. Both an analytic and a graphical solution should be presented for each system, either by a student or by the teacher if all students used the same approach. Circulate while students are working, and take note of which students are approaching the question analytically and which are approaching the question graphically.



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Discussion (10 minutes)

Ask students to explain their reasoning for each of the three systems in the Opening Exercise with both approaches shown for each part, allowing six students the opportunity to present their solutions to the class. It is important to go through both the analytical and graphical approaches for each system so that students draw the connection between graphs that do not intersect and systems that have no analytic solution. Be sure to display the graph of each system of equations as students are led through this discussion.

Part (a):

- Looking at the graphs of the equations in the first system, 2x 4y = -1 and 3x 6y = 4, how can we tell that the system has no solution?
 - D The two lines never intersect.
 - D The two lines are parallel.
- Using an algebraic approach, how can we tell that there is no solution?
 - If we multiply both sides of the top equation by 3 and the bottom equation by 2, we see that an equivalent system can be written.

$$6x - 12y = -3$$

$$6x - 12y = 8$$

Subtracting the first equation from the second results in the false number sentence

Thus, there are no real numbers x and y that satisfy both equations.



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- The graphs of these equations are lines. What happens if we put them in slope-intercept form?
 - Rewriting both linear equations in slope-intercept form, the system from part (a) can be written as

$$y = \frac{1}{2}x + \frac{1}{4}$$
$$y = \frac{1}{2}x - \frac{2}{3}.$$

From what we know about graphing lines, the lines associated to these equations have the same slope and different y-intercepts, so they will be parallel. Since parallel lines do not intersect, the lines have no points in common and, therefore, this system has no solution.

Part (b):

- Looking at the graphs of the equations in the second system, $y = x^2 2$ and = 2x 5, how can we tell that the system has no solution?
 - The line and the parabola never intersect.
- Can we confirm, algebraically, that the system in part (b) has no real solution?
 - Yes. Since $y = x^2 2$ and y = 2x 5, we must have $x^2 2 = 2x 5$, which is equivalent to the quadratic equation $x^2 2x + 3 = 0$. Solving for x using the quadratic formula, we get

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(3)}}{2(1)} = 1 \pm \frac{\sqrt{-8}}{2}.$$

Since the square root of a negative real number is not a real number, there is no real number x that satisfies the equation $x^2 - 2x + 3 = 0$; therefore, there is no point in the plane with coordinates (x, y) that satisfies both equations in the original system.

Part (c):

- Looking at the graphs of the equations in the final system, $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$, how can we tell that there the system has no solution?
 - The circles are concentric, meaning that they have the same center and different radii. Thus, they never intersect, and there are no points that lie on both circles.
- Can we algebraically confirm that the system in part (c) has no solution?
 - Yes. If we try to solve this system, we could subtract the first equation from the second, giving the false number sentence 0 = 3. Since this statement is false, we know that there are no values of x and y that satisfy both equations simultaneously; thus, the system has no solution.

At this point, ask students to summarize in writing or with a partner what they have learned so far. Use this brief exercise as an opportunity to check for understanding.

Exercise 1 (4 minutes)

Have students work individually and then check their answers with a partner. Make sure they write out their steps as they did in the sample solutions. After a few minutes, invite students to share one or two solutions on the board.



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Discussion (7 minutes)

This lesson does not mention complex numbers or complex solutions; those are introduced in the next lesson. Make sure students understand that analytical findings can be confirmed graphically and vice-versa. Students turn their focus to quadratic equations in one variable x without real solutions and to how the absence of any real solution x can be confirmed by graphing a system of equations with two variables x and y.



Scaffolding:

 $x^2 + 5 = 0$

 $x^2 - 4 = 0$

 $x^2 + 1 = 0$

Feel free to assign an optional extension exercise, such as: "Which of these equations will have no solution? Explain how you know in terms of a graph."



Present students with the following graphs of parabolas:

- Remember that a parabola with a vertical axis of symmetry is the graph of an equation of the form y = ax² + bx + c for some real number coefficients a, b, and c with a ≠ 0. We can consider the solutions of the quadratic equation ax² + bx + c = 0 to be the x-coordinates of solutions to the system of equations y = ax² + bx + c and y = 0. Thus, when we are investigating whether a quadratic equation ax² + bx + c = 0 has a solution, we can think of this as finding the x-intercepts of the graph of y = ax² + bx + c.
- Which of these three parabolas are represented by a quadratic equation $y = ax^2 + bx + c$ that has no solution to $ax^2 + bx + c = 0$? Explain how you know.
 - Because the parabola has no x-intercepts, we know that there are no solutions to the associated equation $ax^2 + bx + c = 0$.



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- Now, consider Parabola 2, which is the graph of the equation $y = 8 (x + 1)^2$. How many solutions are there to the equation $8 (x + 1)^2 = 0$? Explain how you know.
 - Because Parabola 2 intersects the x-axis twice, the system consisting of $y = 8 (x + 1)^2$ and y = 0 has two real solutions. The graph suggests that the system will have one positive solution and one negative solution.
- Now, consider Parabola 3, which is the graph of the equation $y = x^2$. How does the graph tell us how many solutions there are to the equation $x^2 = 0$? Explain how you know.
 - Parabola 3 touches the x-axis only at (0,0), so the parabola and the line with equation y = 0 intersect at only one point. Accordingly, the system has exactly one solution, and there is exactly one solution to the equation $x^2 = 0$.

Pause, and ask students to again summarize what they have learned, either in writing or orally to a neighbor. Students should be making connections between the graph of the quadratic equation $y = ax^2 + bx + c$ (which is a parabola), the number of *x*-intercepts of the graph, and the number of solutions to the system consisting of y = 0 and $y = ax^2 + bx + c$.

Exercises 2-4 (12 minutes)

Students should work individually or in pairs on these exercises. To solve these problems analytically, they need to understand that they can determine the x-coordinates of the intersection points of the graphs of these geometric figures by solving an equation. Make sure students are giving their answers to these questions as coordinate pairs. Encourage students to solve the problems analytically and verify the solutions graphically.



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Before moving on, discuss these results as a whole class. Have students put both graphical and analytical solutions to each exercise on the board. Start to reinforce the connection that when the graphs intersect, the related system of equations has real solutions, and when the graphs do not intersect, there are no real solutions to the related system of equations.



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Closing (2 minutes)

Have students discuss with their neighbors the key points from today's lesson. Encourage them to discuss the relationship between the solution(s) to a quadratic equation of the form $ax^2 + bx + c = 0$ and the system

$$y = ax^2 + bx + c$$
$$y = 0.$$

They should discuss an understanding of the relationship between any solution(s) to a system of two equations and the x-coordinate of any point(s) of intersection of the graphs of the equations in the system.

The Lesson Summary below contains key findings from today's lesson.

Lesson Summary An equation or a system of equations may have one or more solutions in the real numbers, or it may have no real number solution. Two graphs that do not intersect in the coordinate plane correspond to a system of two equations without a real solution. If a system of two equations does not have a real solution, the graphs of the two equations do not intersect in the coordinate plane. A quadratic equation in the form $ax^2 + bx + c = 0$, where a, b, and c are real numbers and $a \neq 0$, that has no real solution indicates that the graph of $y = ax^2 + bx + c$ does not intersect the x-axis.

Exit Ticket (4 minutes)

In this Exit Ticket, students show that a particular system of two equations has no real solutions. They demonstrate this both analytically and graphically.



Overcoming a Third Obstacle to Factoring—What If There are No Real Number Solutions?







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Lesson 36: Overcoming a Third Obstacle—What If There Are No Real Number Solutions?

Exit Ticket

Solve the following system of equations or show that it does not have a real solution. Support your answer analytically and graphically.

 $y = x^2 - 4$ y = -(x + 5)



Overcoming a Third Obstacle to Factoring—What If There are No Real Number Solutions?





Exit Ticket Sample Solutions

Solve the following system of equations, or show that it does not have a real solution. Support your answer analytically and graphically. $y = x^2 - 4$ y = -(x + 5)We distribute over the set of parentheses in the second equation and rewrite the system. $y = x^2 - 4$ y = -x - 5The graph of the system shows a parabola and a line that do not intersect. As such, we know that the system does not have a real solution. 2 0 x -3 Algebraically, $x^2 - 4 = -x - 5$ $x^2 + x + 1 = 0.$ Using the quadratic formula with a = 1, b = 1, and c = 1, $x = \frac{-1 + \sqrt{1^2 - 4(1)(1)}}{2(1)}$ or $x = \frac{-1 - \sqrt{1^2 - 4(1)(1)}}{2(1)}$, which indicates that the solutions would be $\frac{-1+\sqrt{-3}}{2}$ and $\frac{-1-\sqrt{-3}}{2}$. Since the square root of a negative number is not a real number, there is no real number x that solves this equation. Thus, the system has no solution (x, y) where x and y are real numbers.

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Problem Set Sample Solutions



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3. Find the value of k so that the graph of the following system of equations has no solution. 3x - 2y - 12 = 0kx + 6y - 10 = 0First, we rewrite the linear equations in the system in slope-intercept form. $y=\frac{3}{2}x-6$ $y = -\frac{k}{6}x + \frac{10}{6}$ There is no solution to this system when the lines are parallel. Two lines are parallel when they share the same slope and have different y-intercepts. Here, the first line has slope $\frac{3}{2}$ and y-intercept -6, and the second line has slope $-\frac{k}{6}$ and y-intercept $\frac{10}{6}$. The lines have different y-intercepts and will be parallel when $-\frac{k}{6} = \frac{3}{2}$ $\frac{3}{2} = -\frac{k}{6}$ 2k = -18k = -9Thus, there is no solution only when k = -9. Offer a geometric explanation to why the equation $x^2 - 6x + 10 = 0$ has no real solutions. 4. The graph of $y = x^2 - 6x + 10$ opens upward (since the leading coefficient is positive) and takes on its lowest value at the vertex (3, 1). Hence, it does not intersect the x-axis, and, therefore, the equation has no real solutions. Without his pencil or calculator, Joey knows that $2x^3 + 3x^2 - 1 = 0$ has at least one real solution. How does he 5. know? The graph of every cubic polynomial function intersects the x-axis at least once because the end behaviors are opposite: one end goes up and the other goes down. This means that the graph of any cubic equation $y = ax^3 + bx^2 + cx + d$ must have at least one x-intercept. Thus, every cubic equation must have at least one real solution.



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(translated) graph would have exactly two x-intercepts. Support or refute Gia's claim. If you agree with her, in which direction did she translate the original graph? Draw graphs to illustrate. By translating the graph of $y = x^2 + 1$ DOWN by 1 unit, the new graph has equation $y = x^2$, which has one x-intercept at x = 0. When translating the original graph DOWN by more than 1 unit, the new graph will cross the x-axis exactly twice. = f(x)y = f(x) -= f(x) - 2.5y -1 In the previous problem, we mentioned that the graph of $y = x^2 + 1$ has no *x*-intercepts. Suppose that $y = x^2 + 1$ 7. is one of two equations in a system of equations and that the other equation is linear. Give an example of a linear equation such that this system has exactly one solution. The line with equation y = 1 is tangent to $y = x^2 + 1$ only at (0, 1); so there would be exactly one real solution to the system. $y = x^2 + 1$ y = 1Another possibility is an equation of any vertical line, such as x = -3 or x = 4, or x = a for any real number a. In prior problems, we mentioned that the graph of $y = x^2 + 1$ has no x-intercepts. Does the graph of $y = x^2 + 1$ 8. intersect the graph of $y = x^3 + 1$? Setting these equations together, we can rearrange terms to get $x^3 - x^2 = 0$, which is an equation we can solve by factoring. We have $x^2(x-1) = 0$, which has solutions at 0 and 1. Thus, the graphs of these equations intersect when x = 0 and when x = 1. When x = 0, y = 1, and when x = 1, y = 2. Thus, the two graphs intersect at the points (0, 1) and (1, 2). The quick answer: The highest term in both equations has degree 3. The third-degree term does not cancel when setting the two equations (in terms of x) equal to each other. All cubic equations have at least one real solution, so

The graph of the quadratic equation $y = x^2 + 1$ has no x-intercepts. However, Gia claims that when the graph of

 $y = x^2 + 1$ is translated by a distance of 1 in a certain direction, the new (translated) graph would have exactly one x-intercept. Further, if $y = x^2 + 1$ is translated by a distance greater than 1 in the same direction, the new

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the two graphs intersect at least at one point.



Lesson 37: A Surprising Boost from Geometry

Student Outcomes

- Students write a complex number in the form a + bi, where a and b are real numbers and the imaginary unit i satisfies $i^2 = -1$. Students geometrically identify i as a multiplicand effecting a 90° counterclockwise rotation of the real number line. Students locate points corresponding to complex numbers in the complex plane.
- Students understand complex numbers as a superset of the real numbers (i.e., a complex number a + bi is real when b = 0). Students learn that complex numbers share many similar properties of the real numbers: associative, commutative, distributive, addition/subtraction, multiplication, etc.

Lesson Notes

Students first receive an introduction to the imaginary unit *i* and develop an algebraic and geometric understanding of the complex numbers (**N-CN.A.1**). Notice that at this level, mathematical tools needed to define the complex numbers are unavailable just as they are unavailable for defining the real numbers; however, students can describe them, understand them, and use them. The lesson then underscores that complex numbers satisfy the same properties of operations as real numbers (**N-CN.A.2**). Finally, students perform exercises to reinforce their understanding of and facility with complex numbers algebraically. This lesson ties into the work in the next lesson, which involves complex solutions to quadratic equations (**N-CN.C.7**).

Students first encounter complex numbers when they classify equations such as $x^2 + 1 = 0$ as having no real number solutions. At that point, the possibility that a solution exists within a superset of the real numbers called the complex numbers is not introduced. At the end of this module, the idea is briefly introduced that every polynomial *P* of degree *n* has *n* values r_i for which $P(r_i) = 0$, where *n* is a whole number and r_i is a real or complex number. Further, in preparation for students' work in Precalculus and Advanced Topics, it is stated (but students are not expected to know) that *P* can be written as the product of *n* linear factors, a result known as the fundamental theorem of algebra. The usefulness of complex numbers as solutions to polynomial equations comes with a cost: While real numbers can be ordered (put in order from smallest to greatest), complex numbers cannot be compared. For example, the complex number $\frac{1}{2} + i \frac{\sqrt{3}}{2}$ is not larger or smaller than $\frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2}$. However, this is a small price to pay. Students begin to see just how important complex numbers are to geometry and computer science in Modules 1 and 2 in Precalculus and Advanced Topics. In college-level science and engineering courses, complex numbers are used in conjunction with differential equations to model circular motion and periodic phenomena in two dimensions.

Classwork

Opening (1 minute)

We introduce a geometric context for complex numbers by demonstrating the analogous relationship between rotations in the plane and multiplication. The intention is for students to develop a deep understanding of *i* through geometry.

 Today, we encounter a new number system that allows us to identify solutions to some equations that have no real number solutions. The complex numbers share many properties with the real numbers with which you are familiar. We take a geometric approach to introducing complex numbers.







Opening Exercise (5 minutes)

Have students work alone on this motivating Opening Exercise. This exercise provides the context and invites the necessity for introducing an alternative number system, namely the complex numbers. Go over parts (a), (b), and (c) with the class; then, suggest that part (d) may be solvable using an alternative number system. Have students table this thought while beginning a geometrically-oriented discussion.

		Seajjolanig.
Opening Exercise Solve each equation for <i>x</i> .		There were times in the past
		when people would have said that an equation such as
a. $x - 1 = 0$	1	$x^2 = 2$ also had no solution.
b. $x + 1 = 0$	-1	
c. $x^2 - 1 = 0$	1, -1	
d. $x^2 + 1 = 0$	No real solution	

Discussion (20 minutes)

Before beginning, allow students to prepare graph paper for drawing images as the discussion unfolds. At the close of this discussion, have students work with partners to summarize at least one thing they learned; then, provide time for some teacher-guided note-taking to capture the definition of the imaginary unit and its connection to geometric rotation.

Recall that multiplying by -1 rotates the number line in the plane by 180° about the point 0.





Demonstrate the rotation concept by drawing the number line carefully on a piece of white paper, drawing an identical number line on a transparency, putting a pin at zero, and rotating the transparency to show that the number line is rotating. For example, rotate from 2 to -2. This, of course, is the same as multiplying by -1.

Pose this interesting thought question to students: Is there a number we can multiply by that corresponds to a 90° rotation?











Students may find that this is a strange question. First, such a number *does not* map the number line to itself, so we have to *imagine* another number line that is a 90° rotation of the original:



This is like the coordinate plane. However, how should we label the points on the vertical axis?

Well, since we *imagined* such a number existed, let's call it the imaginary axis and subdivide it into units of something called *i*. Then, the point 1 on the number line rotates to $1 \cdot i$ on the rotated number line and so on, as follows:



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Lesson 37: A Surprising Boost from Geometry

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- What happens if we multiply a point on the vertical number line by *i*?
 - We rotate that point by 90° counterclockwise:



When we perform two 90° rotations, it is the same as performing a 180° rotation, so multiplying by i twice results in the same rotation as multiplying by -1. Since two rotations by 90° is the same as a single rotation by 180°, two rotations by 90° is equivalent to multiplication by i twice, and one rotation by 180° is equivalent to multiplication by -1, we have

$$i^2 \cdot x = -1 \cdot x$$

for any real number *x*; thus,

MP.2

$$i^2 = -1.$$

- Why might this new number *i* be useful?
 - Recall from the Opening Exercise that there are no real solutions to the equation

$$x^2 + 1 = 0.$$

However, this new number i is a solution.

$$(i)^2 + 1 = -1 + 1 = 0$$

In fact, "solving" the equation $x^2 + 1 = 0$, we get

$$x^{2} = -1$$

$$\sqrt{x^{2}} = \sqrt{-1}$$

$$x = \sqrt{-1} \text{ or } x = -\sqrt{-1}$$

However, because we know from above that $i^2 = -1$, and $(-i)^2 = (-1)^2(i)^2 = -1$, we have two solutions to the quadratic equation $x^2 = -1$, which are *i* and -i.

These results suggests that $i = \sqrt{-1}$. That seems a little weird, but this new imagined number *i* already appears to solve problems we could not solve before.







For example, in Algebra I, when we applied the quadratic formula to

$$x^2 + 2x + 5 = 0$$
,

we found that

$$x = \frac{-2 + \sqrt{2^2 - 4(1)(5)}}{2(1)} \text{ or } x = \frac{-2 - \sqrt{2^2 - 4(1)(5)}}{2(1)}$$
$$x = \frac{-2 + \sqrt{-16}}{2} \text{ or } x = \frac{-2 - \sqrt{-16}}{2}.$$

Recognizing the negative number under the square root, we reported that the equation $x^2 + 2x + 5 = 0$ has no real solutions. Now, however, we can write

$$\sqrt{-16} = \sqrt{16 \cdot -1} = \sqrt{16} \cdot \sqrt{-1} = 4i.$$

Therefore, x = -1 + 2i or x = -1 - 2i, which means -1 + 2i and -1 - 2i are the solutions to $x^2 + 2x + 5 = 0$. The solutions -1 + 2i and -1 - 2i are numbers called *complex numbers*, which we can locate in the complex plane.



In fact, all complex numbers can be written in the form

a + bi,

where a and b are real numbers. Just as we can represent real numbers on the number line, we can represent complex numbers in the complex plane. Each complex number a + bi can be located in the complex plane in the same way we locate the point (a, b) in the Cartesian plane. From the origin, translate a units horizontally along the real axis and b units vertically along the imaginary axis.

Since complex numbers are built from real numbers, we should be able to add, subtract, multiply, and divide them. They should also satisfy the commutative, associative, and distributive properties, just as real numbers do.

Let's check how some of these operations work for complex numbers.





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Scaffolding:

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Name a few complex numbers for students to plot on their graph paper. This builds an understanding of their locations in this coordinate system. For example, consider -2i - 3, -i, i, i - 1, and $\frac{3}{2}i + 2$. Make sure students are also cognizant of the fact that real numbers are also complex numbers (e.g., $-\frac{3}{2}$, 0, $1, \pi$).

Examples 1–2 (4 minutes): Addition and Subtraction with Complex Numbers



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Addition of variable expressions is a matter of rearranging terms according to the properties of operations. Often, we call this combining like terms. These properties of operations apply to complex numbers.

(a + bi) + (c + di) = (a + c) + (b + d)i

Example 1: Addition with Complex Numbers Compute (3 + 4i) + (7 - 20i). (3+4i) + (7-20i) = 3+4i+7-20i = (3+7) + (4-20)i = 10-16i**Example 2: Subtraction with Complex Numbers** Compute (3 + 4i) - (7 - 20i). Scaffolding: (3+4i) - (7-20i) = 3+4i - 7+20i = (3-7) + (4+20)i = -4+20iAs needed, do more exercises with addition and multiplication of complex numbers, such as: Examples 3-4 (6 minutes): Multiplication with Complex Numbers • (6-i) + (3-2i) = 9-3iMultiplication is analogous to polynomial multiplication, using the addition, subtraction, and multiplication operations and the fact that $i^2 = -1$. • (3+2i)(-3+2i) = -13• (5+4i)(2-i) = 14+3i $(a + bi) \cdot (c + di) = ac + bci + adi + bdi^{2}$ • $(2 + \sqrt{3}i)(-2 + \sqrt{3}i) = -7$ = (ac - bd) + (bc + ad)i• $(1-6i)^2 = 37 - 12i$ Example 3: Multiplication with Complex Numbers • (-3-i)((2-4i)+(1+3i)) = -10Compute (1 + 2i)(1 - 2i). $(1+2i)(1-2i) = 1 + 2i - 2i - 4i^2$ = 1 + 0 - 4(-1)= 1 + 4= 5 **Example 4: Multiplication with Complex Numbers** Verify that -1 + 2i and -1 - 2i are solutions to $x^2 + 2x + 5 = 0$. -1+2i: $(-1+2i)^2 + 2(-1+2i) + 5 = 1 - 4i + 4i^2 - 2 + 4i + 5$ $=4i^2-4i+4i+1-2+5$ = -4 + 0 + 4= 0-1 - 2i: $(-1-2i)^2 + 2(-1-2i) + 5 = 1 + 4i + 4i^2 - 2 - 4i + 5$ $=4i^{2}+4i-4i+1-2+5$ = -4 + 0 + 4= 0So, both complex numbers -1 - 2i and -1 + 2i are solutions to the quadratic equation $x^2 + 2x + 5 = 0$.





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Closing (4 minutes)

Ask students to write or discuss with a neighbor some responses to the following prompts:

- What are the advantages of introducing the complex numbers?
- How can we use geometry to explain the imaginary number *i*?

The Lesson Summary box presents key findings from today's lesson.

Lesson Summary		
Multiplication by i rotates every complex number in the complex plane by 90° about the origin.		
Every complex number is in the form $a + bi$, where a is the real part and b is the imaginary part of the number. Real numbers are also complex numbers; the real number a can be written as the complex number $a + 0i$. Numbers of the form bi , for real numbers b , are called imaginary numbers.		
Adding two complex numbers is analogous to combining like terms in a polynomial expression.		
Multiplying two complex numbers is like multiplying two binomials, except one can use $i^2 = -1$ to further write the expression in simpler form.		
Complex numbers satisfy the associative, commutative, and distributive properties.		
Complex numbers allow us to find solutions to polynomial equations that have no real number solutions.		

Exit Ticket (5 minutes)

In this Exit Ticket, students reduce a complex expression to a + bi form and then locate the corresponding point on the complex plane.









Name

Date _____

Lesson 37: A Surprising Boost from Geometry

Exit Ticket

Express the quantities below in a + bi form, and graph the corresponding points on the complex plane. If you use one set of axes, be sure to label each point appropriately.

$$(1+i) - (1-i)$$

 $(1+i)(1-i)$
 $i(2-i)(1+2i)$











Exit Ticket Sample Solutions

Express the quantities below in a + bi form, and graph the corresponding points on the complex plane. If you use one set of axes, be sure to label each point appropriately. (1+i) - (1-i)(1+i)(1-i)i(2-i)(1+2i)(1+i) - (1-i) = 0 + 2iImaginary = 2*i* -3+4i $(1+i)(1-i) = 1+i-i-i^2$ 3 $= 1 - i^2$ **= 1 + 1** 0+2i 21 = 2 + 0i= 2 2+0i Real $i(2-i)(1+2i) = i(2+4i-i-2i^2)$ 0 $\overrightarrow{}_{4}$ =i(2+3i-2(-1))-3 -2 -1 0 = i(2 + 3i + 2)= i(4 + 3i) $= 4i + 3i^2$ = -3 + 4i







A Surprising Boost from Geometry

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This problem set offers students an opportunity to practice and gain facility with complex numbers and complex number arithmetic.





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3. Express each of the following in a + bi form. a. (2+5i) + (4+3i)(2+5i) + (4+3i) = (2+4) + (5+3)i= 6 + 8*i* b. (-1+2i) - (4-3i)(-1+2i) - (4-3i) = -1 + 2i - 4 + 3i= -5 + 5i(4+i)+(2-i)-(1-i)c. (4+i) + (2-i) - (1-i) = 4 + i + 2 - i - 1 + i= 5 + id. (5+3i)(3+5i) $(5+3i)(3+5i) = 5 \cdot 3 + 3 \cdot 3i + 5 \cdot 5i + 3i \cdot 5i$ $= 15 + 9i + 25i + 15i^2$ = 15 + 34i - 15= 0 + 34i= **34***i* e. -i(2-i)(5+6i) $-i(2-i)(5+6i) = -i(10-5i+12i-6i^2)$ = -i(10 + 7i + 6)= -i(16 + 7i) $= -16i - 7i^2$ = -16i + 7= 7 - 16if. (1+i)(2-3i) + 3i(1-i) - i $(1+i)(2-3i) + 3i(1-i) - i = (2+2i-3i-3i^2) + 3i - 3i^2 - i$ = 2 + 2i - 3i + 3 + 3i + 3 - i= 8 + i4. Find the real values of x and y in each of the following equations using the fact that if a + bi = c + di, then a = cand b = d. 5x + 3yi = 20 + 9iа. 5x = 203yi = 9ix = 4y = 3









2(5x+9) = 0

2(5x+9) = (10-3y)i

 $x = -\frac{9}{5}$

2(5x+9) + 0i = 0 + (10 - 3y)i

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с.	3(7-2x) - 5(4y-3)	(7-2x) - 5(4y-3)i = x - 2(1+y)i		
	3(7-2x)=x	-5(4y-3)i = -2(1+y)i		
	21-6x=x	-5(4y-3) = -2(1+y)		
	21 = 7x	-20y + 15 = -2 - 2y		
	x = 3	17 = 18y		
		v – ¹⁷		
		$y = \frac{1}{18}$		

5. Since $i^2 = -1$, we see that

$$i^{3} = i^{2} \cdot i = -1 \cdot i = -i$$

 $i^{4} = i^{2} \cdot i^{2} = -1 \cdot -1 = 1$

Plot *i*, i^2 , i^3 , and i^4 on the complex plane, and describe how multiplication by each rotates points in the complex plane.

0i = (10 - 3y)i

10 - 3y = 0 $y=\frac{10}{3}$

Multiplying by i rotates points by 90° counterclockwise around (0, 0). Multiplying by $i^2 = -1$ rotates points by 180° about (0,0). Multiplying by $i^3 = -i$ rotates points counterclockwise by 270° about the origin, which is equivalent to rotation by 90° clockwise about the origin. Multiplying by i^4 rotates points counterclockwise by 360° , which is equivalent to not rotating at all. The points i, i^2 , i^3 , and i^4 are plotted below on the complex plane.



Express each of the following in a + bi form. 6.

a.	i ⁵	0+i
b.	i^6	-1 + 0i
c.	i ⁷	$0-\mathbf{i}$
d.	i ⁸	1 + 0 <i>i</i>
e.	i ¹⁰²	-1 + 0i

A simple approach is to notice that every 4 multiplications by i result in four 90° rotations, which takes i^4 back to 1. Therefore, divide 102 by 4, which is 25 with remainder 2. So, 102 90° rotations is equivalent to 25 360° rotations and a 180° rotation, and thus $i^{102} = -1$.



MP.8

Lesson 37: A Surprising Boost from Geometry





7. Express each of the following in a + bi form. $(1+i)(1+i) = 1+i+i+i^2$ $(1+i)^2$ а. = 1 + 2i - 1= 2i $(1+i)^4 = ((1+i)^2)^2$ $(1+i)^4$ b. $= (2i)^2$ $= 4i^2$ = -4 $(1+i)^6 = (1+i)^2(1+i)^4$ $(1+i)^{6}$ c. = (2i)(-4)= -8*i* Evaluate $x^2 - 6x$ when x = 3 - i. 8. -10 Evaluate $4x^2 - 12x$ when $x = \frac{3}{2} - \frac{i}{2}$. 9. -10 10. Show by substitution that $\frac{5-i\sqrt{5}}{5}$ is a solution to $5x^2 - 10x + 6 = 0$. $5\left(\frac{5-i\sqrt{5}}{5}\right)^2 - 10\left(\frac{5-i\sqrt{5}}{5}\right) + 6 = \frac{1}{5}\left(5-i\sqrt{5}\right)\left(5-i\sqrt{5}\right) - 2\left(5-i\sqrt{5}\right) + 6$ $=\frac{1}{5}(25-10i\sqrt{5}+5i^2)-2(5-i\sqrt{5})+6$ $=\frac{1}{5}(25-10i\sqrt{5}-5)-2(5-i\sqrt{5})+6$ $= 5 - 2i\sqrt{5} - 1 - 10 + 2i\sqrt{5} + 6$ = 0 11. Evaluate the four products below. a. Evaluate $\sqrt{9} \cdot \sqrt{4}$. $3 \cdot 2 = 6$ Evaluate $\sqrt{9} \cdot \sqrt{-4}$. $3 \cdot 2i = 6i$ Evaluate $\sqrt{-9} \cdot \sqrt{4}$. $3i \cdot 2 = 6i$ Evaluate $\sqrt{-9} \cdot \sqrt{-4}$. $3i \cdot 2i = 6i^2 = -6$ Suppose a and b are positive real numbers. Determine whether the following quantities are equal or not b. equal. $\sqrt{a} \cdot \sqrt{b}$ and $\sqrt{-a} \cdot \sqrt{-b}$ not equal $\sqrt{-a} \cdot \sqrt{b}$ and $\sqrt{a} \cdot \sqrt{-b}$ equal



MP.7







Lesson 38: Complex Numbers as Solutions to Equations

Student Outcomes

Students solve quadratic equations with real coefficients that have complex solutions (N-CN.C.7). They
recognize when the quadratic formula gives complex solutions and write them as a + bi for real numbers a
and b. (A-REI.B.4b)

Lesson Notes

This lesson models how to solve quadratic equations over the set of complex numbers. Students relate the sign of the discriminant to the nature of the solution set for a quadratic equation. Continue to encourage students to make connections between graphs of a quadratic equation, $y = ax^2 + bx + c$, and the number and type of solutions to the equation $ax^2 + bx + c = 0$.

Classwork

Opening (2 minutes)

In Algebra I, students learned that when the quadratic formula resulted in an expression that contained a negative number in the radicand, the equation would have no real solution. Now, students understand the imaginary unit *i* as a number that satisfies $i^2 = -1$, which allows them to solve quadratic equations over the complex numbers. Thus, they can see that every quadratic equation has at least one solution.

Opening Exercises (5 minutes)

Have students work on this opening exercise alone or in pairs. In this exercise, students apply the quadratic formula to three different relatively simple quadratic equations: one with two real roots, one with one real repeated root, and one with two complex roots. Students are then asked to explain the results in terms of the discriminant. Afterward, go over the answers with the class.

Scaffolding:

Advanced students may be able to handle a more abstract framing of—in essence—the same exercise. The exercise below offers the advanced student an opportunity to discover the discriminant and its significance on his or her own.

"Recall that a quadratic equation can have exactly two distinct real solutions, exactly one distinct real solution, or exactly two distinct complex solutions. What is the quadratic formula that we can use to solve an equation in the form $ax^2 + bx + x = 0$, where a, b, and c are real numbers and $a \neq 0$? Analyze this formula to decide when the equation will have two, one, or no real solutions."

Solution:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The *type* of solutions to a quadratic equation hinges on the expression under the radical in the quadratic formula, namely, $b^2 - 4ac$. When $b^2 - 4ac < 0$, both solutions will have imaginary parts. When $b^2 - 4ac > 0$, the quadratic equation has two distinct real solutions. When $b^2 - 4ac = 0$, the quadratic formula simplifies to $x = -\frac{b}{2a}$. In this case, there is only one real solution, which we call a zero of multiplicity two.



: Complex Numbers as Solutions to Equations







Review the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ before beginning this exercise, and define the *discriminant* as the number under the radical; that is, the discriminant is the quantity $b^2 - 4ac$.



Discussion (8 minutes)

- Why do you think we call $b^2 4ac$ the discriminant?
 - In English, a discriminant is a characteristic that allows something (e.g., an object, a person, a function) among a group of other somethings to be distinguished.
 - In this case, the discriminant distinguishes a quadratic equation by its number and type of solutions: one real solution (repeated), two real solutions, or two complex solutions.
- Let's examine the situation when the discriminant is zero. Why does a quadratic equation with discriminant zero have only one real solution?
 - When the discriminant is zero, the quadratic formula gives the single solution $-\frac{b\pm 0}{2a} = -\frac{b}{2a}$
- Why is the solution when $b^2 4ac = 0$ a repeated zero?
 - If $b^2 4ac = 0$, then $c = \frac{b^2}{4a}$ and we can factor the quadratic expression $ax^2 + bx + c$ as follows:

$$ax^{2} + bx + c = ax^{2} + bx + \frac{b^{2}}{4a} = a\left(x^{2} + \frac{b}{a}x + \frac{b^{2}}{4a^{2}}\right) = a\left(x + \frac{b}{2a}\right)^{2}.$$

From what we know of factoring quadratic expressions from Lesson 11, $-\frac{b}{2a}$ is a repeated zero. Analytically, the solutions can be thought of as $-\frac{b+0}{2a}$ and $-\frac{b-0}{2a}$, which are both $-\frac{b}{2a}$. So, there are two solutions that are the same number.





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Scaffolding:

English language learners may benefit from a Frayer diagram or other vocabulary exercise for the word *discriminant*. ^D Geometrically, a quadratic equation represents a parabola. If the discriminant is zero, then the equation of the parabola is $y = a\left(x + \frac{b}{2a}\right)^2$, so the vertex of this parabola is $\left(-\frac{b}{2a}, 0\right)$, meaning the vertex of the parabola lies on the x-axis. Thus, the parabola is tangent to the x-axis and intersects the x-axis only at the point $\left(-\frac{b}{2a}, 0\right)$.

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• For example, the graph of $y = x^2 + 6x + 9$ intersects the *x*-axis only at (-3, 0), as follows.



- Describe the graph of a quadratic equation with positive discriminant.
 - ^a If the discriminant is positive, then the quadratic formula gives two different real solutions.
 - Two real solutions mean the graph intersects the *x*-axis at two distinct real points.
- For example, the graph of $y = x^2 + x 6$ intersects the *x*-axis at (-3, 0) and (2, 0), as follows.





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: Complex Numbers as Solutions to Equations





- Describe the graph of a quadratic equation with negative discriminant.
 - ^a Since the discriminant is negative, the quadratic formula will give two different complex solutions.
 - ^a Since there are no real solutions, the graph does not cross or touch the *x*-axis in the real plane.
- For example, the graph of $y = x^2 + 4$, shown below, does not intersect the *x*-axis.



Example 1 (5 minutes)

Example 1

Consider the equation $3x + x^2 = -7$.

What does the value of the discriminant tell us about number of solutions to this equation?

Solve the equation. Does the number of solutions match the information provided by the discriminant? Explain.

Consider the equation $3x + x^2 = -7$.

- What does the value of the discriminant tell us about number of solutions to this equation?
 - The equation in standard form is $x^2 + 3x + 7 = 0$, so we have a = 1, b = 3, c = 7.
 - ^a The discriminant is $3^2 4(1)(7) = -19$. The negative discriminant indicates that no real solutions exist. There are two complex solutions.
- Solve the equation. Does the number of solutions match the information provided by the discriminant? Explain.
 - Using the quadratic formula,

$$x = \frac{-3 + \sqrt{-19}}{2} \text{ or } x = \frac{-3 + \sqrt{-19}}{2}.$$

• The solutions, in a + bi form, are $-\frac{3}{2} + \frac{\sqrt{19}}{2}i$ and $-\frac{3}{2} - \frac{\sqrt{19}}{2}i$.

• The two complex solutions are consistent with the rule for a negative discriminant.









Exercise (15 minutes)

MP.3

Have students work individually on this exercise; then, have them work with a partner or in a small group to check their solutions. This exercise could also be conducted by using personal white boards and having students show their answers to each question after a few minutes. If many students struggle, invite them to exchange papers with a partner to check for errors. Having students identify errors in their work or the work of others will help them to build fluency when working with these complicated expressions. Debrief this exercise by showing the related graph of the equation in the coordinate plane, and verify that the number of solutions corresponds to the number of *x*-intercepts.

Exercise

Compute the value of the discriminant of the quadratic equation in each part. Use the value of the discriminant to predict the number and type of solutions. Find all real and complex solutions.

a.
$$x^2 + 2x + 1 = 0$$

We have $a = 1, b = 2, and c = 1$. Then
 $b^2 - 4ac = 2^2 - 4(1)(1) = 0$.

Note that the discriminant is zero, so this equation has exactly one real solution.

$$x = \frac{-(2) \pm \sqrt{0}}{2(1)} = -1$$

Thus, the only solution is -1.

b. $x^2 + 4 = 0$

We have a = 1, b = 0, and c = 4. Then

$$b^2 - 4ac = -16$$
.

Note that the discriminant is negative, so this equation has two complex solutions.

$$x=\frac{-0\pm\sqrt{-16}}{2(1)}$$

Thus, the two complex solutions are 2i and -2i.

c.
$$9x^2 - 4x - 14 = 0$$

We have a = 9, b = -4, and c = -14. Then

$$b^2 - 4ac = (-4)^2 - 4(9)(-14)$$

= 16 + 504
= 520.

Note that the discriminant is positive, so this equation has two distinct real solutions.

Using the quadratic formula,

$$x = \frac{-(-4) \pm 2\sqrt{130}}{2(9)}.$$

So, the two real solutions are $\frac{2+\sqrt{130}}{9}$ and $\frac{2-\sqrt{130}}{9}$.



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3: Complex Numbers as Solutions to Equations


d. $3x^2 + 4x + 2 = 0$ We have a = 3, b = 4, and c = 2. Then $b^2 - 4ac = 4^2 - 4(3)(2)$ = 16 - 24= -8. The discriminant is negative, so there will be two complex solutions. Using the quadratic formula, $x = \frac{-4 \pm \sqrt{-8}}{2(3)}.$ So, the two complex solutions are $-\frac{2}{3} + \frac{\sqrt{2}}{3}i$ and $-\frac{2}{3} - \frac{\sqrt{2}}{3}i$. $x = 2x^2 + 5$ e. We can rewrite this equation in standard form with a = 2, b = -1, and c = 5: $2x^2 - x + 5 = 0.$ Then $b^2 - 4ac = (-1)^2 - 4(2)(5)$ = 1 - 40= -39. The discriminant is negative, so there will be two complex solutions. Using the quadratic formula, $x = \frac{-(-1) \pm \sqrt{-39}}{2(2)}$ $x=\frac{1\pm i\sqrt{39}}{4}$ The two solutions are $\frac{1}{4} + \frac{\sqrt{39}}{4}i$ and $\frac{1}{4} - \frac{\sqrt{39}}{4}i$. Scaffolding: $8x^2 + 4x + 32 = 0$ f. Consider assigning advanced We can factor 4 from the left side of this equation to obtain $4(2x^2 + x + 8) = 0$, and students to create quadratic we know that a product is zero when one of the factors are zero. Since $4 \neq 0$, we equations that have specific must have $2x^2 + x + 8 = 0$. This is a quadratic equation with a = 2, b = 1, and solutions. For example, c = 8. Then request a quadratic equation $b^2 - 4ac = 1^2 - 4(2)(8) = -63.$ that has only the solution -5. The discriminant is negative, so there will be two complex solutions. Using the Answer: One such equation is quadratic formula, $x^2 - 10x + 25 = 0$. This $x=\frac{-1\pm\sqrt{-63}}{2(2)}$ $x=\frac{-1\pm 3i\sqrt{7}}{4}.$

follows from the expansion of the left side of $(x + 5)^2 = 0$. Also consider requesting a quadratic equation with solutions 3 + i and 3 - i. One

answer is $x^2 - 6x + 10 = 0$.

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Complex Numbers as Solutions to Equations

The complex solutions are $-rac{1}{4}+rac{3\sqrt{7}}{4}i$ and $-rac{1}{4}-rac{3\sqrt{7}}{4}i$.

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Closing (5 minutes)

As this lesson is summarized, ask students to create a graphic organizer that allows them to compare and contrast the nature of the discriminant, the number and types of solutions to $ax^2 + bx + c = 0$, and the graphs of the equation $y = ax^2 + bx + c$. Have them record a problem of each type from the previous exercise as an example in their graphic organizer.



Exit Ticket (5 minutes)

The Exit Ticket gives students the opportunity to demonstrate their mastery of this lesson's content.



Complex Numbers as Solutions to Equations







Name

Date _____

Lesson 38: Complex Numbers as Solutions to Equations

Exit Ticket

Use the discriminant to predict the nature of the solutions to the equation $4x - 3x^2 = 10$. Then, solve the equation.



Complex Numbers as Solutions to Equations







Exit Ticket Sample Solutions

Use the discriminant to predict the nature of the solutions to the equation $4x - 3x^2 = 10$. Then, solve the equation. $3x^2 - 4x + 10 = 0$ We have a = 3, b = -4, and c = 10. Then $b^2 - 4ac = (-4)^2 - 4(3)(10)$ = 16 - 120 = -104. The value of the discriminant is negative, indicating that there are two complex solutions. $x = \frac{-(-4) \pm \sqrt{-104}}{2(3)}$ $x = \frac{4 \pm 2i\sqrt{26}}{6}$ Thus, the two solutions are $\frac{2}{3} + \frac{\sqrt{26}}{3}i$ and $\frac{2}{3} - \frac{\sqrt{26}}{3}i$.

Problem Set Sample Solutions

The Problem Set offers students more practice solving quadratic equations with complex solutions.

```
Give an example of a quadratic equation in standard form that has ...
1.
            Exactly two distinct real solutions.
      а.
            Since (x + 1)(x - 1) = x^2 - 1, the equation x^2 - 1 = 0 has two distinct real solutions, 1 and -1.
           Exactly one distinct real solution.
      b.
            Since (x + 1)^2 = x^2 + 2x + 1, the equation x^2 + 2x + 1 = 0 has only one real solution. 1.
           Exactly two complex (non-real) solutions.
      c.
            Since x^2 + 1 = 0 has no solutions in the real numbers, this equation must have two complex solutions. They
            are i and -i.
     Suppose we have a quadratic equation ax^2 + bx + c = 0 so that a + c = 0. Does the quadratic equation have one
2.
     solution or two distinct solutions? Are they real or complex? Explain how you know.
     If a + c = 0, then either a = c = 0, a > 0 and c < 0, or a < 0 and c > 0.
     The definition of a quadratic polynomial requires that a \neq 0, so either a > 0 and c < 0 or a < 0 and c > 0.
     In either case, 4ac < 0. Because b^2 is positive and 4ac is negative, we know b^2 - 4ac > 0.
     Therefore, a quadratic equation ax^2 + bx + c = 0 always has two distinct real solutions when a + c = 0.
```



: Complex Numbers as Solutions to Equations







3. Solve the equation
$$5x^2 - 4x + 3 = 0$$
.
We have a quadratic equation with $a = 5$, $b = -4$, and $c = 3$.
 $x = \frac{-(-4) \pm 2\sqrt{-11}}{2(5)}$
So, the solutions are $\frac{2}{5} + \frac{i\sqrt{11}}{5}$ and $\frac{2}{5} - \frac{i\sqrt{11}}{5}$.
4. Solve the equation $2x^2 + 8x = -9$.
In standard form, this is the quadratic equation $2x^2 + 8x + 9 = 0$ with $a = 2$, $b = 8$, and $c = 9$.
 $x = \frac{-8 \pm 2\sqrt{-2}}{2(2)} = \frac{-4 \pm i\sqrt{2}}{2}$
Thus, the solutions are $2 + \frac{i\sqrt{2}}{2}$ and $2 - \frac{i\sqrt{2}}{2}$.
5. Solve the equation $9x - 9x^2 = 3 + x + x^2$.
In standard form, this is the quadratic equation $10x^2 - 8x + 3 = 0$ with $a = 10$, $b = -8$, and $c = 3$.
 $x = -\frac{-(-8) \pm 2\sqrt{-14}}{2(10)} = \frac{8 \pm 2i\sqrt{14}}{20}$
Thus, the solutions are $\frac{2}{5} + \frac{i\sqrt{14}}{10}$ and $\frac{2}{5} - \frac{i\sqrt{14}}{10}$.
6. Solve the equation $3x^2 - x + 1 = 0$.
This is a quadratic equation with $a = 3$, $b = -1$, and $c = 1$.
 $x = -\frac{-(-1) \pm \sqrt{-11}}{2(3)} = \frac{1 \pm i\sqrt{11}}{6}$
Thus, the solutions are $\frac{1}{6} + \frac{i\sqrt{11}}{6}$ and $\frac{1}{6} - \frac{i\sqrt{11}}{6}$.
7. Solve the equation $6x^4 + 4x^2 - 3x + 2 = 2x^2(3x^2 - 1)$.
When expanded, this is a quadratic equation with $a = 6$, $b = -3$, and $c = 2$.
 $6x^4 + 4x^2 - 3x + 2 = 0$
 $x = \frac{-(-3) \pm \sqrt{(-39)}}{2(6)}$
So, the solutions are $\frac{1}{4} + \frac{i\sqrt{39}}{12}$ and $\frac{1}{4} - \frac{i\sqrt{39}}{12}$.

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Solve the equation $25x^2 + 100x + 200 = 0$. 8. We can factor 25 from the left side of this equation to obtain $25(x^2 + 4x + 8) = 0$, and we know that a product is zero when one of the factors is zero. Since $25 \neq 0$, we must have $x^2 + 4x + 8 = 0$. This is a quadratic equation with a = 1, b = 4, and c = 8. Then $x=\frac{-4\pm 4\sqrt{-1}}{2},$ and the solutions are -2 + 2i and -2 - 2i. Write a quadratic equation in standard form such that -5 is its only solution. 9. $(x + 5)^2 = 0$ $x^2 + 10x + 25 = 0$ 10. Is it possible that the quadratic equation $ax^2 + bx + c = 0$ has a positive real solution if a, b, and c are all positive real numbers? No. The solutions are $\frac{-b+\sqrt{b^2-4ac}}{2a}$ and $\frac{-b-\sqrt{b^2-4ac}}{2a}$. If *b* is positive, the second one of these will be negative. So, we need to think about whether or not the first one can be positive. If $-b + \sqrt{b^2 - 4ac} > 0$, then $\sqrt{b^2 - 4ac} > b$; so, $b^2 - 4ac > b^2$, and -4ac > 0. This means that either a or c must be negative. So, if all three coefficients are positive, then there cannot be a positive solution to $ax^2 + bx + c = 0$. 11. Is it possible that the quadratic equation $ax^2 + bx + c = 0$ has a positive real solution if a, b, and c are all negative real numbers? No. If a, b, and c are all negative, then -a, -b, and -c are all positive. The solutions of $ax^2 + bx + c = 0$ are the same as the solutions to $-ax^2 - bx - c = 0$, and by Problem 10, this equation has no positive real solution since it has all positive coefficients. Extension: 12. Show that if k > 3.2, the solutions of $5x^2 - 8x + k = 0$ are not real numbers. We have a = 5, b = -8, and c = k; then $b^2 - 4ac = (-8)^2 - 4 \cdot 5 \cdot k$ = 64 - 20k. When the discriminant is negative, the solutions of the quadratic function are not real numbers. $b^2 - 4ac = 64 - 20k$ *k* < 3.2 $b^2 - 4ac < 64 - 20(3.2)$ $b^2-4ac<0$

Thus, if k > 3.2, then the discriminant is negative and the solutions of $5x^2 - 8x + k = 0$ are not real numbers.

k > 3.2



MP.2

: Complex Numbers as Solutions to Equations









8: Complex Numbers as Solutions to Equations







- 14. We can develop two formulas that can help us find errors in calculated solutions of quadratic equations. Find a formula for the sum *S* of the solutions of the quadratic equation $ax^2 + bx + c = 0$. а. The zeros of the quadratic equation are given by $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. Then $S = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a}$ $=\frac{-b+\sqrt{b^2-4ac}+-b-\sqrt{b^2-4ac}}{2a}$ $=\frac{-b+-b+\sqrt{b^2-4ac}-\sqrt{b^2-4ac}}{2a}$ $=\frac{-2b}{2a}$ Thus, $S = -\frac{b}{a}$.
 - Find a formula for the product *R* of the solutions of the quadratic equation $ax^2 + bx + c = 0$. b.

$$R=\frac{-b+\sqrt{b^2-4ac}}{2a}\cdot\frac{-b-\sqrt{b^2-4ac}}{2a}$$

Note that the numerators differ only in that one is a sum, and one is a difference. The difference of squares formula applies where m = -b and $n = \sqrt{b^2 - 4ac}$. Then,

$$R = \frac{(-b)^2 - \left(\sqrt{b^2 - 4ac}\right)^2}{2a \cdot 2a}$$
$$= \frac{b^2 - b^2 + 4ac}{4a^2}$$
$$= \frac{4ac}{4a^2}$$
$$= \frac{c}{a}.$$

So, the product is $R = \frac{c}{a}$.

June calculated the solutions 7 and -1 to the quadratic equation $x^2 - 6x + 7 = 0$. Do the formulas from c. parts (a) and (b) detect an error in her solutions? If not, determine if her solution is correct.

The sum formula agrees with June's calculations. From June's zeros,

$$7 + -1 = 6$$
,

and from the formula,

$$S=\frac{6}{1}=6.$$

However, the product formula does not agree with her calculations. From June's zeros,

$$\mathbf{7}\cdot -\mathbf{1}=-\mathbf{7},$$

and from the formula,

$$R=\frac{7}{1}=7.$$

June's solutions are not correct: $(7)^2 - 6(7) + 7 = 49 - 42 + 7 = 14$; so, 7 is not a solution to this quadratic equation. Likewise, 1-6+7=2, so 1 is also not a solution to this equation. Thus, the formulas caught her error.

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Lesson 38:

Complex Numbers as Solutions to Equations





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8: Complex Numbers as Solutions to Equations







Lesson 39: Factoring Extended to the Complex Realm

Student Outcomes

- Students solve quadratic equations with real coefficients that have complex solutions. Students extend
 polynomial identities to the complex numbers.
- Students note the difference between solutions to a polynomial equation and the *x*-intercepts of the graph of that equation.

Lesson Notes

This lesson extends the factoring concepts and techniques covered in Topic B of this module to the complex numbers and specifically addresses **N-CN.C.7**. Students will learn how to solve and express the solutions to any quadratic equation. Students observe that complex solutions to polynomial equations with real coefficients occur in conjugate pairs and that only real solutions to polynomial equations are also the *x*-intercepts of the graph of the related polynomial function. In essence, this is the transition lesson to the next lesson on factoring all polynomials into linear factors.

Classwork

Opening (1 minute)

Since the complex numbers have the same arithmetic properties as the real numbers, we should be able to extend our processes for factoring polynomials and finding solutions to polynomial equations to the complex numbers. Today, we extend factoring polynomial expressions and finding solutions to polynomial equations to the complex numbers.

Opening Exercise (8 minutes)

Have students individually complete this opening exercise. Students will eventually identify the expressions in this exercise as differences of squares, to which they can apply the identity $(x + ai)(x - ai) = x^2 + a^2$. But, for now, allow them to work through the algebra and to confirm for themselves that the imaginary terms combine to 0 in each example, resulting in polynomials in standard form with real coefficients. Invite students to the board to display their solutions, and let the class have the first opportunity to correct any mistakes, should it be necessary.

Opening Exercise

Rewrite each expression as a polynomial in standard form.

a. (x+i)(x-i) $(x+i)(x-i) = x^2 + ix - ix - i^2$ $= x^2 - i^2$ $= x^2 - (-1)$ $= x^2 + 1$







b. (x+5i)(x-5i) $(x+5i)(x-5i) = x^2 + 5ix - 5ix - 25i^2$ $= x^2 - 25i^2$ $= x^2 - 25(-1)$ $= x^2 + 25$ c. (x - (2+i))(x - (2-i)) $(x - (2+i))(x - (2-i)) = x^2 - (2+i)x - (2-i)x + [(2+i)(2-i)]$ $= x^2 - 2x - ix - 2x + ix + [4-i^2]$ $= x^2 - 4x + [4 - (-1)]$ $= x^2 - 4x + 5$

Discussion (5 minutes)

Here we begin a dialogue that discusses patterns and regularity observed in the Opening Exercise. As each question is posed, give students time to discuss them with a partner or in their small groups. To encourage students to be accountable for responding to questions during discussion, have them write answers on personal white boards, show a thumbs-up when they have an idea, whisper their idea to a partner before asking for a response from the whole group, or show their agreement or disagreement to a question by showing a thumbs-up/thumbs-down.

- Do you observe any patterns among parts (a)–(c) in the Opening Exercise?
 - After each expression is expanded and like terms are collected, we have quadratic polynomials with real coefficients. The imaginary terms were opposites and combined to 0.
- How could you generalize the patterns into a rule (or identity)?
 - Parts (a) and (b) are instances of the identity

 $(x + ai)(x - ai) = x^2 + a^2.$

- What about part (c)? Do you notice an instance of the same identity?
 - Yes

$$(x - (2 + i))(x - (2 - i)) = ((x - 2) - i)((x - 2) + i)$$

- Where have we seen a similar identity to $(x + ai)(x ai) = x^2 + a^2$?
 - Recall the polynomial identity $(x + a)(x a) = x^2 a^2$ from Lesson 6.
- Recall the quick mental arithmetic we learned in Lesson 7. Can you compute (3 + 2i)(3 2i) mentally?

Yes
$$(3+2i)(3-2i) = 3^2 + 2^2 = 9 + 4 = 13$$

- How about (9 + 4i)(9 4i)?
 - $(9+4i)(9-4i) = 9^2 + 4^2 = 81 + 16 = 97$

Exercises 1–2 (5 minutes)

Students understand that the expansion of (x + ai)(x - ai) is a polynomial with real coefficients; the imaginary terms disappear when working through the algebra. Now, students are expected to understand this process in reverse; in other words, they factor polynomials with real coefficients but complex factors.





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MP.7

Exercises 1–4 Factor the following polynomial expression into products of linear terms. 1. $x^2 + 9$ $x^2 + 9 = (x + 3i)(x - 3i)$ 2. $x^2 + 5$ $x^2 + 5 = (x + i\sqrt{5})(x - i\sqrt{5})$

Discussion (6 minutes)

This discussion is the introduction to conjugate pairs in the context of complex numbers. Relate this idea back to the idea of conjugate pairs for radical expressions from Lesson 29.

• In Lesson 29, we saw that the conjugate of a radical expression such as $x + \sqrt{5}$ is the expression $x - \sqrt{5}$, and if we multiply a radical expression by its conjugate, the result is a rational expression—the radical part disappears.

$$(x + \sqrt{5})(x - \sqrt{5}) = x^2 - x\sqrt{5} + x\sqrt{5} - 5 = x^2 - 5$$

• Analogously, complex numbers a + bi have conjugates. The conjugate of a + bi is a - bi. Then, we see that

$$(a + bi)(a - bi) = a^2 - abi + abi - b^2i^2 = a^2 + b^2.$$

We have observed, for real values of x and a, that the expression

 (x + ai)(x - ai) is a real number. The factors (x + ai) and (x - ai) form a conjugate pair.

Quadratic expressions with real coefficients, as we have seen, can be decomposed into real factors or non-real complex factors. However, non-real factors must be members of a conjugate pair; hence, a quadratic expression with real coefficients cannot have exactly one complex factor.

- Similarly, quadratic equations can have real or non-real complex solutions. If there are complex solutions, they will be conjugates of each other.
- Can a polynomial equation with real coefficients have just one complex solution?
 - No. If there is a complex solution, then the conjugate is also a solution. Complex solutions come in pairs.
- Now, can a polynomial equation have real and non-real solutions?
 - Yes, as long as all non-real complex solutions occur in conjugate pairs.
 - For example, the polynomial equation $(x^2 + 1)(x^2 1) = 0$ has two real solutions, 1 and -1, and two complex solutions. The complex solutions, i and -i, form a conjugate pair.
- If you know that 3 2i is a solution to the polynomial equation P(x) = 0, can you tell me another solution?
 - Complex solutions come in conjugate pairs, so if 3 2i is a solution to the equation, then its conjugate, 3 + 2i, is also a solution.

At this point, have students write down or discuss with their neighbors what they have learned so far. The teacher should walk around the room and check for understanding.





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Scaffolding:

- Show conjugate pairs graphically by graphing (as in the previous lessons) a parabola with 0, 1, and 2 solutions and cubic curves with a various number of solutions. Let students determine visually what is possible.
- Additionally, consider having students complete a Frayer diagram for the term conjugate.
- As an extension, ask students to generate conjugate pairs.



Consider having groups work

Exercise 3 with different

Scaffolding:



Exercise 3 (6 minutes)

Students should work in groups of 2–4 on this exercise. Invite students to the board to present their solutions.

```
polynomials:
     Consider the polynomial P(x) = x^4 - 3x^2 - 4.
3.
                                                                                                           4x^3 - x
           What are the solutions to x^4 - 3x^2 - 4 = 0?
                                                                                                           x^3 - 4x^2 + 29x
                                                    x^4 - 3x^2 - 4 = 0
                                                                                                           x^3 - 6x^2 + 25x
                                                 (x^2)^2 - 3x^2 - 4 = 0
                                                                                                           x^4 - 1
                                                (x^2 + 1)(x^2 - 4) = 0
                                                                                                           x^4 + 3x^2 - 4
                                    (x+i)(x-i)(x+2)(x-2) = 0
                                                                                                           x^4 - 4x^3 + 5x^2
                                                                                                           x^4 - 4x^3 + 13x^2
           The solutions are -i, i, -2, and 2.
                                                                                                           x^4 - 2x^3 - 10x^2
                                                                                                           x^4 + 13x^2 + 36
           How many x-intercepts does the graph of the equation y = x^4 - 3x^2 - 4 have?
     b.
           What are the coordinates of the x-intercepts?
           The graph of y = x^4 - 3x^2 - 4 has two x-intercepts: (-2, 0) and (2, 0).
           Are solutions to the polynomial equation P(x) = 0 the same as the x-intercepts of the graph of y = P(x)?
     c.
           Justify your reasoning.
           No. Only the real solutions to the equation are x-intercepts of the graph. By comparing the graph of the
           polynomial in part (b) to the equation's solutions from part (c), you can see that only the real number
           solutions to the equation correspond to the x-intercepts in the Cartesian plane.
                                                     1
                         0
                                                        0
                                                                      x
```

Exercise 4 (5 minutes)

MP.3

Transition students to the next exercise by announcing that they now want to reverse their thinking. In the previous problem, they solved an equation to find the solutions. Now, pose the question: Can we construct an equation if we know its solutions? Remind students that when a polynomial equation is written in factored form $a(x - r_1)(x - r_2) \cdots (x - r_n) = 0$, the solutions to the equation are r_1, r_2, \dots, r_n . Students will apply what they learned in the previous exercise to create a polynomial equation given its solutions. The problems scaffold from easier to more difficult. Students are encouraged to rewrite the factored form to show the polynomial in standard form for additional practice with complex number arithmetic, but consider having them leave the polynomial in factored form if time is a concern. Have students work with a partner on this exercise.



Factoring Extended to the Complex Realm





4. Write a polynomial P with the lowest possible degree that has the given solutions. Explain how you generated each answer. -2, 3, -4i, 4iа. The polynomial P has two real zeros and two complex zeros. Since the two complex zeros are members of a conjugate pair, P may have as few as four total factors. Therefore, P has degree at least 4. P(x) = (x+2)(x-3)(x+4i)(x-4i) $=(x^2-x-6)(x^2-16i^2)$ $=(x^2-x-6)(x^2+16)$ $= x^4 - x^3 - 6x^2 + 16x^2 - 16x - 96$ $= x^4 - x^3 + 10x^2 - 16x - 96$ -1, 3i b. The polynomial P has one real zero and two complex zeros because complex zeros come in pairs. Since 3i and -3i form a conjugate pair, P has at least three total factors. Therefore, P has degree at least 3. P(x) = (x+1)(x-3i)(x+3i) $= (x+1)(x^2 - 9i^2)$ $= (x+1)(x^2+9)$ $= x^3 + x^2 + 9x + 9$ 0, 2, 1 + i, 1 - ic. Since 1 + i and 1 - i are complex conjugates, P is at least a 4^{th} degree polynomial. P(x) = x(x-2)(x-(1+i))(x-(1-i))= x(x-2)[(x-1)-i][(x-1)+i] $= x(x-2)[(x-1)^2 - i^2]$ $= x(x-2)[(x^2-2x+1)+1]$ $= x(x-2)(x^2-2x+2)$ $= x(x^3 - 2x^2 + 2x - 2x^2 + 4x - 4)$ $= x(x^3 - 4x^2 + 6x - 4)$ $= x^4 - 4x^3 + 6x^2 - 4x$ d. $\sqrt{2}, -\sqrt{2}, 3, 1+2i$ Since 1 + 2i is a complex solution to P(x) = 0, its conjugate, 1 - 2i, must also be a complex solution. Thus, P is at least a fifth-degree polynomial. $P(x) = (x - \sqrt{2})(x + \sqrt{2})(x - 3)(x - (1 + 2i))(x - (1 - 2i))$ $= (x^2 - 2)(x - 3)[(x - 1) - 2i][(x - 1) + 2i]$ $= (x^2 - 2)(x - 3)[(x - 1)^2 - 4i^2]$ $= (x^2 - 2)(x - 3)[(x^2 - 2x + 1) + 4]$ $=(x^2-2)(x-3)(x^2-2x+5)$ $=(x^3-3x^2-2x+6)(x^2-2x+5)$ $= x^5 - 5x^4 + 9x^3 - 5x^2 - 22x + 30$

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e.	2i, 3-i		
	The complex conjugates of $2i$ and $3 - i$ are $-2i$ and $3 + i$, respectively. So, P is at least a fourth-degree polynomial.		
	P(x) = (x - 2i)(x + 2i)(x - (3 - i))(x - (3 + i))		
	$= (x^2 - 4i^2)[(x - 3) + i][(x - 3) - i]$		
	$= (x^2 + 4)[(x - 3)^2 - i^2]$		
	$= (x^2 + 4)[(x^2 - 6x + 9) + 1]$		
	$= (x^2 + 4)(x^2 - 6x + 10)$		
	$= x^4 - 6x^3 + 14x^2 - 24x + 40$		

Closing (3 minutes)

Have students break into small groups to discuss what they learned today. Today's lesson is summarized in the box below.

Lesson Summary

- Polynomial equations with real coefficients can have real or complex solutions or they can have both.
- If a complex number is a solution to a polynomial equation, then its conjugate is also a solution.
- Real solutions to polynomial equations correspond to the *x*-intercepts of the associated graph, but complex solutions do not.

Exit Ticket (6 minutes)

In this Exit Ticket, students solve quadratic equations with real and complex solutions.



9: Factoring Extended to the Complex Realm









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Lesson 39: Factoring Extended to the Complex Realm

Exit Ticket

1. Solve the quadratic equation $x^2 + 9 = 0$. What are the *x*-intercepts of the graph of the function $f(x) = x^2 + 9$?

2. Find the solutions to $2x^5 - 5x^3 - 3x = 0$. What are the *x*-intercepts of the graph of the function $f(x) = 2x^5 - 5x^3 - 3x$?



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Exit Ticket Sample Solutions

Solve the quadratic equation $x^2 + 9 = 0$. What are the x-intercepts of the graph of the function $f(x) = x^2 + 9$? 1. $x^2 + 9 = 0$ $x^2 = -9$ $x = \sqrt{-9}$ or $x = -\sqrt{-9}$ $x = 3\sqrt{-1}$ or $x = -3\sqrt{-1}$ x = 3i or x = -3iThe *x*-intercepts of the graph of the function $f(x) = x^2 + 9$ are the real solutions to the equation $x^2 + 9 = 0$. However, since both solutions to $x^2 + 9 = 0$ are not real, the function $f(x) = x^2 + 9$ does not have any xintercepts. 2. Find the solutions to $2x^5 - 5x^3 - 3x = 0$. What are the *x*-intercepts of the graph of the function $f(x) = 2x^5 - 5x^3 - 3x?$ $(2x^4 - 5x^2 - 3) = 0$ $x(x^2 - 3)(2x^2 + 1) = 0$ $x(x+\sqrt{3})(x-\sqrt{3})(2x^{2}+1)=0$ $x(x+\sqrt{3})(x-\sqrt{3})\left(x+\frac{i\sqrt{2}}{2}\right)\left(x-\frac{i\sqrt{2}}{2}\right)=0$ Thus, x = 0, $x = -\sqrt{3}$, $x = \sqrt{3}$, $x = -\frac{i\sqrt{2}}{2}$, or $x = \frac{i\sqrt{2}}{2}$. The solutions are 0, $\sqrt{3}$, $-\sqrt{3}$, $\frac{i\sqrt{2}}{2}$, and $-\frac{i\sqrt{2}}{2}$. The x-intercepts of the graph of the function $f(x) = 2x^5 - 5x^3 - 3x$ are the real solutions to the equation $2x^{5} - 5x^{3} - 3x = 0$, so the *x*-intercepts are 0, $\sqrt{3}$, and $-\sqrt{3}$.

Problem Set Sample Solutions

1. Rewrite each expression in standard form.
a.
$$(x + 3i)(x - 3i)$$

 $x^{2} + 3^{2} = x^{2} + 9$
b. $(x - a + bi)(x - (a + bi))$
 $(x - a + bi)(x - (a + bi)) = ((x - a) + bi)((x - a) - bi)$
 $= (x - a)^{2} + b^{2}$
 $= x^{2} - 2ax + a^{2} + b^{2}$
c. $(x + 2i)(x - i)(x + i)(x - 2i)$
 $(x + 2i)(x - 2i)(x + i)(x - i) = (x^{2} + 2^{2})(x^{2} + 1^{2})$
 $= (x^{2} + 4)(x^{2} + 1)$
 $= x^{4} + 5x^{2} + 4$



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d. $(x+i)^2 \cdot (x-i)^2$ $(x+i)(x-i) \cdot (x+i)(x-i) = (x^2+1)(x^2+1)$ $= x^4 + 2x^2 + 1$ Suppose in Problem 1 that you had no access to paper, writing utensils, or technology. How do you know that the 2. expressions in parts (a)-(d) are polynomials with real coefficients? In part (a), the identity $(x + ai)(x - ai) = x^2 + a^2$ can be applied. Since the number a is real, the resulting polynomial will have real coefficients. The remaining three expressions can all be rearranged to take advantage of the conjugate pairs identity. In parts (c) and (d), regrouping terms will produce products of polynomial expressions with real coefficients, which will again have real coefficients. 3. Write a polynomial equation of degree 4 in standard form that has the solutions i_1 , $-i_2$, 1_2 , -1_3 . The first step is writing the equation in factored form: (x+i)(x-i)(x+1)(x-1) = 0.Then, use the commutative property to rearrange terms and apply the difference of squares formula: $(x+i)(x-i)(x+1)(x-1) = (x^2+1)(x^2-1)$ $= x^4 - 1$. So, the standard form of the equation is $x^4 - 1 = 0.$ Explain the difference between x-intercepts and solutions to an equation. Give an example of a polynomial with 4. real coefficients that has twice as many solutions as x-intercepts. Write it in standard form. The x-intercepts are the real solutions to a polynomial equation with real coefficients. The solutions to an equation can be real or not real. The previous problem is an example of a polynomial with twice as many solutions than xintercepts. Or, we could consider the equation $x^4 - 6x^3 + 13x^2 - 12x + 4 = 0$, which has zeros of multiplicity 2 at both 1 and 2. Find the solutions to $x^4 - 5x^2 - 36 = 0$ and the *x*-intercepts of the graph of $y = x^4 - 5x^2 - 36$. 5. $(x^2 + 4)(x^2 - 9) = 0$ (x+2i)(x-2i)(x+3)(x-3) = 0Since the solutions are 2i, -2i, 3, and -3, and only real solutions to the equation are x-intercepts of the graph, the x-intercepts are 3 and -3. Find the solutions to $2x^4 - 24x^2 + 40 = 0$ and the x-intercepts of the graph of $y = 2x^4 - 24x^2 + 40$. 6. $2(x^4 - 12x^2 + 20) = 0$ $2(x^2 - 10)(x^2 - 2) = 0$ Since all of the solutions $\sqrt{10}$, $-\sqrt{10}$, $\sqrt{2}$, and $-\sqrt{2}$ are real numbers, the x-intercepts of the graph are $\sqrt{10}$, $-\sqrt{10}$, $\sqrt{2}$, and $-\sqrt{2}$.



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7. Find the solutions to $x^4 - 64 = 0$ and the *x*-intercepts of the graph of $y = x^4 - 64$.

$$(x^{2}+8)(x^{2}-8) = 0$$

(x+ $\sqrt{8}i$)(x- $\sqrt{8}i$)(x+ $\sqrt{8}$)(x- $\sqrt{8}$) = 0

The x-intercepts are $2\sqrt{2}$ and $-2\sqrt{2}$.

8. Use the fact that $x^4 + 64 = (x^2 - 4x + 8)(x^2 + 4x + 8)$ to explain how you know that the graph of $y = x^4 + 64$ has no *x*-intercepts. You need not find the solutions.

The x-intercepts of $y = x^4 + 64$ are solutions to $(x^2 - 4x + 8)(x^2 + 4x + 8) = 0$. Both $x^2 - 4x + 8 = 0$ and $x^2 + 4x + 8 = 0$ have negative discriminant values of -16, so the equations $x^2 - 4x + 8 = 0$ and $x^2 + 4x + 8 = 0$ have no real solutions. Thus, the equation $x^4 + 64 = 0$ has no real solutions, and the graph of $y = x^4 + 64$ has no x-intercepts.

Since $x^4 + 64 = 0$ has no real solutions, the graph of $y = x^4 + 64$ has no x-intercepts.



9: Factoring Extended to the Complex Realm







Lesson 40: Obstacles Resolved—A Surprising Result

Student Outcomes

 Students understand the fundamental theorem of algebra and that all polynomial expressions factor into linear terms in the realm of complex numbers.

Lesson Notes

There is no real consensus in the literature about what exactly constitutes the fundamental theorem of algebra; it is stated differently in different texts. The two-part theorem stated in this lesson encapsulates the main ideas of the theorem and its corollaries while remaining accessible to students. The first part of what is stated here as the fundamental theorem of algebra is the one that students are not mathematically equipped to prove or justify at this level; this part states that every polynomial equation has at least one solution in the complex numbers and will need to be accepted without proof. The consequence of this first part is what is really interesting—that every polynomial expression factors into the same number of linear factors as its degree. Justification for this second part of the fundamental theorem of algebra is accessible for students as long as they can accept the first part without needing proof. Since every polynomial of degree $n \ge 1$ will factor into n linear factors, then any polynomial function of degree n will have n zeros (including repeated zeros).

Classwork

Opening Exercise (5 minutes)

At the beginning of the lesson, focus on the most familiar of polynomial expressions, the quadratic equations. Ensure that students understand the link provided by the remainder theorem between solutions of polynomial equations and factors of the associated polynomial expression. Allow students time to work on the Opening Exercise, and then debrief.

Opening Exercise				
Write each of the following quadratic expressions as a product of linear factors. Verify that the factored form is equivalent.				
a.	$x^2 + 12x + 27$	(x+3)(x+9)		
b.	$x^2 - 16$	(x+4)(x-4)		
c.	$x^2 + 16$	(x+4i)(x-4i)		
d.	$x^2 + 4x + 5$	(x+2+i)(x+2-i)		



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Discussion (7 minutes)

Remind students about the remainder theorem studied in Lesson 19 in this module. The remainder theorem states that if *P* is a polynomial function, and P(a) = 0 for some value of *a*, then x - a is a factor of *P*. The remainder theorem plays an important role in the development of this lesson, linking the solutions of a polynomial equation to the factors of the associated polynomial expression.

- With a partner, describe any patterns you see in the Opening Exercise.
 - Can every quadratic polynomial be written in terms of linear factors? If so, how many linear factors?
 - Yes; two
- How do you know?
 - Every quadratic equation has two solutions that can be found using the quadratic formula. These solutions of the equation lead to linear factors of the quadratic polynomial.
- What types of solutions can a quadratic equation have? What does this mean about the graph of the corresponding function?
 - The equation has either two real solutions, one real solution, or two complex solutions. These situations correspond to the graph having two x-intercepts, one x-intercept, or no x-intercepts.

Be sure that students realize that real numbers are also complex numbers; a real number a can be written as a + 0i.

Example 1 (8 minutes)

The purpose of this example is to help students move fluently between the concepts of x-intercepts of the graph of a polynomial equation y = P(x), the solutions of the polynomial equation P(x) = 0, and the factors in the factored form of the associated polynomial P. Talk students through parts (a)–(e), and then allow them time to work alone or in pairs on part (f) before completing the discussion.





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- We established earlier in the lesson that all quadratic expressions can be written in terms of two linear factors. How many factors did our cubic expression have?
 - Three
- Is it true that every cubic expression can be factored into three linear factors?
 - Yes, because a cubic equation will always have at least one real solution that corresponds to a linear factor of the expression. What is left over will be a quadratic expression, which can be written in terms of two linear factors.

If students don't seem ready to answer the last question or are unsure of the answer, let them work through Exercise 1 and then readdress it.

Exercises 1–2 (6 minutes)

Give students time to work through the two exercises and then lead the discussion that follows.



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has three real and two

complex solutions.





Discussion (3 minutes)

- Do your results from Exercises 1 and 2 agree with our conclusions from Example 1?
 - Yes, each cubic function could be written as a product of three linear factors.
- Make a conjecture about what might happen if we factored a degree 4 polynomial. What about a degree 5 polynomial? Explain your reasoning.
 - A degree 4 polynomial should have 4 linear factors. Based on the previous examples, it seems that a polynomial has as many linear factors as its degree. Similarly, a degree 5 polynomial should be able to be written as a product of 5 linear factors.
- Our major conclusion in this lesson is a two-part theorem known as the fundamental theorem of algebra (FTA). Part 1 of the fundamental theorem of algebra says that every polynomial equation has at least one solution in the complex numbers. Does that agree with our experience?

Scaffolding:

To meet a variety of student needs, ask students to:

- Restate the FTA in their own words (either in writing or verbally).
- Illustrate the FTA with an example. For instance, show that the FTA is true for some polynomial function.
- Apply the FTA to some examples. For instance, how many linear factors would be in the factored form of x⁵ - 3x + 1?

P Yes

- Think about how we factor a polynomial expression P: We find one solution a to P(x) = 0, then we factor out the term (x a). We are left with a new polynomial expression of one degree lower than P, so we look for another solution, and repeat until we have factored everything into linear parts.
- Consider the polynomial $P(x) = x^4 3x^3 + 6x^2 12x + 8$ in the next example.

Example 2 (8 minutes)

While we do not have the mathematical tools or experience needed to prove the fundamental theorem of algebra (either part), this example illustrates how the logic of the second part of the FTA works. Lead students through this example, allowing time for factoring and discussion at each step.



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The logic students just followed leads to part 2 of the fundamental theorem of algebra, which is the result that was already conjectured: A polynomial of degree $N \ge 1$ will factor into N linear factors with complex coefficients. Collectively, these two results are often just referred to as the fundamental theorem of algebra. Although students have only worked with polynomials with real coefficients, the FTA applies to polynomial functions with real coefficients, such as $P(x) = x^3 + 2x^2 - 4$ as well as to polynomial functions with non-real coefficients, such as $P(x) = x^3 + 3ix^2 + 4 - 2i$. Students have not attempted to justify the first part, but they should be able to justify the second part of the theorem.

Fundamental Theorem of Algebra

- 1. Every polynomial function of degree $N \ge 1$ with real or complex coefficients has at least one real or complex zero.
- 2. Every polynomial of degree $N \ge 1$ with real or complex coefficients factors into N linear terms with real or complex coefficients.
- Why is the fundamental theorem of algebra so "fundamental" to mathematics?
 - The fundamental theorem says that the complex number system contains every zero of every polynomial function. We do not need to look anywhere else to find zeros to these types of functions.
- Notice that the fundamental theorem just tells us that the factorization of the polynomial exists; it does not help us actually find it. If we had been given a polynomial function that did not have any real zeros, it would have been very hard to start the factorization process.

Closing (3 minutes)

- With a partner, summarize the key points of this lesson.
- What does the fundamental theorem of algebra guarantee?
 - A polynomial of degree $N \ge 1$ will factor into N linear factors, and the associated function will have N zeros, some of which may be repeated.
- Why is this important?
 - The fundamental theorem of algebra ensures that there are as many zeros as we'd expect for a polynomial function and that factoring will always (in theory) work to find solutions to polynomial equations.
- Illustrate the fundamental theorem of algebra with an example.



Obstacles Resolved—A Surprising Result



Lesson 40

ALGEBRA II



Lesson Summary

Every polynomial function of degree n, for $n \ge 1$, has n roots over the complex numbers, counted with multiplicity. Therefore, such polynomials can always be factored into n linear factors, and the obstacles to factoring we saw before have all disappeared in the larger context of allowing solutions to be complex numbers.

The Fundamental Theorem of Algebra:

- 1. If P is a polynomial function of degree $n \ge 1$, with real or complex coefficients, then there exists at least one number r (real or complex) such that P(r) = 0.
- 2. If P is a polynomial function of degree $n \ge 1$, given by $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ with real or complex coefficients a_i , then P has exactly n zeros r_1, r_2, \dots, r_n (not all necessarily distinct), such that $P(x) = a_n(x r_1)(x r_2) \dots (x r_n)$.

Exit Ticket (5 minutes)









Name

Lesson 40: Obstacles Resolved—A Surprising Result

Exit Ticket

Consider the degree 5 polynomial function $P(x) = x^5 - 4x^3 + 2x^2 + 3x - 5$, whose graph is shown below. You do not need to factor this polynomial to answer the questions below.

1. How many linear factors is *P* guaranteed to have? Explain.



2. How many zeros does *P* have? Explain.

3. How many real zeros does *P* have? Explain.

4. How many complex zeros does *P* have? Explain.



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Exit Ticket Sample Solutions



Problem Set Sample Solutions



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3. Express each cubic function below in terms of linear factors. $f(x) = x^3 - 6x^2 - 27x$ b. $f(x) = x^3 - 16x^2$ c. $f(x) = x^3 + 16x$ f(x) = x(x-9)(x+3) $f(x) = x^2(x-16)$ f(x) = x(x+4i)(x-4i)а. 4. For each cubic function below, one of the zeros is given. Express each cubic function in terms of linear factors. $f(x) = 2x^3 - 9x^2 - 53x - 24; f(8) = 0$ b. $f(x) = x^3 + x^2 + 6x + 6; f(-1) = 0$ a. f(x) = (x - 8)(2x + 1)(x + 3) $f(x) = (x+1)(x+i\sqrt{6})(x-i\sqrt{6})$ Determine if each statement is always true or sometimes false. If it is sometimes false, explain why it is not always 5. true. A degree 2 polynomial function will have two linear factors. a. Always true b. The graph of a degree 2 polynomial function will have two x-intercepts. False. It is possible for the solutions to a degree 2 polynomial to be complex, in which case the graph would not cross the x-axis. It is also possible for the graph to have only one x-intercept if the vertex lies on the xaxis. c. The graph of a degree 3 polynomial function might not cross the x-axis. False. A degree 3 polynomial must cross the x-axis at least one time. A polynomial function of degree *n* can be written in terms of *n* linear factors. d. Always true Consider the polynomial function $f(x) = x^6 - 9x^3 + 8$. 6. How many linear factors does $x^6 - 9x^3 + 8$ have? Explain. a. Since the degree is 6, the polynomial must have 6 linear factors. How is this information useful for finding the zeros of f? b. We know that the function has 6 zeros since there are 6 linear factors. Each factor corresponds to a zero of the function. Find the zeros of f. (Hint: Let $Q = x^3$. Rewrite the equation in terms of Q to factor.) c. 1, 2, $-1 + i\sqrt{3}$, $-1 - i\sqrt{3}$, $\frac{-1 + i\sqrt{3}}{2}$, $\frac{-1 - i\sqrt{3}}{2}$



0: Obstacles Resolved—A Surprising Result









 $x^{2} - 12x + 36 = (x - 6)(x - 6)$, so the equation $x^{2} - 12x + 36 = 0$ has only one solution, which is 6.



MP.3

Lesson 40: Obstacles Resolved—A Surprising Result



