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Relationships Between Quantities and Reasoning with Equations and Their Graphs

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[1] Each lesson is ONE day, and ONE day is considered a 45-minute period.

Algebra I • Module 1

Relationships Between Quantities and Reasoning with Equations and Their Graphs

OVERVIEW

By the end of Grade 8, students have learned to solve linear equations in one variable and have applied graphical and algebraic methods to analyze and solve systems of linear equations in two variables. Now, students are introduced to non-linear equations and their graphs. Students formalize their understanding of equivalent algebraic expressions and begin their study of polynomial expressions. Further, they learn that there are some actions that, when applied to the expressions on both sides of an equal sign, will not result in an equation with the same solution set as the original equation. Finally, they encounter problems that induce the full modeling cycle, as it is described in the Common Core Learning Standards for Mathematics.

In Topic A, students explore the main functions that they will work with in Algebra I: linear, quadratic, and exponential. The goal is to introduce students to these functions by having them make graphs of situations (usually based upon time) in which the functions naturally arise (**A-CED.A.2**). As they graph, they reason abstractly and quantitatively as well as choose and interpret units to solve problems related to the graphs they create (**N-Q.A.1**, **N-Q.A.2**, **N-Q.A.3**).

In middle school, students applied the properties of operations to add, subtract, factor, and expand expressions (**6.EE.A.3**, **6.EE.A.4**, **7.EE.A.1**, **8.EE.A.1**). Now, in Topic B, students use the structure of expressions to define what it means for two algebraic expressions to be equivalent. In doing so, they discern that the commutative, associative, and distributive properties help link each of the expressions in the collection together, even if the expressions look very different themselves (**A-SSE.A.2**). They learn the definition of a polynomial expression and build fluency in identifying and generating polynomial expressions as well as adding, subtracting, and multiplying polynomial expressions (**A-APR.A.1**). The Mid-Module Assessment follows Topic B.

Throughout middle school, students practice the process of solving linear equations (**6.EE.B.5**, **6.EE.B.7**, **7.EE.B.4**, **8.EE.C.7**) and systems of linear equations (**8.EE.C.8**). Now, in Topic C, instead of just solving equations, they formalize descriptions of what they learned before (variable, solution sets, etc.) and are able to explain, justify, and evaluate their reasoning as they strategize methods for solving linear and non-linear equations (**A-REI.A.1**, **A-REI.B.3**, **A-CED.A.4**). Students take their experience solving systems of linear equations further as they prove the validity of the addition method, learn a formal definition for the graph of an equation and use it to explain the reasoning of solving systems graphically, and represent the solution to systems of linear inequalities graphically (**A-CED.A.3**, **A-REI.C.5**, **A-REI.C.6**, **A-REI.D.10**, **A-REI.D.12**).

In Topic D, students are formally introduced to the modeling cycle (see page 61 of the CCLS) through problems that can be solved by creating equations and inequalities in one variable, systems of equations, and graphing (**N-Q.A.1**, **A-SSE.A.1**, **A-CED.A.1**, **A-CED.A.2**, **A-REI.B.3**). The End-of-Module Assessment follows Topic D.

Focus Standards

Reason quantitatively and use units to solve problems.

N-Q.A.1 Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays.★

N-Q.A.2[2] Define appropriate quantities for the purpose of descriptive modeling.★

N-Q.A.3 Choose a level of accuracy appropriate to limitations on measurement when reporting quantities.★

Interpret the structure of expressions.

A-SSE.A.1 Interpret expressions that represent a quantity in terms of its context.★

- a. Interpret parts of an expression, such as terms, factors, and coefficients.
- b. Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret* $P(1 + r)^n$ *as the product of P and a factor not depending on P.*

A-SSE.A.2 Use the structure of an expression to identify ways to rewrite it. *For example, see* $x^4 – y^4$ *as* $(x^2)^2 – (y^2)^2$*, thus recognizing it as a difference of squares that can be factored as* $(x^2 – y^2)(x^2 + y^2)$*.*

Perform arithmetic operations on polynomials.

A-APR.A.1 Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.

Create equations that describe numbers or relationships.

A-CED.A.1[3] Create equations and inequalities in one variable and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions*.★

A-CED.A.2 Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.★

A-CED.A.3 Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.*★

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[2] This standard will be assessed in Algebra I by ensuring that some modeling tasks (involving Algebra I content or securely held content from Grades 6-8) require the student to create a quantity of interest in the situation being described.

[3] In Algebra I, tasks are limited to linear, quadratic, or exponential equations with integer exponents.

A-CED.A.4 Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. *For example, rearrange Ohm's law V = IR to highlight resistance R.*★

Understand solving equations as a process of reasoning and explain the reasoning.

A-REI.A.1 Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method.

Solve equations and inequalities in one variable.

A-REI.B.3 Solve linear equations and inequalities in one variable, including equations with coefficients represented by letters.

Solve systems of equations.

A-REI.C.5 Prove that, given a system of two equations in two variables, replacing one equation by the sum of that equation and a multiple of the other produces a system with the same solutions.

A-REI.6[4] Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables.

Represent and solve equations and inequalities graphically.

A-REI.D.10 Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line).

A-REI.D.12 Graph the solutions to a linear inequality in two variables as a half-plane (excluding the boundary in the case of a strict inequality), and graph the solution set to a system of linear inequalities in two variables as the intersection of the corresponding half-planes.

Foundational Standards

Apply and extend previous understandings of numbers to the system of rational numbers.

6.NS.C.7 Understand ordering and absolute value of rational numbers.

a. Interpret statements of inequality as statements about the relative position of two numbers on a number line diagram. *For example, interpret –3 > –7 as a statement that –3 is located to the right of –7 on a number line oriented from left to right.*

b. Write, interpret, and explain statements of order for rational numbers in real-world contexts. *For example, write –3°C > –7°C to express the fact that –3°C is warmer than –7°C.*

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[4] Tasks have a real-world context. In Algebra I, tasks have hallmarks of modeling as a mathematical practice (less defined tasks, more of the modeling cycle, etc.).

Apply and extend previous understandings of arithmetic to algebraic expressions.

6.EE.A.3 Apply the properties of operations to generate equivalent expressions. *For example, apply the distributive property to the expression 3(2 + x) to produce the equivalent expression 6 + 3x; apply the distributive property to the expression 24x + 18y to produce the equivalent expression 6(4x + 3y); apply properties of operations to y + y + y to produce the equivalent expression 3y.*

6.EE.A.4 Identify when two expressions are equivalent (i.e., when the two expressions name the same number regardless of which value is substituted into them). *For example, the expressions y + y + y and 3y are equivalent because they name the same number regardless of which number y stands for.*

Reason about and solve one-variable equations and inequalities.

6.EE.B.5 Understand solving an equation or inequality as a process of answering a question: which values from a specified set, if any, make the equation or inequality true? Use substitution to determine whether a given number in a specified set makes an equation or inequality true.

6.EE.B.6 Use variables to represent numbers and write expressions when solving a real-world or mathematical problem; understand that a variable can represent an unknown number, or depending on the purpose at hand, any number in a specified set.

6.EE.B.7 Solve real-world and mathematical problems by writing and solving equations of the form $x + p = q$ and $px = q$ for cases in which p, q and x are all nonnegative rational numbers.

6.EE.B.8 Write an inequality of the form $x > c$ or $x < c$ to represent a constraint or condition in a real-world or mathematical problem. Recognize that inequalities of the form $x > c$ or $x < c$ have infinitely many solutions; represent solutions of such inequalities on number line diagrams.

Use properties of operations to generate equivalent expressions.

7.EE.A.1 Apply properties of operations as strategies to add, subtract, factor, and expand linear expressions with rational coefficients.

7.EE.A.2 Understand that rewriting an expression in different forms in a problem context can shed light on the problem and how the quantities in it are related. *For example, a + 0.05a = 1.05a means that "increase by 5%" is the same as "multiply by 1.05."*

Solve real-life and mathematical problems using numerical and algebraic expressions and equations.

7.EE.B.3 Solve multi-step real-life and mathematical problems posed with positive and negative rational numbers in any form (whole numbers, fractions, and decimals), using tools strategically. Apply properties of operations to calculate with numbers in any form; convert between forms as appropriate; and assess the reasonableness of answers using mental computation and estimation strategies. *For example: If a woman making $25 an hour gets a 10% raise, she will make an additional 1/10 of her salary an hour, or $2.50, for a new salary of $27.50. If you want to place a towel bar 9 3/4 inches long in the center of a door that is 27 1/2 inches wide, you will need to place the bar about 9 inches from each edge; this estimate can be used as a check on the exact computation.*

7.EE.B.4 Use variables to represent quantities in a real-world or mathematical problem, and construct simple equations and inequalities to solve problems by reasoning about the quantities.

a. Solve word problems leading to equations of the form $px + q = r$ and $p(x + q) = r$, where p, q, and r are specific rational numbers. Solve equations of these forms fluently. Compare an algebraic solution to an arithmetic solution, identifying the sequence of the operations used in each approach. *For example, the perimeter of a rectangle is 54 cm. Its length is 6 cm. What is its width?*

b. Solve word problems leading to inequalities of the form $px + q > r$ or $px + q < r$, where p, q, and r are specific rational numbers. Graph the solution set of the inequality and interpret it in the context of the problem. *For example: As a salesperson, you are paid $50 per week plus $3 per sale. This week you want your pay to be at least $100. Write an inequality for the number of sales you need to make, and describe the solutions.*

Work with radicals and integer exponents.

8.EE.A.1 Know and apply the properties of integer exponents to generate equivalent numerical expressions. *For example,* $3^2 \times 3^{-5} = 3^{-3} = 1/3^3 = 1/27$.

8.EE.A.2 Use square root and cube root symbols to represent solutions to equations of the form $x^2 = p$ and $x^3 = p$, where p is a positive rational number. Evaluate square roots of small perfect squares and cube roots of small perfect cubes. Know that $\sqrt{2}$ is irrational.

Analyze and solve linear equations and pairs of simultaneous linear equations.

8.EE.C.7 Solve linear equations in one variable.

a. Give examples of linear equations in one variable with one solution, infinitely many solutions, or no solutions. Show which of these possibilities is the case by successively transforming the given equation into simpler forms, until an equivalent equation of the form $x = a$, $a = a$, or $a = b$ results (where a and b are different numbers).

b. Solve linear equations with rational number coefficients, including equations whose solutions require expanding expressions using the distributive property and collecting like terms.

8.EE.C.8 Analyze and solve pairs of simultaneous linear equations.

a. Understand that solutions to a system of two linear equations in two variables correspond to points of intersection of their graphs, because points of intersection satisfy both equations simultaneously.

b. Solve systems of two linear equations in two variables algebraically, and estimate solutions by graphing the equations. Solve simple cases by inspection. *For example,* $3x + 2y = 5$ *and* $3x + 2y = 6$ *have no solution because* $3x + 2y$ *cannot simultaneously be 5 and 6.*

c. Solve real-world and mathematical problems leading to two linear equations in two variables. *For example, given coordinates for two pairs of points, determine whether the line through the first pair of points intersects the line through the second pair.*

Focus Standards for Mathematical Practice

MP.1 **Make sense of problems and persevere in solving them.** Students are presented with problems that require them to try special cases and simpler forms of the original problem to gain better understanding of the problem.

MP.2 **Reason abstractly and quantitatively.** Students analyze graphs of non-constant rate measurements and reason from the shape of the graphs to infer what quantities are being displayed and consider possible units to represent those quantities.

MP.3 **Construct viable arguments and critique the reasoning of others.** Students reason about solving equations using "if-then" moves based on equivalent expressions and properties of equality and inequality. They analyze when an "if-then" move is not reversible.

MP.4 **Model with mathematics.** Students have numerous opportunities in this module to solve problems arising in everyday life, society, and the workplace from modeling bacteria growth to understanding the federal progressive income tax system.

MP.6 **Attend to precision.** Students formalize descriptions of what they learned before (variables, solution sets, numerical expressions, algebraic expressions, etc.) as they build equivalent expressions and solve equations. Students analyze solution sets of equations to determine processes (e.g., squaring both sides of an equation) that might lead to a solution set that differs from that of the original equation.

MP.7 **Look for and make use of structure.** Students reason with and about collections of equivalent expressions to see how all the expressions in the collection are linked together through the properties of operations. They discern patterns in sequences of solving equation problems that reveal structures in the equations themselves: $2x + 4 = 10$, $2(x - 3) + 4 = 10$, $2(3x - 4) + 4 = 10$, etc.

MP.8 **Look for and express regularity in repeated reasoning.** After solving many linear equations in one variable (e.g., $3x + 5 = 8x - 17$), students look for general methods for solving a generic linear equation in one variable by replacing the numbers with letters: $ax + b = cx + d$. They have opportunities to pay close attention to calculations involving the properties of operations, properties of equality, and properties of inequality as they find equivalent expressions and solve equations, noting common ways to solve different types of equations.

Terminology

New or Recently Introduced Terms

- **Piecewise-Linear Function** (Given a finite number of non-overlapping intervals on the real number line, a *(real) piecewise-linear function* is a function from the union of the intervals to the set of real numbers such that the function is defined by (possibly different) linear functions on each interval.)
- **Numerical Symbol** (A *numerical symbol* is a symbol that represents a specific number.)

- **Variable Symbol** (A *variable symbol* is a symbol that is a placeholder for a number. It is possible that a question may restrict the type of number that a placeholder might permit, maybe integers only or a positive real number, for instance.)
- **Numerical Expression** (A *numerical expression* is an algebraic expression that contains only numerical symbols (no variable symbols) and that evaluates to a single number.)
- **Algebraic Expression** (An *algebraic expression* is either: (1) a numerical symbol or a variable symbol or (2) the result of placing previously generated algebraic expressions into the two blanks of one of the four operators ((__) + (__), (__) − (__), (__) × (__), (__) ÷ (__)) or into the base blank of an exponentiation with an exponent that is a rational number.)
- **Equivalent Numerical Expressions** (Two numerical expressions are *equivalent* if they evaluate to the same number.)
- **Equivalent Algebraic Expressions** (Two algebraic expressions are *equivalent* if we can convert one expression into the other by repeatedly applying the commutative, associative, and distributive properties and the properties of rational exponents to components of the first expression.)
- **Polynomial Expression** (A *polynomial expression* is either: (1) a numerical expression or a variable symbol or (2) the result of placing two previously generated polynomial expressions into the blanks of the addition operator (__ + __) or the multiplication operator (__ × __).)
- **Monomial** (A *monomial* is a polynomial expression generated using only the multiplication operator (__ × __). Monomials are products whose factors are numerical expressions or variable symbols.)
- **Degree of a Monomial** (The *degree of a non-zero monomial* is the sum of the exponents of the variable symbols that appear in the monomial.)
- **Standard Form of a Polynomial Expression in One Variable** (A polynomial expression with one variable symbol x is in *standard form* if it is expressed as $a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$, where n is a non-negative integer, and $a_0, a_1, a_2, \dots, a_n$ are constant coefficients with $a_n \neq 0$. A polynomial expression in x that is in standard form is often called a *polynomial in* x.)
- **Degree of a Polynomial in Standard Form** (The *degree of a polynomial in standard form* is the highest degree of the terms in the polynomial, namely n.)
- **Leading Term and Leading Coefficient of a Polynomial in Standard Form** (The term a_nx^n is called the *leading term,* and a_n is called the *leading coefficient.*)
- **Constant Term of a Polynomial in Standard Form** (The *constant term* is the value of the numerical expression found by substituting 0 into all the variable symbols of the polynomial, namely a_0.)
- **Solution** (A *solution* to an equation with one variable is a number in the domain of the variable that, when substituted for all instances of the variable in both expressions, makes the equation a true number sentence.)
- **Solution Set** (The set of solutions of an equation is called its *solution set.*)
- **Graph of an Equation in Two Variables** (The set of all points in the coordinate plane that are solutions to an equation in two variables is called the *graph of the equation.*)
- **Zero Product Property** (The Zero Product Property states that given real numbers, a and b, if $a \cdot b = 0$ then either $a = 0$ or $b = 0$, or both a and $b = 0$.)

Familiar Terms and Symbols[5]

- Equation
- Identity
- Inequality
- System of Equations
- Properties of Equality
- Properties of Inequality
- Solve
- Linear Function
- Formula
- Term

Suggested Tools and Representations

- Coordinate Plane
- Equations and Inequalities

Assessment Summary

Assessment Type	Administered	Format	Standards Addressed
Mid-Module Assessment Task	After Topic B	Constructed response with rubric	N-Q.A.1, N-Q.A.2, N-Q.A.3, A-APR.A.1, A-SSE.A.2
End-of-Module Assessment Task	After Topic D	Constructed response with rubric	N-Q.A.1, A-SSE.A.1, A-SSE.A.2, A-APR.A.1, A-CED.A.1, A-CED.A.2, A-CED.A.3, A-CED.A.4, A-REI.A.1, A-REI.B.3, A-REI.C.5, A-REI.C.6, A-REI.D.10, A-REI.D.12

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[5] These are terms and symbols students have seen previously.

Topic A:

Introduction to Functions Studied This Year—Graphing Stories

N-Q.A.1, N-Q.A.2, N-Q.A.3, A-CED.A.2

Focus Standard:	N-Q.A.1	Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; and choose and interpret the scale and the origin in graphs and data displays.
	N-Q.A.2	Define appropriate quantities for the purpose of descriptive modeling.
	N-Q.A.3	Choose a level of accuracy appropriate to limitations on measurement when reporting quantities.
	A-CED.A.2	Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.
Instructional Days:	5	
Lesson 1:	Graphs of Piecewise Linear Functions (E)[1]	
Lesson 2:	Graphs of Quadratic Functions (E)	
Lesson 3:	Graphs of Exponential Functions (E)	
Lesson 4:	Analyzing Graphs—Water Usage During a Typical Day at School (E)	
Lesson 5:	Two Graphing Stories (E)	

Students explore the main functions that they will work with in Algebra I: linear, quadratic, and exponential. The goal is to introduce students to these functions by having them make graphs of a situation (usually based upon time) in which these functions naturally arise. As they graph, they reason quantitatively and use units to solve problems related to the graphs they create.

For example, in Lesson 3 they watch a 20-second video that shows bacteria subdividing every few seconds. The narrator of the video states these bacteria are actually subdividing every 20 minutes. After counting the initial number of bacteria and analyzing the video, students are asked to create the graph to describe the number of bacteria with respect to actual time (not the sped-up time in the video) and to use the graph to approximate the number of bacteria shown at the end of the video.

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[1] Lesson Structure Key: **P**-Problem Set Lesson, **M**-Modeling Cycle Lesson, **E**-Exploration Lesson, **S-**Socratic Lesson

Another example of quantitative reasoning occurs in Lesson 4. Students are shown a graph (without labels) of the water usage rate of a high school. The rate remains consistent most of the day but jumps every hour for five minutes, supposedly during the bell breaks between classes. As students interpret the graph, they are asked to choose and interpret the scale and decide on the level of accuracy of the measurements needed to capture the behavior in the graph.

The topic ends with a lesson that introduces the next two topics on expressions and equations. Students are asked to graph two stories that intersect in one point on the same coordinate plane. After students and teachers form linear equations to represent both graphs and use those equations to find the intersection point (**8.EE.C.8**), the question is posed to students: How can we use algebra, in general, to solve problems like this one but for non-linear equations? Topics B and C set the stage for students' understanding of the general procedure for solving equations.

Lesson 1: Graphs of Piecewise Linear Functions

Student Outcomes

- Students define appropriate quantities from a situation (a “graphing story”), choose and interpret the scale and the origin for the graph, and graph the piecewise linear function described in the video. They understand the relationship between physical measurements and their representation on a graph.

Classwork

Exploratory Challenge (20 minutes)

Show the first $1:08$ minutes of the video below, telling the class that our goal will simply be to describe in words the motion of the man. (Note: Be sure to stop the video at 1:08 because after that the answers to the graphing questions are given.)

> Elevation vs. Time #2 [http://www.mrmeyer.com/graphingstories1/graphingstories2.mov. This is the second video under “Download Options” at the site http://blog.mrmeyer.com/?p=213 called “Elevation vs. Time #2.”]

MP.1 After viewing the video, have students share out loud their ideas on describing the motion. Some might speak in terms of speed, distance traveled over time, or change of elevation. All approaches are valid. Help students begin to shape their ideas with precise language.

Direct the class to focus on the change of elevation of the man over time and begin to put into words specific details linking elevation with time.

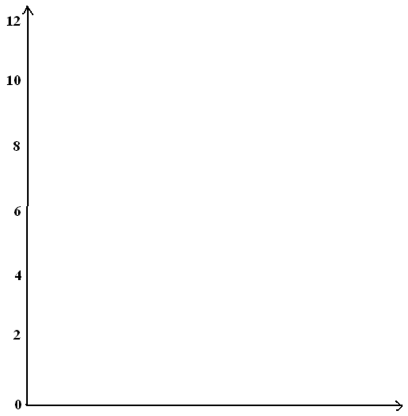
- How high do you think he was at the top of the stairs? How did you estimate that elevation?
- Were there intervals of time when his elevation wasn’t changing? Was he still moving?
- Did his elevation ever increase? When?

Help students discern statements relevant to the chosen variable of elevation.

If students do not naturally do so, suggest representing this information on a graph. As per the discussion that follows, display a set of axes on the board with vertical axis labeled in units relevant to the elevation.

Ask these types of questions:

- How should we label the vertical axis? What unit of measurement should we choose (feet or meters)?
- How should we label the horizontal axis? What unit of measurement should we choose?
- Should we measure the man’s elevation to his feet or to his head on the graph?



- The man starts at the top of the stairs. Where would that be located on the graph?
- Show me with your hand what the general shape of the graph should look like.

MP.6 Give time for students to draw the graph of the story (alone or in pairs). Lead a discussion through the issues of formalizing the diagram: The labels and units of the axes, a title for the graph, the meaning of a point plotted on the graph, a method for finding points to plot on the graph, and so on.

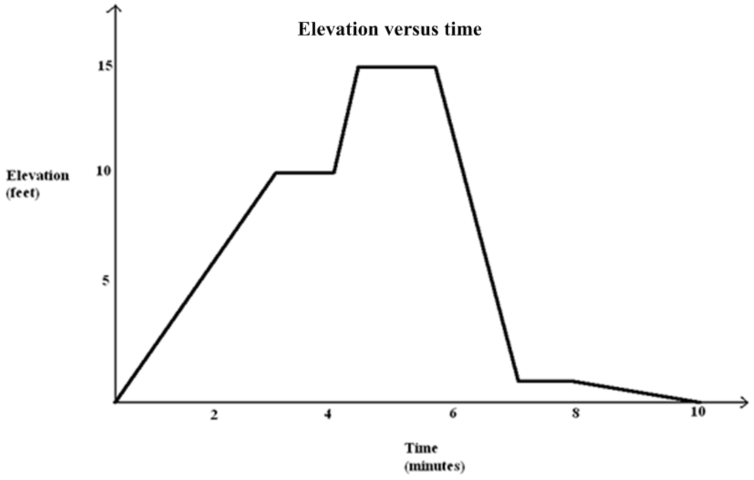
MP.3 Note: The graph shown at the end of the video is incorrect! The man starts at "30 feet above the ground," which is clearly false. You might ask students, "Can you find the error made in the video?"

Example 1 (15 minutes)

Present the following graph and question.

Example 1

Here is an elevation-versus-time graph of a person's motion. Can we describe what the person might have been doing?



Have students discuss this question in pairs or in small groups. It will take some imagination to create a context that matches the shape of the graph, and there will likely be debate.

Additional questions to ask:

- What is happening in the story when the graph is increasing, decreasing, constant over time?
    - *Answers will vary depending on the story: a person is "walking up a hill," etc.*
- What does it mean for one part of the graph to be steeper than another?
    - *The person is climbing or descending faster than in the other part.*
- How does the slope of each line segment relate to the context of the person's elevation?
    - *The slope gives the average change in elevation per minute.*

- Is it reasonable that a person moving up and down a vertical ladder could have produced this elevation versus time graph?
    - *It is unlikely because the speed is too slow:* 2.5 *feet per minute. If the same graph had units in seconds then it would be reasonable.*
- Is it possible for someone walking on a hill to produce this elevation versus time graph <u>and</u> return to her starting point at the 10-minute mark? If it is, describe what the hill might look like.
    - *Yes, the hill could have a long path with a gentle slope that would zigzag back up to the top and then a shorter, slightly steeper path back down to the beginning position.*
- What was the average rate of change of the person's elevation between time 0 minutes and time 4 minutes?
    - $\frac{10}{4}$ *ft/min, or* 2.5 *ft/min.*

These types of questions help students understand that the graph represents only elevation, not speed or horizontal distance from the starting point. This is an important observation.

Closing (5 minutes)

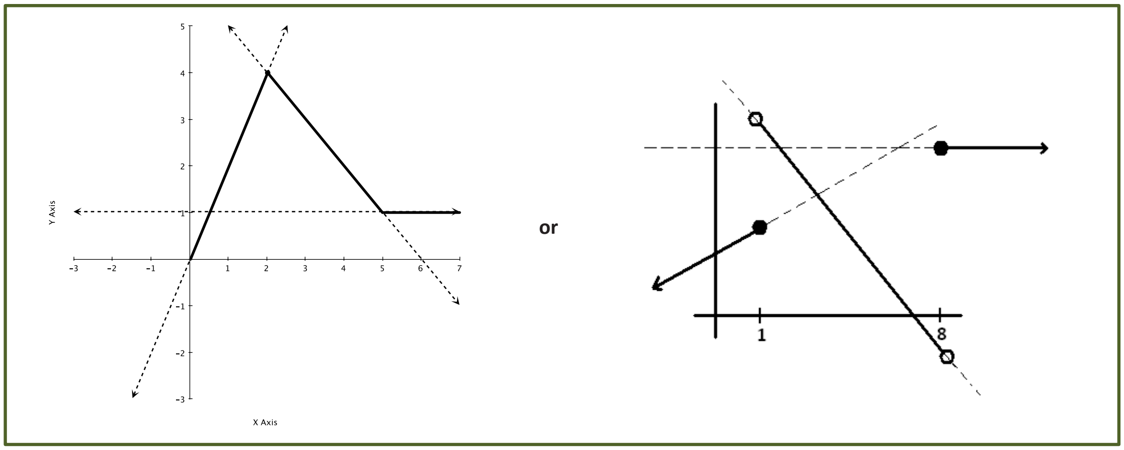
Ask the following:

- How would you describe the graph of Example 1 to a friend?
- What type of equation(s) would be required to create this graph?

Introduce the following definition to your students and discuss briefly. (We will return to this definition later in the year.)

> **<u>Piecewise-Defined Linear Function</u>: Given non-overlapping intervals on the real number line, a *(real) piecewise linear function* is a function from the union of the intervals on the real number line that is defined by (possibly different) linear functions on each interval.**

Point out that all graphs we studied today are graphs of piecewise linear functions. Remind students (see Standard 8.F.A.3) that the graphs of linear functions are straight lines, and show how each segment in one of the graphs studied today is part of a straight line as in:



Also show students the intervals on which each linear function is defined. One may wish to point out there might be ambiguity as to whether or not the endpoints of a given interval belong to that interval. For example, in the first diagram we could argue that three linear functions are defined on the intervals $[0,2)$, $[2,5)$, and $[5,\infty)$, or perhaps on the intervals $[0,2]$, $(2,5)$, and $[5,\infty)$ instead. (Warning: Your students have not been formally introduced to interval notation.) There is no ambiguity in the second example. This point about the interval endpoints is subtle and is not an issue to focus on in a concerted way in this particular lesson.

Exit Ticket (5 minutes)

Name ____ Date______________________

Lesson 1: Graphs of Piecewise Linear Functions

Exit Ticket

The graph in the Exploratory Challenge is made by combining pieces of nine linear functions (it is a piecewise linear function). Each linear function is defined over an interval of time, represented on the horizontal axis. List those nine time intervals.

Exit Ticket Sample Solutions

Students may describe the intervals in words. Do not worry about the endpoints of the intervals in this lesson.

The graph in the Exploratory Challenge is made by combining pieces of nine linear functions (it is a piecewise linear function). Each linear function is defined over an interval of time, represented on the horizontal axis. List those nine time intervals.

Between 0 and 3 seconds;

Between 3 and 5.5 seconds;

Between 5.5 and 7 seconds;

Between 7 and 8.5 seconds;

Between 8.5 and 9 seconds;

Between 9 and 11 seconds;

Between 11 and 12.7 seconds;

Between 12.7 and 13 seconds;

And 13 seconds onwards.

Problem Set Sample Solutions

1. **Watch the video, "Elevation vs. Time #3" (below).**

   http://www.mrmeyer.com/graphingstories1/graphingstories3.mov. (This is the third video under "Download Options" at the site http://blog.mrmeyer.com/?p=213 called "Elevation vs. Time #3.")

   It shows a man climbing down a ladder that is 10 feet high. At time 0 seconds, his shoes are at 10 feet above the floor, and at time 6 seconds, his shoes are at 3 feet. From time 6 seconds to the 8.5 second mark, he drinks some water on the step 3 feet off the ground. After drinking the water, he takes 1.5 seconds to descend to the ground and then he walks into the kitchen. The video ends at the 15 second mark.

   a. **Draw your own graph for this graphing story. Use straight line segments in your graph to model the elevation of the man over different time intervals. Label your x-axis and y-axis appropriately, and give a title for your graph.**

      [See video for one example of a graph of this story.]

   b. **Your picture is an example of a graph of a piecewise linear function. Each linear function is defined over an interval of time, represented on the horizontal axis. List those time intervals.**

      The intervals are $[0,6]$, $(6,8.5]$, $(8.5,10]$, and $(10,15]$, with the understanding that the inclusions of the endpoints may vary. Students may use any notation they want to describe the intervals.

   c. **In your graph in part (a), what does a horizontal line segment represent in the graphing story?**

      It is a period of time when he is neither going up nor down.

   d. **If you measured from the top of the man's head instead (he is 6.2 feet tall), how would your graph change?**

      The whole graph would be shifted up 6.2 feet.

e. Suppose the ladder descends into the basement of the apartment. The top of the ladder is at ground level (0 feet) and the base of the ladder is 10 feet below ground level. How would your graph change in observing the man following the same motion descending the ladder?

The whole graph would be shifted downwards 10 *feet.*

f. What is his average rate of descent between time 0 seconds and time 6 seconds? What was his average rate of descent between time 8.5 seconds and time 10 seconds? Over which interval does he descend faster? Describe how your graph in part (a) can also be used to find the interval during which he is descending fastest.

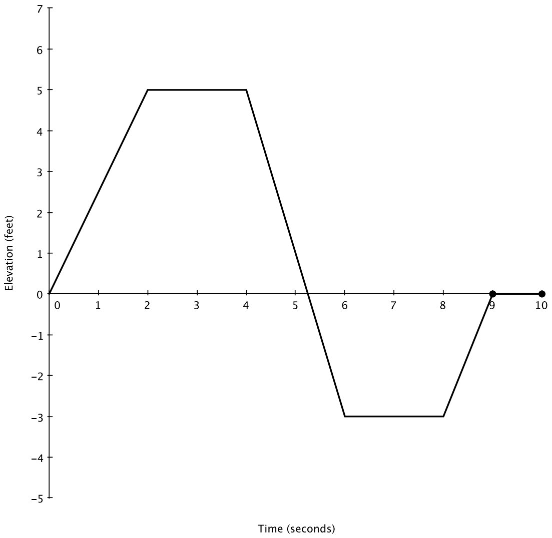
His average rate of descent between 0 *and* 6 *seconds was* $\frac{7}{6}$ *ft/sec.*

His average rate of descent between 8.5 *and* 10 *seconds was* 2 *ft/sec.*

He was descending faster from 8.5 *to* 10 *seconds.*

The interval during which he is descending the fastest corresponds to the line segment with the steepest negative slope.

2. Create an elevation-versus-time graphing story for the following graph:



Answers will vary. A story such as the following fits the graph:

A swimmer climbs a ladder to a waterslide, sits for two seconds at the top of the slide, and then slides down the slide into water. She stays steady at the same position underwater for two seconds before rising to the surface.

Teachers should also accept other contexts, e.g., interpreting "0 *elevation" as the height of a deck* 3 *feet above ground.*

3. Draw an elevation-versus-time graphing story of your own, and then create a story for it.

Answers will vary. Do not be too critical of their graphs and stories.

Lesson 2: Graphs of Quadratic Functions

Student Outcomes

- Students represent graphically a non-linear relationship between two quantities and interpret features of the graph. They will understand the relationship between physical quantities via the graph.

Lesson Notes

Distinctions between $\boldsymbol{h(x)}$ and $\boldsymbol{h(t)}$:

$h(t) = at^2 + bt + c$ is the height h as function of time t; it is not the *actual physical trajectory*. **It has a parabolic "shape or trajectory" in the conceptual $\boldsymbol{t}$-$\boldsymbol{h}$ plane. The $\boldsymbol{t}$-$\boldsymbol{h}$ parabolic trajectory is not directly visible to the human eye.**

$h(x) = ax^2 + bx + c$ is the actual parabolic trajectory in the physical x-h plane. The parabolic trajectory is directly visible to the human eye. This parabolic trajectory is easily confused with the $h(t)$ if we are not careful.

In the special case of projectiles with straight up-and-down vertical motions; e.g., zero horizontal speed,

$h(t)$ is still parabolic when the motion is pure up and down.

$h(x)$ is a "delta function" (only poorly defined at $x = 0$), not a parabola.

In the special case of projectiles with a finite horizontal speed component V_x (throwing a ball at an angle), ***t and x are directly proportional because physics dictates $\boldsymbol{x = V_x \cdot t}$.*** **The two graphs $\boldsymbol{h(t)}$ and $\boldsymbol{h(x)}$ may, therefore, "look similar" to the eyes, but they involve very different concepts.**

Physics of projectiles:

In $h(t) = at^2 + bt + c$, the constants a, b, and c have definite meanings in physics.

- a is $\frac{1}{2}$ of the local gravitational constant. Since g on Earth is 9.8 m/sec^2, or 32 ft/sec^2, the constant a is, numerically, always -4.9 or -16 near Earth's surface, depending on the choice of units for the height. **One does not have the freedom to randomly choose friendly numbers in projectile word problems.**
- b is the initial upward speed.
- c is the initial height above ground.

In $h(x) = ax^2 + bx + c$, the constants a, b, and c also have definite meanings in physics.

- a is $\frac{1}{2}$ of the local gravitational constant divided by the square of the initial horizontal speed.
- b is the initial slope in the x-h plane, or $tan(\theta)$.
- c is the initial height above ground.

Classwork

Example 1 (8 minutes)

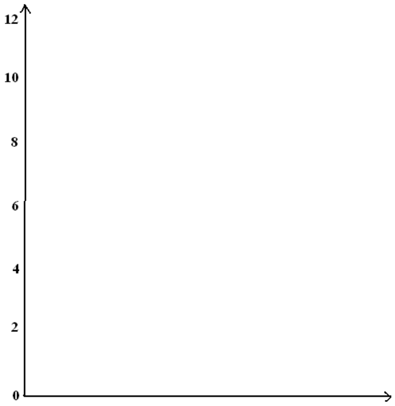
Show the video of a ball rolling down a ramp given at http://youtu.be/xgODzAwxrx8, telling the class that our goal will simply be to describe in words the motion of the ball. (If the link does not work, search for "Algebra I, Module 1 Lesson 2 ball rolling down ramp video.")

After viewing the video, have students share aloud their ideas on describing the motion. Some might speak in terms of speed, distance rolled over time, or change of elevation. All approaches are valid. Help students begin to shape their ideas with language that specifies the names of the quantities being observed and how they are changing over time.

Direct the class to focus on the change of elevation of the ball over time and begin to put into words specific details linking elevation with time.

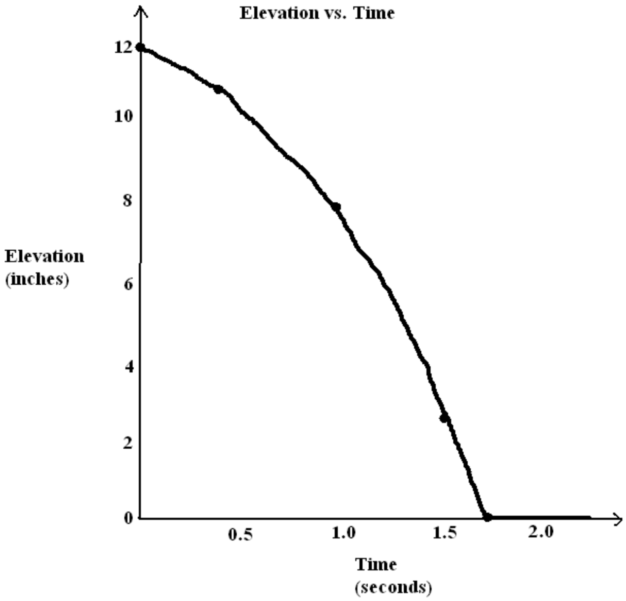
- It first started at the top of the ramp.
- It rolled down the ramp for about 2 seconds.
- After it hit the bottom of the ramp, it rolled across the floor.

If students do not naturally do so, suggest representing this information on a graph. Display a set of axes on the board.



Begin a discussion that leads students through issues of formalizing the diagram: The labels and units of the axes, a title for the graph, the meaning of a point plotted on the graph, the height at time 0 seconds, the time when it reached the bottom of the ramp, etc.

Either individually or in groups of two, have students estimate the general shape of the graph of the elevation-versus-time of the ball rolling down the ramp. At this point, do not worry about engaging students in measuring specific heights and specific times; right now they are just looking for the general shape. Be sure to have students notice the horizontal "tail" of the graph and ask them to interpret its meaning. (The elevation of the ball does not change as it rolls across the floor.) Some students may draw a curved graph. Others will likely draw a straight line for the graph over the interval 0 to 1.7 seconds. Even though this is not correct, allow it at this stage.



After students have created graphs, have them compare their graphs with each other and share with the class. Ask the following questions:

- Should the change in elevation be decreasing at a constant rate? That is, do you think this graph should be a straight line-graph, at least between 0 and about 1.7 seconds?

Some specific questions (below) can help lead to the correct conclusion:

- Where does the elevation change more slowly? Explain.
- If it is changing more slowly at the top and more quickly at the bottom, should the graph look the same at those times?

In Exploratory Challenge, students will see an elevation-versus-time graph that is clearly not a straight line.

Exploratory Challenge (25 minutes)

MP.4

Have students plot a graphical representation of change in elevation over time for the following "graphing story." It is a video of a man jumping from 36 feet above ground into 1 foot of water.

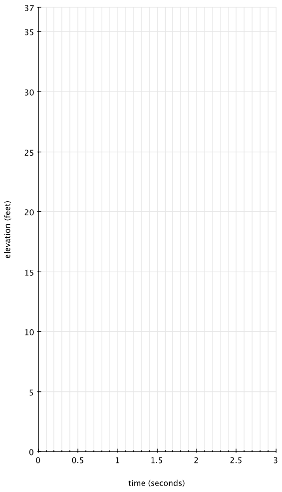
https://www.youtube.com/watch?v=ZCFBC8aXz-g or http://youtu.be/ZCFBC8aXz-g (If neither link works, search for "OFFICIAL Professor Splash World Record Video!")

Exploratory Challenge

Plot a graphical representation of change in elevation over time for the following "graphing story." It is a video of a man jumping from 36 feet above ground into 1 foot of water.

http://www.youtube.com/watch?v=ZCFBC8aXz-g or

http://youtu.be/ZCFBC8aXz-g (If neither link works, search for "OFFICIAL Professor Splash World Record Video!")

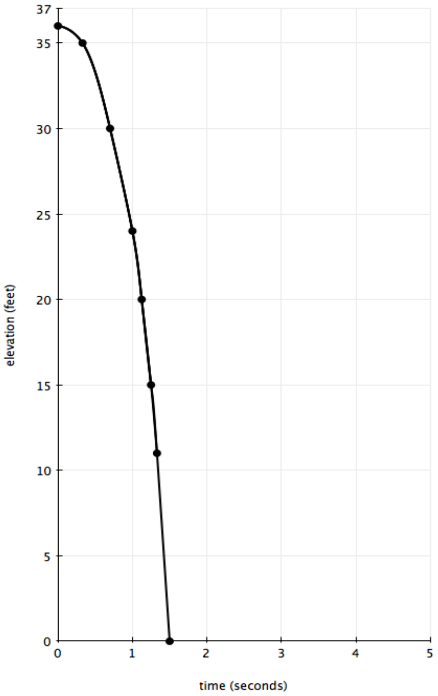


Allow students to analyze the full video initially giving them as much control as possible (if not full control) in choosing when to pause a video, or play it half speed, and so on. Perhaps conduct this part of a class in a computer lab or display the video on an interactive Smart Board. Then direct students to investigate one of the two portions of the clip below, depending on the level of technology you have available:

Portion 1: From time 32 seconds to time 33 seconds. Use this to get data for drawing the graph **if they have access to a powerful movie editor that can show each frame.**

Portion 2: From time 48 seconds to time 53 seconds. Use this for a slow-motion version that can be analyzed easily via YouTube.

Teacher Note: In this lesson, keep students' focus on understanding the relationship between physical measurements and the graph. Students will revisit scenarios like this one in Modules 4 and 5. We know, from physics, that the height function for this example can be modeled by $s(t) = 36 - 16t^2$ and that in this case, the function models the situation very accurately. For example, the function predicts that he will hit the water at exactly 1.5 seconds, which he does. However, quadratic equations and functions are topics for later modules.



Students should start measuring from the top of the man's jump, which is about 36 feet high. (He starts at 35.5 feet and jumps $\frac{1}{2}$ a foot up before falling.)

Solutions are provided below for each portion.

MP.1

Give students the opportunity to share their work with the class as a whole or with a partner. Have students articulate and justify any interesting choices they have made. For example, instead of plotting points for every second of footage, some students may wish to plot the time per distance markers $(36, 30, 25, 20, 15$, etc.$)$. Both methods will lead to a graph of a quadratic function.

Time (sec.)	Elevation (ft.)
0	36
0.33	35
0.7	30
1	24
1.125	20
1.25	15
1.33	11
1.5	0

Students should produce graphs that look similar to the following, according to which clip they used:

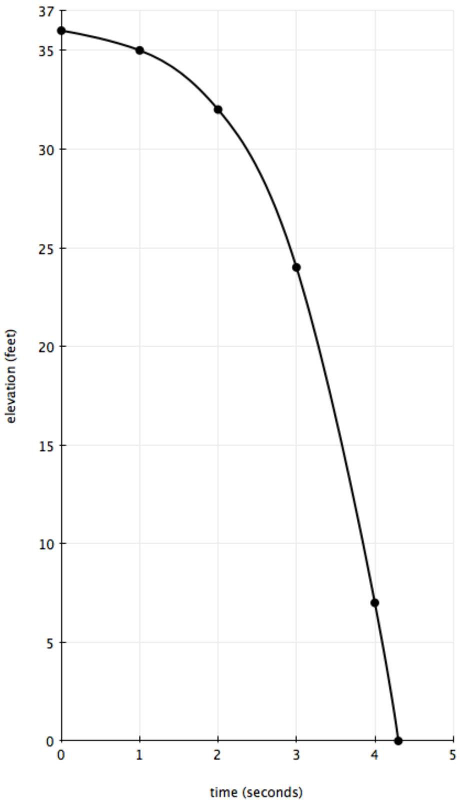
Portion 1 Results: If your students used Portion 1, they should get something that looks similar to the table (left) and graph (above).

The estimates of the heights in the table are based upon the markers in the video. Since the video is taken from the ground, reading height directly from the video can be hampered by the parallax effect. Ideally, for accurate reading, the camera should be at the same height as the jumper AT ALL TIMES so the height reading from the ruler in the background could be accurate. Remind the students to make some necessary corrections or adjustments instead of just *reading horizontally across the frozen video frame*. Regardless, you should not expect your students to have "perfect" graphs, but their graphs should NOT be straight lines.

Portion 2 Results: If your students used Portion 2 of the tape, they should get something that looks similar to the table (below) and graph (right). Have a discussion with them that the results below do not model the actual situation. By specifically talking about film time versus real time, we lay the foundation to talk about transformations of functions. Guide students by asking these questions:

- What would you need to do to this graph to correctly model the jump with a graph?
    - *You need to know how long it took for him to complete the jump.*
- It takes the jumper exactly 1.5 seconds from the top of his jump until he splashes in the water.
- Then we should shrink the graph horizontally until the x-intercept is at 1.5 seconds.
- Learning how to "shrink" graphs in this way will be an important topic in Modules 3 and 5.

As an extension, encourage students to try adjusting the graph in this way (shrinking it horizontally such that the x-intercept is at 1.5 seconds), using their intuition to guide them, and compare their results with their peers.



Time (sec.)	Elevation (ft.)
0	36
1	35.5
2	32
3	24
4	7
4.3	0

Example 2 (5 minutes)

Example 2

The table below gives the area of a square with sides of whole number lengths. Have students plot the points in the table on a graph and draw the curve that goes through the points.

Side **(cm)**	**0**	**1**	**2**	**3**	**4**
Area **(cm^2)**	**0**	**1**	**4**	**9**	**16**

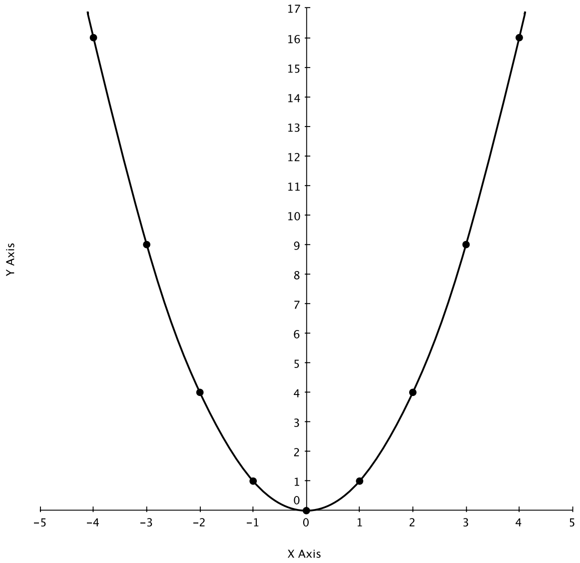
On the same graph, reflect the curve across the y-axis. This graph is an example of a "graph of a quadratic function."

Bring up or ask:

- On the graph, what do the points between the plotted points from the table represent?
  - *Areas of squares with non-whole number side lengths.*

Closing (3 minutes)

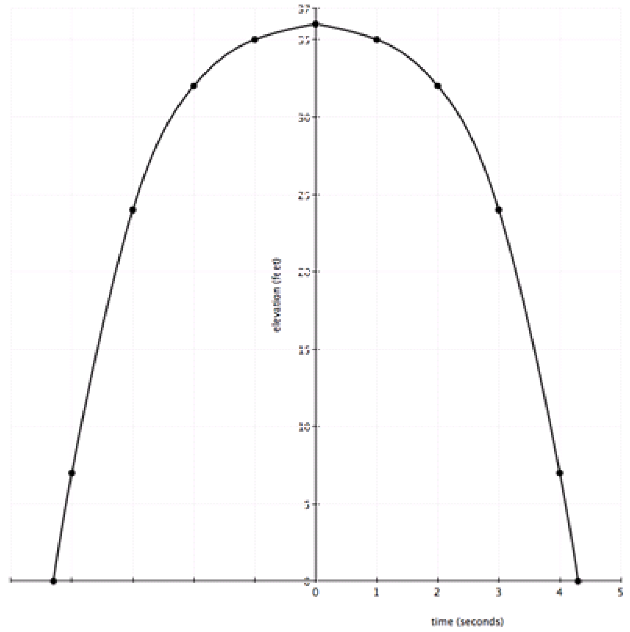
Ask students to use their 8th grade geometry background to reflect the graph from Example 2 across the y-axis on the same graph they just drew (8.G.A.3). They should get a picture like this:



Mention to them that this is an example of a "graph of a quadratic function." Students have seen functions of the form $A = s^2$ in their eighth-grade studies of area as an example of a function that is not linear (8.F.A.3). Tell them that they will be studying these functions throughout the year and that they will learn different ways to recognize and represent quadratic functions and equations.

Extension: Show students that the graph in the Exploratory Challenge is also *part of* a graph of a quadratic function by plotting the reflection across the y-axis (e.g., points with negative x-values).

MP.4



Bring up or ask:

- Does this extended graph I have just drawn from the Exploratory Challenge model the elevation of the man jumping into the shallow pool?
    - *No, the part of the graph before time* 0 *was not good. He went up to the platform very slowly compared to how fast he came down.*

If time allows, have students draw a graph of the man's elevation versus time that incorporates his climbing the ladder. Bring up or ask:

- Can you imagine an elevation versus time situation where the graph would look like the extended graph?
    - *Yes, my elevation when I jump up into the air*

If you need to, show them or have a student demonstrate the answer to this question. Finally, ask:

- Does the reflection of the graph along the y-axis in Example 2 make sense?
    - *No, squares cannot have negative side lengths.*

Exit Ticket (4 minutes)

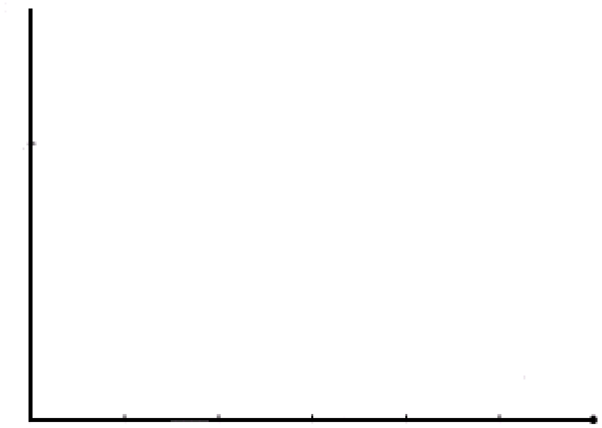
Name ____ Date____________________

Lesson 2: Graphs of Quadratic Functions

Exit Ticket

If you jumped in the air three times, what might the elevation versus time graph of that story look like?

Label the axes appropriately.



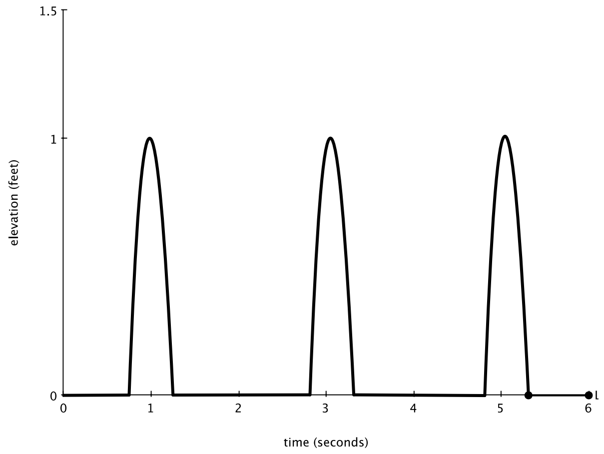
Exit Ticket Sample Solution

If you jumped in the air three times, what might the elevation versus time graph of that story look like?

Label the axes appropriately.

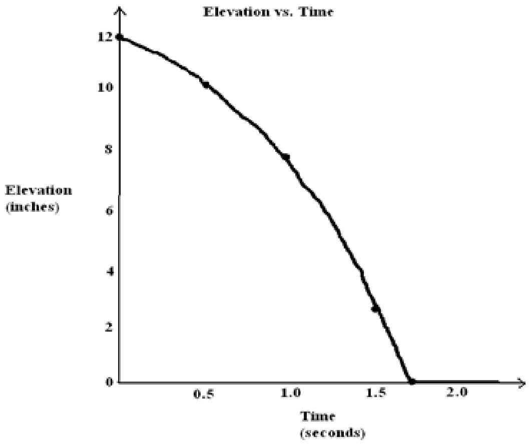
Student answers will vary.

I jump about 1 foot high every other second.



Problem Set Sample Solutions

1. **Here is an elevation versus time graph of a ball rolling down a ramp. The first section of the graph is slightly curved.**



   a. **From the time of about 1.7 seconds onwards, the graph is a flat horizontal line. If Ken puts his foot on the ball at time 2 seconds to stop the ball from rolling, how will this graph of elevation versus time change?**

   Even if the ball is at rest on the floor, its elevation remains 0 inches and does not change. The elevation versus time graph does not change.

b. Estimate the number of inches of change in elevation of the ball from 0 seconds to 0.5 seconds. Also estimate the change in elevation of the ball between 1.0 seconds and 1.5 seconds.

Between 0 and 0.5 seconds, the change in elevation was about -2 inches. Between 1.0 and 1.5 seconds, the change in elevation was about -5.5 inches.

c. At what point is the speed of the ball the fastest, near the top of the ramp at the beginning of its journey or near the bottom of the ramp? How does your answer to part (b) support what you say?

The speed of the ball is the fastest near the bottom of the ramp. During the half-second from 1.0 second to 1.5 seconds, the ball's change in elevation is greater than the half-second at the beginning of its journey. It must have traversed a greater length of the ramp during this half-second and so was traveling faster. Its speed is greater near the bottom of the ramp.

2. Watch the following graphing story:

Elevation vs. Time #4 (http://www.mrmeyer.com/graphingstories1/graphingstories4.mov. This is the second video under "Download Options" at the site http://blog.mrmeyer.com/?p=213 called "Elevation vs. Time #4.")

The video is of a man hopping up and down several times at three different heights (first, five medium-sized jumps immediately followed by three large jumps, a slight pause, and then 11 very quick small jumps).

a. What object in the video can be used to estimate the height of the man's jump? What is your estimate of the object's height?

The stair step (Answers will vary between 4 inches to 8 inches.)

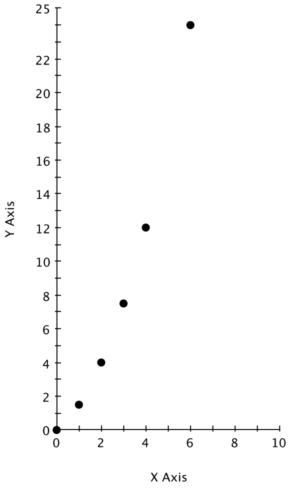
b. Draw your own graph for this graphing story. Use parts of graphs of quadratic functions to model each of the man's hops. Label your x-axis and y-axis appropriately and give a title for your graph.

Answers will vary but should reflect the estimate made in part (a). See the end of the video for a picture of the graph.

3. Use the table below to answer the following questions.

x	0	1	2	3	4	5	6
y	0	3/2	4	15/2	12		24

a. Plot the points (x, y) in this table on a graph (except when x is 5).



b. **The y-values in the table follow a regular pattern that can be discovered by computing the differences of consecutive y-values. Find the pattern and use it to find the y-value when x is 5.**

The y-values have differences that increase by one, suggesting that we next have a y-value of $12 + \frac{11}{2} = \frac{35}{2}$ when x is 5.

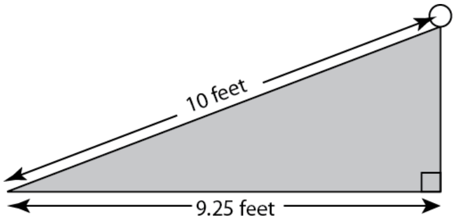
c. **Plot the point you found in part (b). Draw a curve through the points in your graph. Does the graph go through the point you plotted?**

Yes.

d. **How is this graph similar to the graphs you drew in Examples 1 and 2 and the Exploratory Challenge? Different?**

Answers will vary. This graph is similar to the graphs from Examples 1 and 2the Exploratory Challenge in that the curve has the same basic U-shape. (Students may mention that they are all graphs of quadratic functions, although at this point that is not necessary.) This graph is different from the graphs from Example 1 and the Exploratory Challenge because this curve is increasing and the graphs in Example 1 and the Exploratory Challenge are decreasing. This graph is similar to the graph from Example 2 because they are both increasing. However, this graph increases at a slower rate than the one from Example 2. For example this graph has value 12 when x is 4 (the Example 2 graph had value 16), and this graph has value 24 when x is 6 (the Example 2 graph had value 36), and so on.

4. **A ramp is made in the shape of a right triangle using the dimensions described in the picture below. The ramp length is 10 feet from the top of the ramp to the bottom, and the horizontal width of the ramp is 9.25 feet.**



A ball is released at the top of the ramp and takes 1.6 seconds to roll from the top of the ramp to the bottom. Find each answer below to the nearest 0.1 feet/sec.

a. **Find the average speed of the ball over the 1.6 seconds.**

$\frac{10}{1.6}$ ft/sec, or 6.3 ft/sec

b. **Find the average rate of horizontal change of the ball over the 1.6 seconds.**

$\frac{9.25}{1.6}$ ft/sec or 5.8 ft/sec

c. **Find the average rate of vertical change of the ball over the 1.6 seconds.**

By the Pythagorean Theorem, the vertical length is approximately $\sqrt{10^2 - 9.25^2} \approx 3.8$ feet. Hence, the average rate of vertical change is $\frac{3.8}{1.6}$ ft/sec or 2.4 ft/sec

d. **What relationship do you think holds for the values of the three average speeds you found in parts (a), (b), and (c)? (Hint: Use the Pythagorean Theorem.)**

The sum of the squares of the horizontal and vertical rates of change is equal to the square of the speed of the ball.

Lesson 3: Graphs of Exponential Functions

Student Outcomes

- Students choose and interpret the scale on a graph to appropriately represent an exponential function. Students plot points representing the number of bacteria over time, given that bacteria grow by a constant factor over evenly spaced time intervals.

Classwork

Example 1 (10 minutes)

MP.4 Have students sketch a graph that depicts Darryl's change in elevation over time.

Example 1

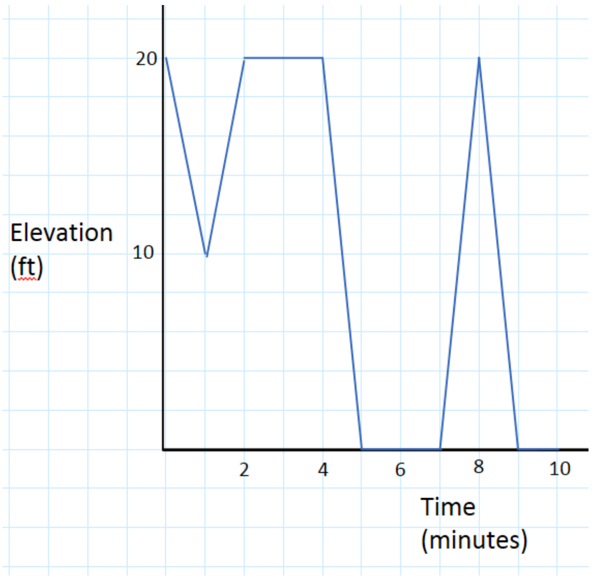
Consider the story:

> ***Darryl lives on the third floor of his apartment building. His bike is locked up outside on the ground floor. At 3:00 p.m., he leaves to go run errands, but as he is walking down the stairs, he realizes he forgot his wallet. He goes back up the stairs to get it and then leaves again. As he tries to unlock his bike, he realizes that he forgot his keys. One last time, he goes back up the stairs to get his keys. He then unlocks his bike, and he is on his way at 3:10 p.m.***

Sketch a graph that depicts Darryl's change in elevation over time.

There will be variations in the students' graphs, but the graph students produce should appear as follows:

Elevation vs Time



Exploratory Challenge (25 minutes)

In this example, students will plot a graph for the following (exponential) graphing story:

https://www.youtube.com/watch?v=gEwzDydciWc

If the link does not work, google "Bacteria Growth video." The video shows bacteria doubling every second. Later in the video, the narrator says, "Just one bacterium, dividing every 20 minutes…" This video is sped up so that 1 second corresponds to 20 minutes. The goal is for students to initially graph the number of bacteria versus time in seconds (a graph that corresponds to the time-lapse video), and later graph the number of bacteria versus time in minutes (a graph that corresponds to how the bacteria grew in real time).

Exploratory Challenge

Watch the following graphing story:

https://www.youtube.com/watch?v=gEwzDydciWc

The video shows bacteria doubling every second.

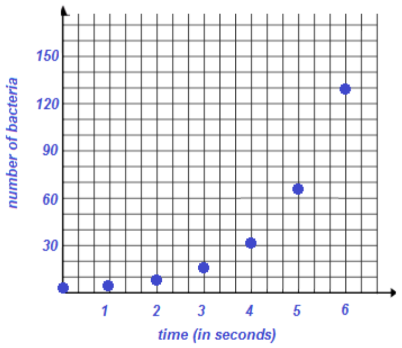
Get students started by creating a table of the first few seconds. Stop the video as close as possible to the 0-second mark, 1-second mark, 2-second mark, 3-second mark, and so on. Have them count the bacteria at each of those times. Note that some of the bacteria have not fully divided, so expect a discussion on how to count them (guide them to make the choice: approximately the length of a small bacterium). Do not suggest any model just yet as there are some great number sense problems to explore first. Below is a fairly accurate account of the number of bacteria up to the 3-second mark:

Time (sec)	0	1	2	3
Number of bacteria	2	4	8	16

The 4-second mark is already tricky to count (~31 or 32), as well as at the 5-second mark (~58–62). Ask students what they might expect to see at time 6 seconds (~116–130). Do not have students count the bacteria at time 6 seconds but instead estimate the number (e.g., the radius of the circular region of bacteria doubles roughly every second).

Students might first produce a graph similar to this:

1. **Graph the number of bacteria versus time in seconds. Begin by counting the number of bacteria present at each second and plotting the appropriate points on the set of axes below. Consider how you might handle estimating these counts as the population of the bacteria grows.**



Ask for, or eventually suggest, the following model to describe the bacteria shown in the video:

- At time 0 seconds, there are 2 bacteria, and the number of bacteria in the Petri dish doubles every second. Stress the point to your students that this model does not describe the *exact* number of bacteria at a given time in the video but that it does reasonably represent an estimate of the number of bacteria at a given time.

MP.6 Ask students to create a graph that represents bacterial population growth beyond the 6-second mark using this model. Two challenges will quickly arise for students: Identifying an appropriate scale for the vertical axis and dealing with the extraordinarily large numbers that arise from this mark onward. Suggest that students first describe the shape of the graph after the 6-second mark. To help with larger values, a table can be expanded beyond the 6-second mark using scientific notation.

A good question to ask:

- Will the curve ever be perfectly vertical?

To develop the real-time exponential growth of the bacteria, state:

- Listen to what the narrator says about the situation, "Just one bacterium, dividing every 20 minutes…"

Then ask:

- If one bacterium divides once every 20 minutes, how much "real time" is passed in what we watched in the time-lapse video?

To help students answer this question, return to the beginning of the video, with two bacteria, and notice how many bacteria there are after 1 second. There are four. And at the 2-second mark, there are eight. Have them conclude that one second of "video time" matches about 20 minutes of "real time."

- We first created a graph with the unit scale for the horizontal axis as seconds. This time scale is according to the video's clock. How could we change the scale of the horizontal axis so that it represents the real time of bacteria growth?

Students will likely suggest "crossing out the marks 1 second, 2 seconds, 3 seconds, …." on the horizontal axis and replacing them with the numbers "20 minutes, 40 minutes, 60 minutes, …." While this is acceptable, suggest being careful and deliberate by drawing a table of values akin to the following:

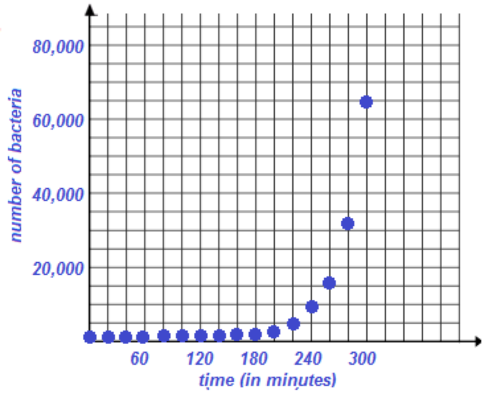
Original time (in seconds)	New time (in minutes)	Number of bacteria
0	0	2
1	20	4
2	40	8
3	60	16
4	80	32

Use this table as a discussion point for justifying the change of scale for the horizontal axis.

Have them sketch a graph of the count of bacteria versus (real) time in minutes for the first five-hour period. Lead students in a discussion to decide on appropriate scales for the vertical and horizontal axes. If needed, encourage the students to extend their tables to determine the number of bacteria at the end of the five-hour period before they decide on the scale. The students' graphs do not need to match exactly the sample provided below but should accurately depict the points over the first 300 minutes.

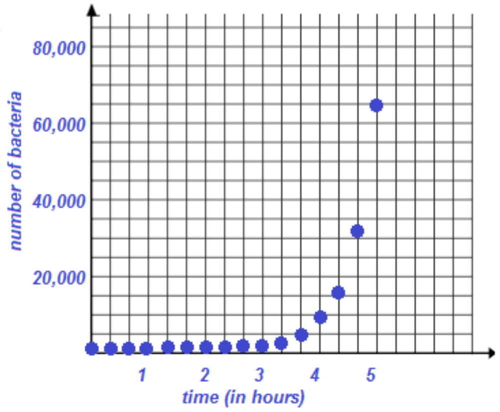
Students may instinctively connect the points with a smooth curve. It is acceptable to use the curve as a model of the reality of the discrete points on the graph; however, encourage students to recognize the difference between the points supported by the context and a continuous curve that is a model of the situation.

2. **Graph the number of bacteria versus time in minutes.**



Now, have students redraw this graph with the unit scale of the horizontal axis as hours.

3. **Graph the number of bacteria versus time in hours (for the first five hours).**



Closing (5 minutes)

Tell students that the graph of the count of bacteria is an example of a "graph of an exponential function." Have students share the differences between linear, quadratic, and exponential graphs, first with a partner and then as a class. Ask students to share any ideas of how to differentiate between the graph of a quadratic function and an exponential function. Let them brainstorm and do not expect any well-formed statements about how to differentiate; *one of the goals of the year is to understand these issues*. To summarize, tell your students the following:

- The three types of graphs (linear, quadratic, and exponential) we have looked at over the past few days are the "pictures" of the main types of equations and functions we will be studying throughout this year. One of our main goals for the year is to be able to recognize linear, quadratic, and exponential relationships in real-life situations and develop a solid understanding of these functions to model those real-life situations.

Exit ticket (5 minutes)

Name ____ Date____________________

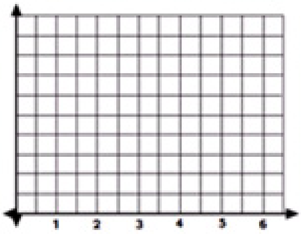
Lesson 3: Graphs of Exponential Functions

Exit Ticket

Assume that a bacteria population doubles every hour. Which of the following three tables of data, with x representing time in hours and y the count of bacteria, could represent the bacteria population with respect to time? For the chosen table of data, plot the graph of that data. Label the axes appropriately with units.

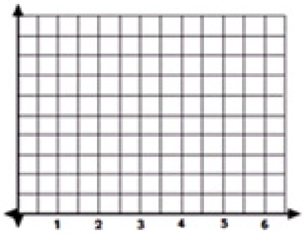
a)

x	0	1	2	3	4	5	6
y	4	7	10	13	16	19	22



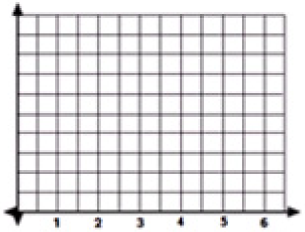
b)

x	0	1	2	3	4	5	6
y	3	6	12	24	48	96	192



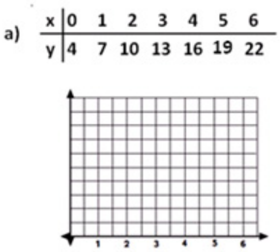
c)

x	0	1	2	3	4	5	6
y	1	3	7	13	21	31	43



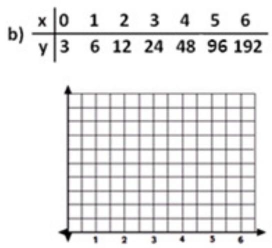
Exit Ticket Sample Solution

Assume that a bacteria population doubles every hour. Which of the following three tables of data, with x representing time in hours and y the count of bacteria, could represent the bacteria population with respect to time? For the chosen table of data, plot the graph of that data. Label the axes appropriately with units.



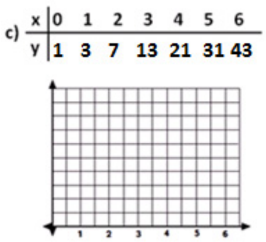
a)

x	0	1	2	3	4	5	6
y	4	7	10	13	16	19	22



b)

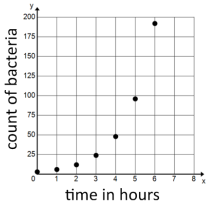
x	0	1	2	3	4	5	6
y	3	6	12	24	48	96	192



c)

x	0	1	2	3	4	5	6
y	1	3	7	13	21	31	43

The answer is b.



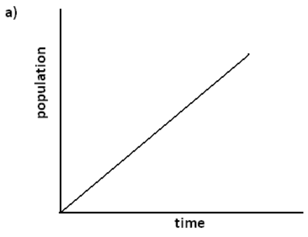
Problem Set Sample Solutions

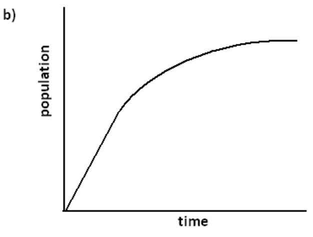
1. **Below are three stories about the population of a city over a period of time and four population-versus-time graphs. <u>Two</u> of the stories each correspond to a graph. Match the two graphs and the two stories. Write stories for the other two graphs, and draw a graph that matches the third story.**

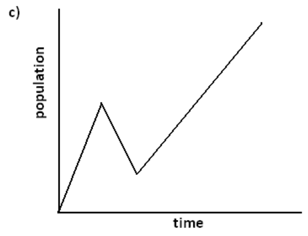
   Story 1: The population size grows at a constant rate for some time, then doesn't change for a while, and then grows at a constant rate once again.

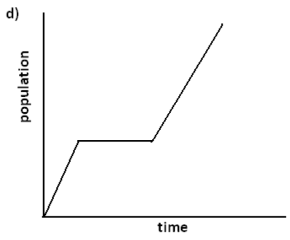
   Story 2: The population size grows somewhat fast at first, and then the rate of growth slows.

   Story 3: The population size declines to zero.









Story 1 corresponds to (d), and Story 2 corresponds to (b). For Story 3 answers will vary. The graph can begin at any positive population value and decrease to 0 in any manner.

Sample story for (a): The population starts out at 0 and grows at a constant rate.

Sample story for (c): The population size grows at a constant linear rate for some time, then decreases at a constant linear rate for a while, then increases at a constant linear rate slower than the original rate of increase.

2. In the video, the narrator says:

   "Just one bacterium, dividing every 20 minutes, could produce nearly $5,000$ billion billion bacteria in one day. That is $5,000,000,000,000,000,000,000$ bacteria."

   This seems WAY too big. Could this be correct, or did she make a mistake? (Feel free to experiment with numbers using a calculator.)

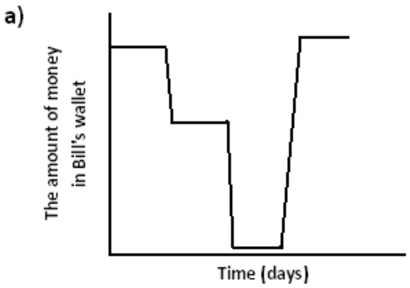
   Yes, this is correct. Do not be too critical of justifications. Accept any explanation that reasonably explains why she is correct.

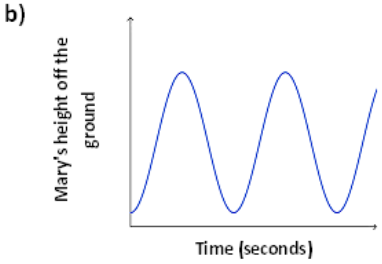
   For the teacher: The function $f(t) = 2^{(t/20)}$ models the number of bacteria after t minutes starting from a single bacterium. There are $1,440$ minutes in a 24-hour period, which means that since the bacteria divide every 20 minutes, the count of the bacteria will double 72 times during a 24-hour period. Thus, the answer is $f(1440) = 2^{72}$, which is approximately 4.72×10^{21}. This number is nearly $5,000$ billion billion bacteria.

3. *Bacillus cereus* is a soil-dwelling bacterium that sometimes causes food poisoning. Each cell divides to form two new cells every 30 minutes. If a culture starts out with exactly 100 bacterial cells, how many bacteria will be present after 3 hours?

   The result is $2^6(100) = 6400$ bacteria. Students can do this by repeated multiplication, without much knowledge of exponential functions.

4. Create a story to match each graph below:





Answers will vary.

Sample story for (a): Bill received his paycheck and did not touch it for a few days. Then, he bought groceries and gas and stopped spending money. After a few more days, he spent almost all of his remaining money on a new jet ski. A few days later he received his paycheck.

Sample story for (b): Mary is riding a Ferris wheel at a theme park.

5. **Consider the following story about skydiving:**

Julie gets into an airplane and waits on the tarmac for 2 minutes before it takes off. The airplane climbs to $10,000$ feet over the next 15 minutes. After 2 minutes at that constant elevation, Julie jumps from the plane and free falls for 45 seconds until she reaches a height of $5,000$ feet. Deploying her chute, she slowly glides back to Earth over the next 7 minutes where she lands gently on the ground.

   a. **Draw an elevation versus time graph to represent Julie's elevation with respect to time.**

   The graph should depict the horizontal line $y = 0$ from 0 minutes to 2 minutes, a linear increase to $10,000$ feet from 2 minutes to 17 minutes, a brief free fall (concave down) curve for 45 seconds, and then a leveled out concave up curve for the slow fall at the end.

   b. **According to your graph, describe the manner in which the plane climbed to its elevation of $10,000$ feet.**

   Answers will vary depending on the graph. Example: I assumed that the plane climbed at a constant rate.

   c. **What assumption(s) did you make about falling after she opened the parachute?**

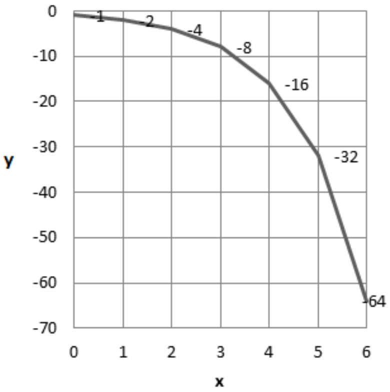
   Answers may vary. Example: I assumed that her change in elevation slowed down suddenly when she opened her parachute and then was an almost constant rate of change until she reached the ground.

6. **Draw a graph of the number of bacteria versus time for the following story: Dave is doing an experiment with a type of bacteria that he assumes divides in half exactly every 30 minutes. He begins at 8:00 a.m. with 10 bacteria in a Petri dish and waits for 3 hours. At 11:00 a.m., he decides this is too large of a sample and adds Chemical A to the dish, which kills half of the bacteria almost immediately. The remaining bacteria continue to grow in the same way. At noon, he adds Chemical B to observe its effects. After observing the bacteria for two more hours, he observes that Chemical B seems to have cut the growth rate in half.**

Answers vary somewhat, but the graph should include the information in the table below connected by some smooth curve. There should be a sudden decrease from 640 to 320 after time 3 (i.e., 3 hours), say around time 3.01 or sooner, giving a nearly vertical line.

Time (hours)	*Number of Bacteria*
0	10
1	40
2	160
3	640
≤ 3.01	320
4	1280
5	2560
6	5120

7. Decide how to label the vertical axis so that you can graph the data set on the axes below. Graph the data set and draw a curve through the data points.



x	y
0	-1
1	-2
2	-4
3	-8
4	-16
5	-32
6	-64

(A sample graph is shown above.)

Lesson 4: Analyzing Graphs—Water Usage During a Typical Day at School

Student Outcomes

- Students develop the tools necessary to discern units for quantities in real-world situations and choose levels of accuracy appropriate to limitations on measurement. They refine their skills in interpreting the meaning of features appearing in graphs.

Classwork

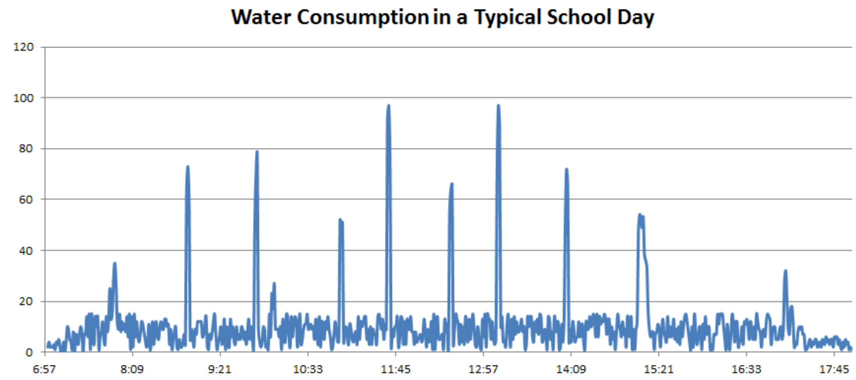
Exploratory Challenge (27 minutes)

Example 1

Direct students to the following graph that appears in their student materials, and ask questions about the kind of data it represents.

- The title of the graph is "Water Consumption in a Typical School Day." For what purposes is water used at a school?
  - *Primarily through bathroom use.*
- What do you think the numbers along the horizontal axis represent?
  - *Time of day.*
- What might the numbers along the vertical axis represent? Do we have any indication of the units being used?
  - *Answers will vary. We have no indication of the units being used. We would assume that these numbers are related, in some way, to a volume of water.*
- What could be the reason for the "spikes" in the graph?
  - *Student bathroom use between class periods.*
- What might be the reason for the smaller spikes between the large ones?
  - *Some student use during class time.*

Example 1



Now, offer the following further information about the typical school day for the school from which this data was recorded.

- Regular school day hours: 8:00 a.m.–3:04 p.m.
- After school activities: 3:15 p.m. –5:15 p.m.
- Around 10:00 a.m. there is a 13 minute advisory or homeroom period.

Ask the students:

- Can you see features of this information appearing in the graph?

Be sure students notice the large peaks at the times just before 8:00 a.m., near 10:00 a.m., just after 3:00 p.m., and near 5:15 p.m., which can be explained by bathroom use just before and just after activities.

Ask the students:

- Is it possible to deduce the time of lunch at this school?

Students might conjecture that lunch comes in two or three shifts so as to explain the multiple large peaks in the middle of the day.

Now, return to the question of the numbers on the vertical axis:

- Around 10:00 a.m. the graph indicates a peak of 80 units. What is the number 80 representing?

Students will likely answer “gallons of water used” or “volume of water used.” But what do we mean by “used”? Lead to the idea that the amount “used” is measured by the volume of water that drains through the pipes and leaves the school.

Then ask the question:

- How does one actually measure the amount of water flowing out through the pipes precisely at 10:00 a.m.? What does “80 units of water leaving the school” right at 10:00 a.m. mean?
    - *The issue to be discussed next is that 80 units of water must take some period of time to flow out through the pipes.*

Ask students to think about the bell for the end of class. At this time, they will all rise and begin to walk out of the classroom, "flowing" through the door of the classroom. If there are 25 students in the class, would I say that "a volume of 25 students is flowing through the door" right at an instant? No, we have instead a "flow of 25 students" over a short period of time.

Ask students to consider again what the number 80 at the 10:00 a.m. mark might mean.

Now add the following information:

- The researchers who collected this data watched the school's water meter during a 12-hour period. The meter shows the total amount of water (in gallons) that has left the school since the time the meter was last set to zero. Since the researchers did not know when this resetting last occurred, they decided, at each minute mark during the day, to measure how much the meter reading increased over the next minute of time. Thus, the value "80" at the $10{:}00$ a.m. mark on the graph means that 80 gallons of water flowed through the meter and thus left the school during the period from 10:00 a.m. to 10:01 a.m.

MP.2

Ask the students:

- What are the units for the numbers on the vertical axis?
  - *The units for the numbers on the vertical axis are gallons per minute.*
- Ignoring the large spikes in the graph, what seems to be the typical range of values for water use during the school day?

Help students notice that the value of the graph between the large spikes seems to oscillate between the flow of 0 gallons of water per minute and about 15 gallons of water per minute.

MP.1

Now have students complete Exercises 1 and 2 independently and then work in pairs or in small groups to discuss approaches and compare answers. Ask students to volunteer their answers to a general class discussion. Discuss any assumptions that were made to arrive at answers.

Exercises 1–2

Exercises

1. **The bulk of water usage is due to the flushing of toilets. Each flush uses 2.5 gallons of water. Samson estimates that 2% of the school population uses the bathroom between 10:00 a.m. and 10:01 a.m. right before homeroom. What is a good estimate of the population of the school?**

   We know that 80 gallons of water are used during that minute. This corresponds to $\frac{80}{2.5} = 32$ flushes. Assuming that each student flushes just once, we can say that 32 students used the bathroom during this time. This represents 2% of the school population, which is one fiftieth of the school population. Thus, there are about $1,600$ people at the school.

2. **Samson then wonders this: If everyone at the school flushed a toilet at the same time, how much water would go down the drain (if the water-pressure of the system allowed)? Are we able to find an answer for Samson?**

   Answers will vary based on assumptions made. If there were enough toilets that everyone could flush a toilet at the same time, and using the estimated school population of $1,600$ (there would have to be $1,600$ toilets), then $1,600 \times 2.5$ gallons $= 4,000$ gallons of water.

Exercise 3 (15 minutes): Estimation Exercise

MP.3 Have students work in pairs to draw a graph of the water usage at their school based on their answers to the following questions:

3. Estimation Exercise

- **Make a guess as to how many toilets are at the school.**

  80

- **Make a guess as to how many students are in the school, and what percentage of students might be using the bathroom at break times between classes, just before the start of school, and just after the end of school. Are there enough toilets for the count of students wishing to use them?**

  Of $1,600$ students, at any given break time, 20% are using the restroom.

- **Using the previous two considerations, estimate the number of students using the bathroom during the peak minute of each break.**

  320 students are using the bathroom during a break, but maybe only one fourth of them at any one minute, so 80 at any one minute.

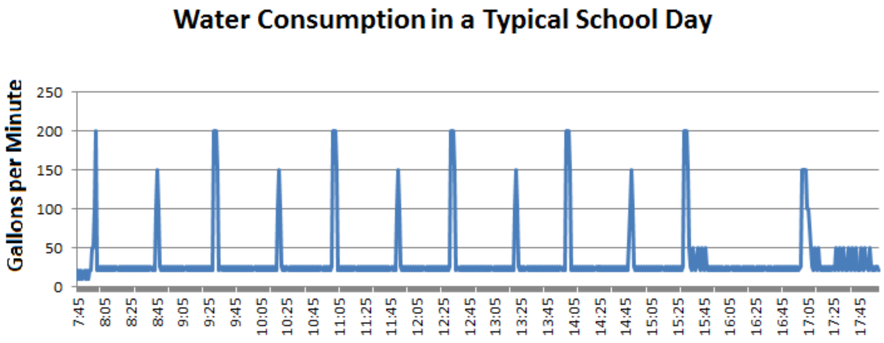
- **Assuming each flush uses 2.5 gallons of water, estimate the amount of water being used during the peak minute of each break.**

  80 flushes $\times$ 2.5 gallons per flush $=$ 200 gallons

- **What time of day do these breaks occur? (If the school schedule varies, consider today's schedule.)**

  Answers will vary by school; an example: $8{:}00$, $9{:}45$, $11{:}30$, $12{:}00$, $1{:}45$, $3{:}30$

- **Draw a graph that could represent the water consumption in a typical school day of your school.**



Exit Ticket (3 minutes)

Name ____ Date____________________

Lesson 4: Analyzing Graphs—Water Usage During a Typical Day at School

Exit Ticket

Suppose the researchers collecting data for water consumption during a typical school day collected data through the night too.

1. For the period between the time the last person leaves the building for the evening and the time of the arrival of the first person the next morning, how should the graph of water consumption appear?

2. Suppose the researchers see instead, from the time 1:21 a.m. onwards, the graph shows a horizontal line of constant value, 4. What might have happened during the night?

Exit Ticket Sample Solutions

Suppose the researchers collecting data for water consumption during a typical school day collected data through the night too.

1. **For the period between the time the last person leaves the building for the evening and the time of the arrival of the first person the next morning, how should the graph of water consumption appear?**

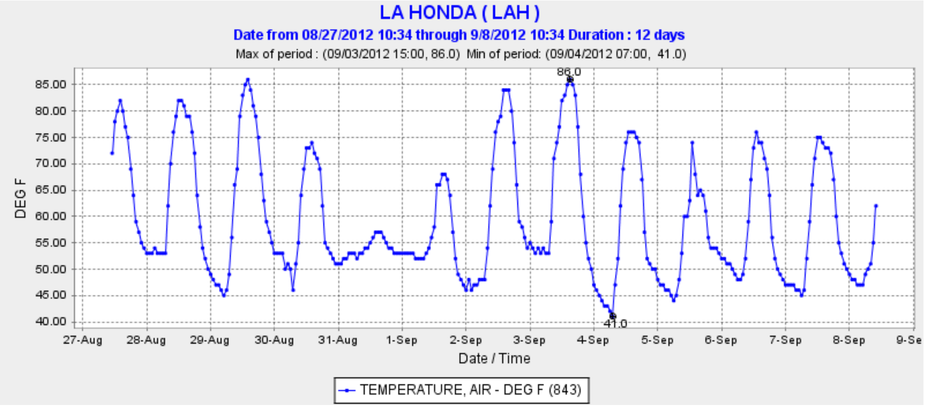
   Since no toilets are being used, the graph should be a horizontal line of constant value zero.

2. **Suppose the researchers see instead, from the time 1:21 a.m. onwards, the graph shows a horizontal line of constant value, 4. What might have happened during the night?**

   Perhaps one toilet started to leak at 1:21 a.m., draining water at a rate of 4 gallons per minute.

Problem Set Sample Solutions

1. **The following graph shows the temperature (in degrees Fahrenheit) of La Honda, CA in the months of August and September of 2012. Answer the questions following the graph.**



- **The graph seems to alternate between peak, valley, peak. Explain why.**

  The temperature increases during the day and drops during the night, and the difference between the high and low temperatures can be very large.

- **When do you think it should be the warmest during each day? Circle the peak of each day to determine if the graph matches your guess.**

  The warmest time should be before sunset, a few hours after noon, since heating occurs throughout the day while the sun is up.

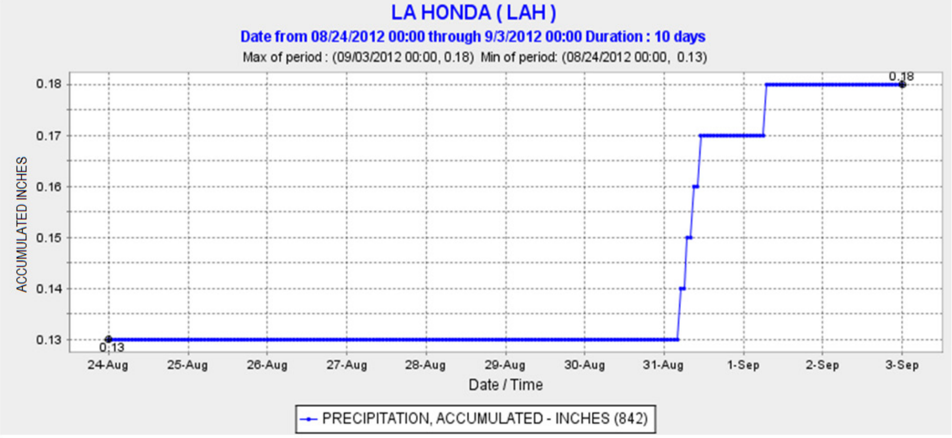
- **When do you think it should be the coldest during each day? Draw a dot at the lowest point of each day to determine if the graph matches your guess.**

  Similarly, the coldest temperature should be before sunrise, since cooling occurs throughout the night.

- **Does the graph do anything unexpected such as not following a pattern? What do you notice? Can you explain why it's happening?**

  The graph seems to follow the expected pattern with natural variations, except for the unusually low daytime temperatures on August 31^{st}.

2. **The following graph shows the amount of precipitation (rain, snow, or hail) that accumulated over a period of time in La Honda, CA.**



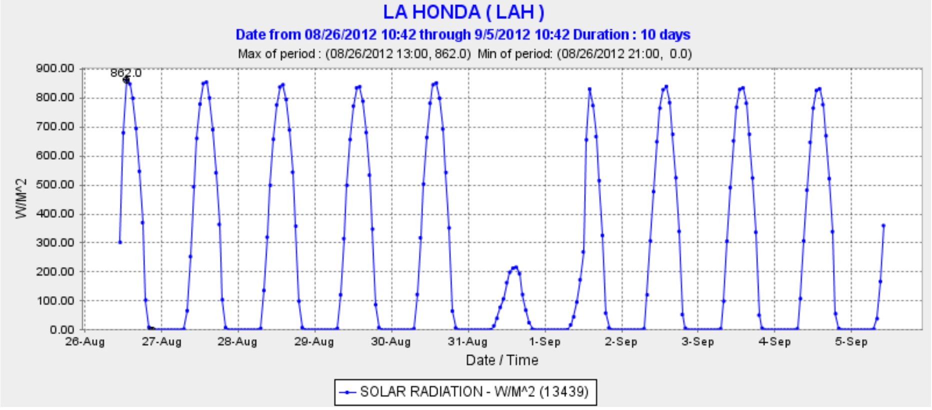
- **Tell the complete story of this graph.**

  On August 24^{th}, the measurement started with an initial cumulative data of 0.13 inches. It stands for the precipitation accumulated before the current measurement. Nothing else happened until August 31^{st}, when there were 0.04 inches of precipitation. On September 1^{st}, there were 0.01 more inches of precipitation.

- **The term "accumulate" in context of the graph means to add up the amounts of precipitation over time. The graph starts on August 24^{th}. Why didn't the graph start at 0 inches instead of starting at 0.13 inches?**

  On August 24^{th}, the measurement started with an initial cumulative data of 0.13 inches. It stands for the precipitation accumulated before the current measurement.

3. **The following graph shows the solar radiation over a period of time in La Honda, CA. Solar radiation is the amount of the sun's rays that reach the Earth's surface.**



- **What happens in La Honda when the graph is flat?**

  This represents time periods during which the solar radiation per unit area is constant. For example during night time, there is no sun light, and hence there are flat regions on the curve.

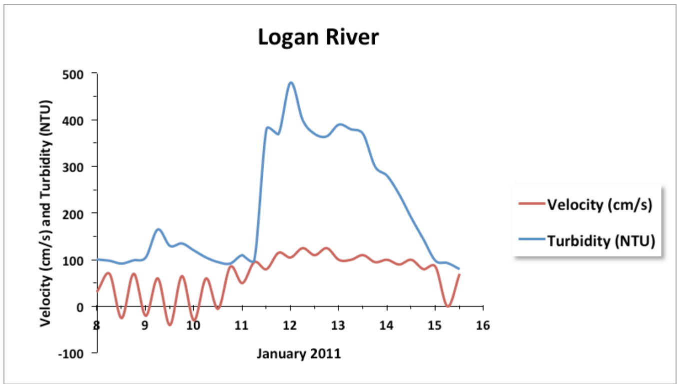
- **What do you think is happening when the peaks are very low?**

  It could be an overcast day. Other less common events such as solar eclipse would also cause this.

- **Looking at all three graphs above, what do you conclude happened on August 31^{st}, 2012 in La Honda, CA?**

  The lower temperature, increase in accumulated precipitation, and the low solar radiation makes me think that August 31^{st}, 2012 was overcast and rainy for most of the day.

4. **The following graph shows the velocity (in centimeters per second) and turbidity of the Logan River in Queensland, Australia during a flood. Turbidity refers to the clarity of the water (higher turbidity means murkier water) and is related to the total amount of suspended solids, such as clay, silt, sand, and phytoplankton, present in the water.**



- **For recreation, Jill visited the river during the month of January and saw clean and beautiful water. On which day do you think she visited?**

  The 15th appears to be the best choice because of the low turbidity. The 8th and 10th are also good choices.

- **What do the negative velocities (below the grey line) that appear periodically at the beginning represent?**

  This shows normal tidal flow, which is disturbed during the flood.

- **The behavior of the river seems to follow a normal pattern at the beginning and at the very end of the time period shown. Approximately when does the flood start? Describe its effects on velocity and turbidity.**

  The flood starts on the 11th. It increases the velocity so that it is always positive, disturbing the tide, and it increases the turbidity of the water.

Lesson 5: Two Graphing Stories

Student Outcomes

- Students interpret the meaning of the point of intersection of two graphs and use analytic tools to find its coordinates.

Classwork

Example 1 (7 minutes)

Have students read the situation and sketch a graphing story. Prompt them to visualize both the story *and* what the graph will look like as they read the situation. Share a few student responses.

Some students may raise questions:

- Are the two people traveling at the same rate? If yes, how would their graphs compare?
  - *If they were, the graphs of the lines would have opposite slopes.*

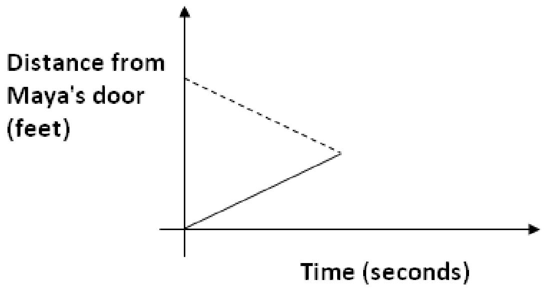
You may also need to clarify what it means to graph Earl's distance from Maya's door. His distance should be decreasing as time passes. Earl's distance is represented by the dotted line. Notice that this graph shows the story ending when the two people meet each other in the hallway, and it assumes they travel at the same rate. Do not spend too much time developing content or equations here. The rest of the lesson will provide a more formal approach.

Example 1

Consider the story:

Maya and Earl live at opposite ends of the hallway in their apartment building. Their doors are 50 feet apart. Each starts at his or her own door and walks at a steady pace toward each other and stop when they meet.

What would their graphing stories look like if we put them on the *same* graph? When the two people meet in the hallway, what would be happening on the graph? Sketch a graph that shows their distance from Maya's door.



Exploratory Challenge/Exercises 1–4 (20 minutes)

The following video is of a man walking up and then back down a flight of stairs. On the way down, a girl starts walking up the stairs. http://youtu.be/X956EvmCevl

Exploratory Challenge

Watch the following graphing story.

http://youtu.be/X956EvmCevI

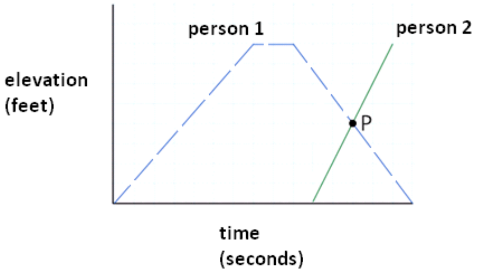
The video shows man and a girl walking on the same stairway.

Exercises 1–4

1. **Graph the man's elevation on the stairway versus time in seconds.**

2. **Add the girl's elevation to the same graph. How did you account for the fact that the two people did not start at the same time?**

   Students' answers should look something like the graph to the right. Student work should also include scales.



Have the students form groups and produce a group graph on whiteboards or poster paper in response to these exercises. Students might have questions about how to incorporate information depicting the motion of the second person. After a few minutes, have students hold up what they have drawn. Give the class further opportunity to revise their own graphs if they wish. Call out groups that have labeled and scaled their axes and ask follow-up questions to elicit their thinking when they created their graphs. The goal should be that all groups have a roughly accurate sketch with axes labeled. Students may struggle with starting the 2nd person at a point along the horizontal axis. Clarify that time 0 represents the time when the first person started walking up the stairs. Work with students by stopping and starting the video to refine and scale their graphs. Estimate that each rise of each stair is 8 inches. There are 16 stairs with a maximum elevation of $10\frac{2}{3}$ ft.

MP.1

Sketch the graph on the board and label the intersection point P. Ask the following questions:

- Does it seem reasonable to say that each graph is composed of linear segments?
  - *Most students will accept that linear functions represent a good model. We might assume that each person is ascending or descending at a constant rate.*

3. **Suppose the two graphs intersect at the point $P(24, 4)$. What is the meaning of this point in this situation?**

   Many students will respond that P is where the two people pass each other on the stairway.

Lead a discussion that highlights these more subtle points before proceeding.

- We have two elevation-versus-time graphs, one for each of the two people (and that time is being measured in the same way for both people).
- The point P lies on the elevation-versus-time graph for the first person, AND it also lies on the elevation-versus-time graph for the second person.

- We know the coordinates of the point P. These coordinates mean that since the first person is at an elevation of 4 feet at 24 seconds, the second person is also at an elevation of 4 feet at 24 seconds.

4. Is it possible for two people, walking in stairwells, to produce the same graphs you have been using and NOT pass each other at time 12 seconds? Explain your reasoning.

Yes, they could be walking in separate stairwells. They would still have the same elevation of 4 feet at time 24 seconds but in different locations.

Give students time to revise their work after you discuss this with the entire class.

Closing (13 minutes)

Example 2/Exercises 5–7

Example 2

Consider the story:

Duke starts at the base of a ramp and walks up it at a constant rate. His elevation increases by three feet every second. Just as Duke starts walking up the ramp, Shirley starts at the top of the same 25 foot high ramp and begins walking down the ramp at a constant rate. Her elevation decreases two feet every second.

Exercises 5–7

5. Sketch two graphs on the same set of elevation-versus-time axes to represent Duke's and Shirley's motions.

6. What are the coordinates of the point of intersection of the two graphs? At what time do Duke and Shirley pass each other?

$(5, 15)$

$t = 5$

7. Write down the equation of the line that represents Duke's motion as he moves up the ramp and the equation of the line that represents Shirley's motion as she moves down the ramp. Show that the coordinates of the point you found in the question above satisfy both equations.

If y represents elevation in feet and t represents time in seconds, then Duke's elevation satisfies $y = 3t$ and Shirley's $y = 25 - 2t$. The lines intersect at $(5, 15)$, and this point does indeed lie on both lines.

Duke: $15 = 3(5)$ *Shirley:* $15 = 25 - 2(5)$

Exercise 6 has a similar scenario to Example 1. After students work this exercise in small groups, have each group share their results as time permits. Circulate around the classroom providing assistance to groups as needed.

Use the results of the exercises in Example 2 to close this session.

MP.3

- How did you figure out the slope of your linear equations from the story? Why was Shirley's slope negative?
  - *The slope is the rate of change, feet per second. Shirley's slope was negative because she was walking down the ramp, and thus her elevation was decreasing over time.*
- Why did Shirley's graph and equation have the y-intercept $(0, 25)$?
  - *The y-intercept was $(0, 25)$ because at time equal to 0, Shirley was at an elevation of 25 ft.*
- What was easier in this problem, determining the intersection point algebraically or graphically? When might an algebraic approach be better?
  - *Students could answer either, depending on the accuracy of their graphs. An algebraic approach would be better when the graphs intersect at non-integer values.*

Lesson Summary

The intersection point of the graphs of two equations is an ordered pair that is a solution to *BOTH* equations. In the context of a distance (or elevation) story, this point represents the fact that both distances (or elevations) are equal at the given time.

Graphing stories with quantities that change at a constant rate can be represented using piecewise linear equations.

Exit Ticket (5 minutes)

Name ____ Date____________________

Lesson 5: Two Graphing Stories

Exit Ticket

Maya and Earl live at opposite ends of the hallway in their apartment building. Their doors are 50 feet apart. Each person starts at his or her own door and walks at a steady pace towards the other. They stop walking when they meet.

Suppose:

- Maya walks at a constant rate of 3 feet every second.
- Earl walks at a constant rate of 4 feet every second.

1. Graph both people's distance from Maya's door versus time in seconds.

2. According to your graphs, approximately how far will they be from Maya's door when they meet?

Exit Ticket Sample Solutions

Maya and Earl live at opposite ends of the hallway in their apartment building. Their doors are 50 feet apart. Each person starts at his or her own door and walks at a steady pace towards the other. They stop walking when they meet.

Suppose:

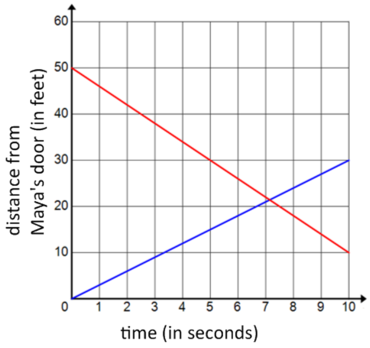
- Maya walks at a constant rate of 3 feet every second.
- Earl walks at a constant rate of 4 feet every second.

1. Graph both people's distance from Maya's door versus time in seconds.

   Graphs should be scaled and labeled appropriately. Maya's graph should start at $(0, 0)$ and have a slope of 3, and Earl's graph should start at $(0, 50)$ and have a slope of -4.

2. According to your graphs, approximately how far will they be from Maya's door when they meet?

   The graphs intersect at approximately 7 seconds at a distance of about 21 feet from Maya's door.



Problem Set Sample Solutions

1. Reread the story about Maya and Earl from Example 1. Suppose that Maya walks at a constant rate of 3 feet every second and Earl walks at a constant rate of 4 feet every second starting from 50 feet away. Create equations for each person's distance from Maya's door and determine exactly when they meet in the hallway. How far are they from Maya's door at this time?

   Maya's Equation: $y = 3t$.

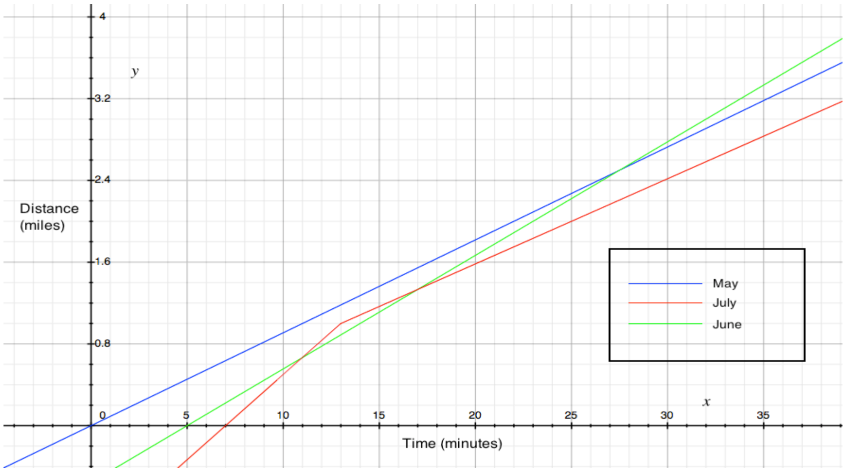
   Earl's Equation: $y = 50 - 4t$.

   Solving the equation $3t = 50 - 4t$ gives the solution $= 7\frac{1}{7}$. The two meet at exactly this time at a distance of $3\left(7\frac{1}{7}\right) = 21\frac{3}{7}$ feet from Maya's door.

2. **Consider the story:**

May, June, and July were running at the track. May started first and ran at a steady pace of 1 *mile every* 11 *minutes. June started* 5 *minutes later than May and ran at a steady pace of* 1 *mile every* 9 *minutes. July started* 2 *minutes after June and ran at a steady pace, running the first lap (*$\frac{1}{4}$ *mile) in* 1.5 *minutes. She maintained this steady pace for* 3 *more laps and then slowed down to* 1 *lap every* 3 *minutes.*

a. **Sketch May, June, and July's distance versus time graphs on a coordinate plane.**



b. **Create linear equations that represent each girl's mileage in terms of time in minutes. You will need two equations for July since her pace changes after 4 laps (1 mile).**

Equations for May, June, and July are shown below. Notice that July has two equations since her speed changes after her first mile, which occurs 13 *minutes after May starts running.*

May: $d = \frac{1}{11}t$

June: $d = \frac{1}{9}(t-5)$

July: $d = \frac{1}{6}(t-7),\ t \le 13$ *and* $d = \frac{1}{12}(t-13)+1,\ t > 13$

c. **Who was the first person to run 3 miles?**

June at time 32 *minutes*

d. **Did June and July pass May on the track? If they did, when and at what mileage?**

Assuming that they started at the same place, June passes May at time 27.5 *minutes at the* 2.5 *mile marker. July does not pass May.*

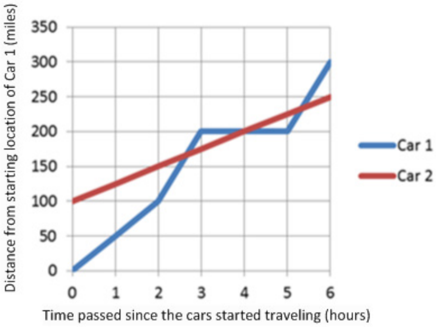
e. **Did July pass June on the track? If she did, when and at what mileage?**

July passes June at time 11 *minutes at the* $\frac{2}{3}$ *mile marker.*

3. **Suppose two cars are travelling north along a road. Car 1 travels at a constant speed of 50 mph for two hours, then speeds up and drives at a constant speed of 100 mph for the next hour. The car breaks down and the driver has to stop and work on it for two hours. When he gets it running again, he continues driving recklessly at a constant speed of 100 mph. Car 2 starts at the same time that Car 1 starts, but Car 2 starts 100 miles farther north than Car 1 and travels at a constant speed of 25 mph throughout the trip.**

a. **Sketch the distance versus time graphs for Car 1 and Car 2 on a coordinate plane. Start with time 0 and measure time in hours.**

A graph is shown below that approximates the two cars traveling north.



b. **Approximately when do the cars pass each other?**

The cars pass after about $2\frac{1}{2}$ *hours, after* 4 *hours, and after about* $5\frac{1}{2}$ *hours.*

c. **Tell the entire story of the graph from the point of view of Car 2. (What does the driver of Car 2 see along the way and when?)**

The driver of Car 2 is carefully driving along at 25 *mph, and he sees Car 1 pass him at* 100 *mph after about* $2\frac{1}{2}$ *hours. About an hour and a half later, he sees Car 1 broken down along the road. After about another hour and a half, Car 1 whizzes past again.*

d. **Create linear equations representing each car's distance in terms of time (in hours). Note that you will need four equations for Car 1 and only one for Car 2. Use these equations to find the exact coordinates of when the cars meet.**

Using the variables, d *for distance (in miles) and t for time (in hours):*

Equation for Car 2: $d = 25t + 100$

Equations for Car 1:

$d = 50t,\ 0 \le t \le 2,$
$d = 100(t-2) + 100 = 100(t-1),\ 2 < t \le 3$
$d = 200,\ 3 < t \le 5$
$d = 100(t-5) + 200 = 100(t-3),\ 5 < t$

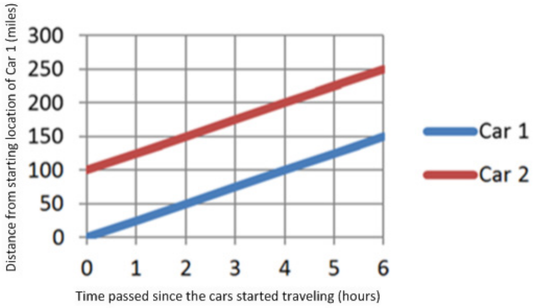
Intersection points:

First: solving $100(t-1) = 25t + 100$ *gives* $\left(\frac{200}{75}, \frac{(25)(200)}{75} + 100\right) \approx (2.7, 166.7)$,
Second: solving $200 = 25t + 100$ *gives* $(4, 200)$, *and*

Third: solving $100(t-3) = 25t + 100$ *gives* $\left(\frac{400}{75}, \frac{(25)(400)}{75} + 100\right) \approx (5.3, 233.3)$.

4. **Suppose that in Problem 3 above, Car 1 travels at the constant speed of 25 mph the entire time. Sketch the distance versus time graphs for the two cars on the graph below. Do the cars ever pass each other? What is the linear equation for Car 1 in this case?**

   A sample graph is shown below. Car 1 never overtakes Car 2, and they are 100 miles apart the entire time. The equation for Car 1 is $y = 25t$.



5. **Generate six distinct random whole numbers between 2 and 9 inclusive and fill in the blanks below with the numbers in the order in which they were generated.**

   A (0 , ______), B (______, ______), C (10 , ______)

   D (0 , ______), E (10 , ______).

   (Link to a random number generator http://www.mathgoodies.com/calculators/random_no_custom.html)

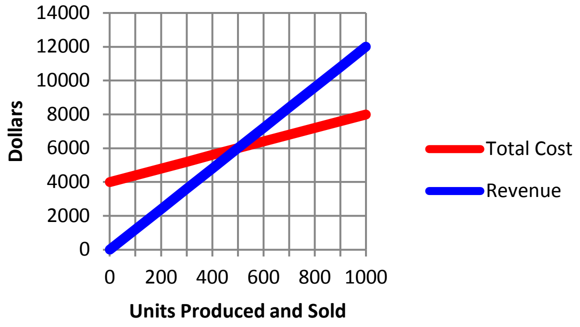
   a. **On a coordinate plane, plot points A, B, and C. Draw line segments from point A to point B, and from point B to point C.**

   b. **On the same coordinate plane, plot points D and E and draw a line segment from point D to point E.**

   c. **Write a graphing story that describes what is happening in this graph. Include a title, x- and y-axis labels, and scales on your graph that correspond to your story.**

   Answers will vary depending on the random points generated.

6. **The following graph shows the revenue (or income) a company makes from designer coffee mugs and the total cost (including overhead, maintenance of machines, etc.) that the company spends to make the coffee mugs.**



a. **How are revenue and total cost related to the number of units of coffee mugs produced?**

Definition: Profit $=$ Revenue $-$ Cost. Revenue is the income from the sales and is directly proportional to the number of coffee mugs actually sold; it does not depend on the units of coffee mugs produced. Total cost is the sum of the fixed costs (overhead, maintaining the machines, rent, etc.) PLUS the production costs associated with the number of coffee mugs produced; it does not depend on the number of coffee mugs sold.

b. **What is the meaning of the point $(0, 4000)$ on the total cost line?**

The overhead costs, the costs incurred regardless of whether 0 or $1,000$ coffee mugs are made or sold, is $\$4,000$.

c. **What are the coordinates of the intersection point? What is the meaning of this point in this situation?**

$(500, 6000)$. *The revenue, $\$6,000$, from selling 500 coffee mugs, is equal to the total cost, $\$6,000$, of producing 500 coffee mugs. This is known as the break-even point. When Revenue $=$ Cost, the Profit is $\$0$. After this point, the more coffee mugs sold, the more the positive profit; before this point, the company loses money.*

d. **Create linear equations for revenue and total cost in terms of units produced and sold. Verify the coordinates of the intersection point.**

If u is a whole number for the number of coffee mugs produced and sold, C is the total cost to produce u mugs, and R is the total revenue when u mugs are sold, then

$$C = 4000 + 4u,$$
$$R = 12u.$$

When $u = 500$, both $C = 4000 + 4 \cdot 500 = 6000$ and $R = 12 \cdot 500 = 6000$.

e. **Profit for selling $1,000$ units is equal to revenue generated by selling $1,000$ units minus the total cost of making $1,000$ units. What is the company's profit if $1,000$ units are produced and sold?**

Profit $=$ Revenue $-$ Total Cost. Hence, $P = R - C = 12 \cdot 1000 - (4000 + 4 \cdot 1000) = 12000 - 8000$

$= 4000$

The company's profit is $\$4,000$.

Topic B:

The Structure of Expressions

A-SSE.A.1, A-SSE.A.2

Focus Standard:	A-SSE.A.2	Use the structure of an expression to identify ways to rewrite it. *For example, see* $x^4 - y^4$ *as* $(x^2)^2 - (y^2)^2$*, thus recognizing it as a difference of squares that can be factored as* $(x^2 - y^2)(x^2 + y^2)$*.*
	A-APR.A.1	Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.
Instructional Days:	4	
Lesson 6:	Algebraic Expressions—The Distributive Property (P)[1]	
Lesson 7:	Algebraic Expressions—The Commutative and Associative Properties (S)	
Lesson 8:	Adding and Subtracting Polynomials (S)	
Lesson 9:	Multiplying Polynomials (P)	

In Lessons 6 and 7 of this topic, students develop a precise understanding of what it means for expressions to be algebraically equivalent. By exploring geometric representations of the distributive, associative, and commutative properties for positive whole numbers and variable expressions assumed to represent positive whole numbers, students confirm their understanding of these properties and expand them to apply to all real numbers. Students use the properties to generate equivalent expressions and formalize that two algebraic expressions are equivalent if we can convert one expression into the other by repeatedly applying the commutative, associative, and distributive properties and the properties of rational exponents to components of the first expression. A goal of this topic is to address a fundamental, underlying question: *Why are the commutative, associative, and distributive properties so important in mathematics?*[2] The answer to the question is, of course, because these three properties help to generate all equivalent algebraic expressions discussed in Algebra I.

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[1] Lesson Structure Key: **P**-Problem Set Lesson, **M**-Modeling Cycle Lesson, **E**-Exploration Lesson, **S**-Socratic Lesson

[2] The *welcomed* removal of parentheses from expressions through the use of order of operations and other tricks greatly simplifies how we write expressions, but those same tricks also make it appear as if the associative property is never used when rewriting expressions.

Lessons 6 and 7 also engage students in their first experience using a recursive definition for building algebraic expressions. Recursive definitions are sometimes confused with being circular in nature because the definition of the term uses the very term one is defining. However, a recursive definition or process is not circular because it has what is referred to as a base case. For example, a definition for algebraic expression is presented as follows:

> An algebraic expression is either:
>
> 1. A numerical symbol or a variable symbol or
> 2. The result of placing previously generated algebraic expressions into the two blanks of one of the four operators ((__) $+$ (__), (__) $-$ (__), (__) $\times$ (__), (__) $\div$ (__)) or into the base blank of an exponentiation with an exponent that is a rational number.

Part (1) of this definition serves as a base case, stating that any numerical or variable symbol is in itself an algebraic expression. The recursive portion of the definition is in part (2) where one can use any previously generated algebraic expression to form new ones using the given operands. Recursive definitions are an important part of the study of sequences in Module 3. Giving students this early experience lays a nice foundation for the work to come.

Having a clear understanding of how algebraic expressions are built and what makes them equivalent provides a foundation for the study of polynomials and polynomial expressions.

In Lessons 8 and 9, students learn to relate polynomials to integers written in base x, rather than our traditional base of 10. The analogies between the system of integers and the system of polynomials continue as they learn to add, subtract, and multiply polynomials and to find that the polynomials for a system that is closed under those operations (e.g., a polynomial added to, subtracted from, or multiplied by another polynomial) always produces another polynomial.

We use the terms polynomial and polynomial expression in much the same way as we use the terms number and numerical expression. Where we would not call $27(3+8)$ a number, we would call it a numerical expression. Similarly, we reserve the word polynomial for polynomial expressions that are written as a sum of monomials.

Lesson 6: Algebraic Expressions—The Distributive Property

Student Outcomes

- Students use the structure of an expression to identify ways to rewrite it.
- Students use the distributive property to prove equivalency of expressions.

Lesson Notes

The previous five lessons introduced the graphs of the functions students will study in this algebra course. These next lessons change the focus from graphs to expressions and their structures. In Grades 5–8, the term "expression" was described but not formally defined, and many subtleties may have been overlooked. For example, the associative property may not have been made explicit; $3 \cdot 5 \cdot 7$, for instance, is really $(3 \cdot 5) \cdot 7$. The lessons that follow will formally define algebraic expression and the equivalency of algebraic expressions and simultaneously introduce students to the notion of a recursive definition, which later becomes a major aspect of Algebra I (recursive sequences). Lesson 6 begins with an expression-building competition. As this is a strange, abrupt change of direction for students from the previous five lessons, it may be worthwhile to mention this change.

- In middle school you learned to find equations for straight line graphs such as the ones that appear in Lesson 1, but as we saw in Lessons 2 and 3, not all graphs are linear. It would be nice to develop the machinery for developing equations for those too, if at all possible. Note that Lesson 4 indicates that graphs still might be very complicated and finding a single equation for them might not be possible. We may, however, be able to find equations that produce graphs that approximate the graphs, or sections thereof. Also, Lesson 5 points out the value of finding the point of intersection of two graphs. Our goal is to develop the algebraic tools to do this.

Classwork

Exercise 1 (13 minutes)

The following is known as the "4-number game." It challenges students to write each positive integer as a combination of the digits 1, 2, 3, and 4; each used at most once, combined via the operations of addition and multiplication only, as well as grouping symbols. For example, 24 can be expressed as $(1 + 3)(2 + 4)$. Students may use parentheses or not, at their own discretion (as long as their expressions evaluate to the given number, following the order of operations). Digits may not be juxtaposed to represent larger whole numbers, so you cannot use the numerals 1 and 2 to create the number 12 for instance.

Scaffold:

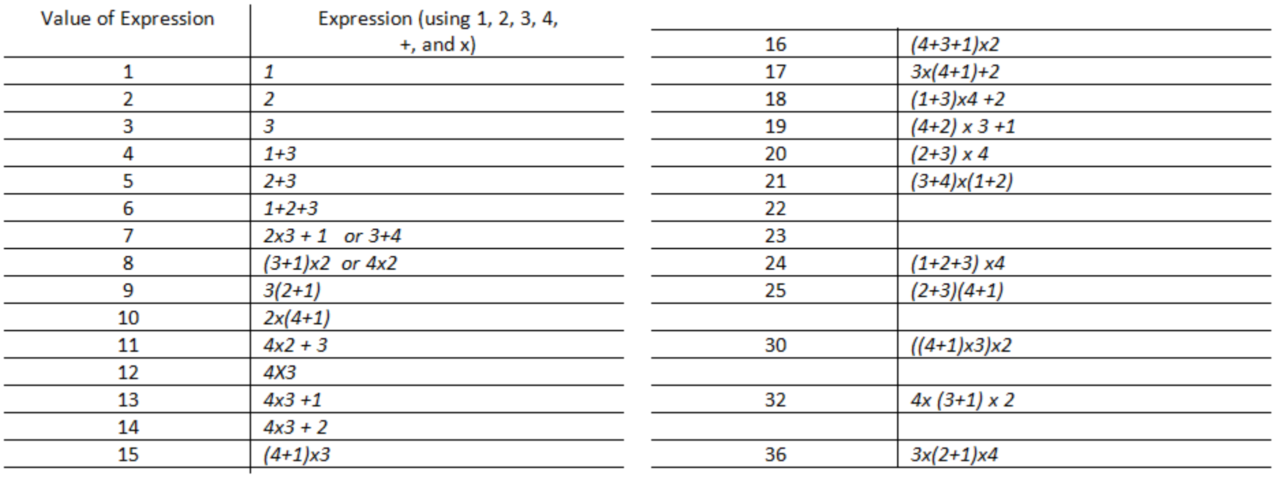
Should you feel your students would benefit from it, you may optionally begin the lesson with the "3-number game" using only the digits 1, 2, and 3.

Play the 4-number game as a competition within pairs. Give the students 3 minutes to express the longest list of numbers they can, each written in terms of the digits 1, 2, 3, and 4. Students may want to use small dry erase boards to play this game or pencil and paper. Optionally, consider splitting up the tasks (e.g., 1–8, 9–20, 21–30, 31–36) and assign them to different groups.

Below are some sample expressions the students may build and a suggested structure for displaying possible expressions on the board as students call out what they have created.

When reviewing the game, it is likely that students will have different expressions for the same number (see answers to rows 7 and 8 given below). Share alternative expressions on the board and discuss as a class the validity of the expressions.

Challenge the students to come up with more than one way to create the number 21.



Value of Expression	Expression (using 1, 2, 3, 4, +, and x)
1	1
2	2
3	3
4	1+3
5	2+3
6	1+2+3
7	2x3 + 1 or 3+4
8	(3+1)x2 or 4x2
9	3(2+1)
10	2x(4+1)
11	4x2 + 3
12	4X3
13	4x3 +1
14	4x3 + 2
15	(4+1)x3
16	(4+3+1)x2
17	3x(4+1)+2
18	(1+3)x4 +2
19	(4+2) x 3 +1
20	(2+3) x 4
21	(3+4)x(1+2)
22	
23	
24	(1+2+3) x4
25	(2+3)(4+1)
30	((4+1)x3)x2
32	4x (3+1) x 2
36	3x(2+1)x4

After students share their results, ask these questions:

- What seems to be the first counting number that cannot be created using only the numbers 1, 2, 3, and 4 and the operations of multiplication and addition?
    - 22
- What seems to be the largest number that can be made in the 4-number game?
    - $(1+2)(3)(4)=36$

We can now launch into an interesting investigation to find structure in the game.

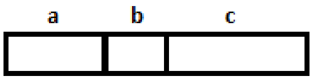
- Suppose we were playing the 2-number game. What seems to be the largest number you could create using the numbers 1 and 2 (each at most once) and the operations of multiplication and addition?
    - $(1+2)=3$
- Suppose we were playing the 3-number game. What seems to be the largest number you could create using the numbers 1, 2, and 3 (each at most once) and the operations of multiplication and addition?
    - $(1+2)(3) \; = \; 9$

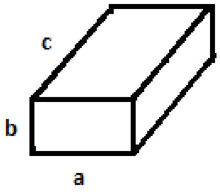
Encourage students to generalize the pattern for the 5-number game, and the N-number game, and think about why this pattern for an expression gives the largest attainable number.

- *Add* 1 *and* 2*, and then multiply the remaining numbers. For the* 5*-number game, the largest number would be* $(1+2)(3)(4)(5)=180$*. For the* N*-number game, the largest number would be* $(1+2)(3)(4)(5)\ldots(n)$

Discussion (5 minutes)

- We have seen that both $1+2+3$ and $1\cdot 2\cdot 3$ evaluate to 6. Does it seem reasonable that for any real numbers, a, b, and c, that $a+b+c$ would be equivalent to $a\cdot b\cdot c$?
  - *No, students will likely quickly think of counterexamples. (Proceed to the next bullet point regardless.)*
- How can we show geometrically that they are not generally equivalent? Assume that a, b, and c, are positive integers. Does it seem intuitive that these two geometric representations are equivalent?
  - *The geometric representations do not suggest equivalency.*





- Therefore, it does not appear that being numerically equivalent in one specific case implies equivalency in other, more general cases.
- Perhaps we should play a game like this with symbols.
- Let's consider first what the effect would be of allowing for repeat use of symbols in the 4-number game.

MP.7 With the repeat use of symbols, we can build larger and larger expressions by making use of expressions we already possess. For example, from $10=1+2+3+4$, we can generate the number 110 by making repeated use of the expression:

$$110=10\cdot 10+10=(1+2+3+4)\cdot(1+2+3+4)+(1+2+3+4).$$

Exercise 2 (1 minute)

Exercise 2

Using the numbers $1, 2, 3, 4$ only once and the operations $+$ or $\times$ as many times as you like, write an expression that evaluates to 16. Use this expression and any combination of those symbols as many times as you like to write an expression that evaluates to 816.

$(4+3+1)\times 2$

$$\big((4+3+1)\times 2\big)\big((4+3+1)\times 2+(4+3+1)\times 2+(4+3+1)\times 2\big)+(4+3+1)\times 2+(4+3+1)\times 2 \\ +(4+3+1)\times 2$$

Exercise 3 (5 minutes)

- Suppose we now alter the 4-number game to be as follows:

Exercise 3

Define the rules of a game as follows:

a. **Begin by choosing an initial set of symbols, variable or numeric, as a starting set of expressions.**

   3, x, y, *and* a

b. Generate more expressions by placing any previously created expressions into the blanks of the addition operator: _____ + _____.

$3+a$ or $x+y$ or $y+3$ or $a+a$

- Let's play the game using 3, x, y, and a as our set of starting expressions.

Write the symbols to be used on the board. (These are not provided in the student materials.)

- Can you see that:

  Part (1) of the game gives us 3, x, y, and a to start.

  Part (2) gives us expressions, such as $3+a$ or $x+y$ or $y+3$ or $a+a$.

  Repeated use of part (2) then gives $(3+a)+3$ or $x+(x+y)$ and $(x+y)+(x+y)$, for example,

  and then $\big((x+y)+(x+y)\big)+a$ for example, etc.

Make sure students are clear that in this version of the game, we are going to limit ourselves to addition (no multiplication).

- Take 1 minute to generate as many expressions as you can, following the rules of the game. (Time the students for 1 minute.)
- Compare your list with your neighbor. Did your neighbor follow the rules of the game?
- If you followed the rules, you end up generating strings of sums. Is that correct?
- Is it possible to create the expression $4x+5y+3a$ from the game?

Ask a student to verbally describe how they did it and show the string of sums on the board.

$$\Big(\big((x+y)+(x+y)\big)+\big((x+y)+(x+y)\big)\Big)+\big((y+a)+(a+a)\big)$$

Exercise 4 (4 minutes)

Scaffold: At the teacher's discretion, begin with the problem $2x+3x=5x$. Or offer the problem just the way it is given, and allow students to ponder and struggle for a bit. Then, suggest "If you are not sure, let's try this one: is $2x+3x=5x$ an application of the distributive property?"

Exercise 4

Roma says that collecting like terms can be seen as an application of the distributive property. Is writing $x+x=2x$ an application of the distributive property?

Roma is correct. By the distributive properties, we have $x+x=1\cdot x+1\cdot x=(1+1)x=2x$.

Exercises 5–7 (7 minutes)

Have students work the following exercises one at a time, taking time to discuss the solutions between each one.

Exercise 5

Leela is convinced that $(a+b)^2 = a^2 + b^2$. Do you think she is right? Use a picture to illustrate your reasoning.

Leela is not right.

	a	b
a	a^2	ab
b	ab	b^2

$(a+b)^2 = a^2 + b^2 + 2ab$

Exercise 6

Draw a picture to represent the expression $(a+b+1) \times (b+1)$.

Answer:

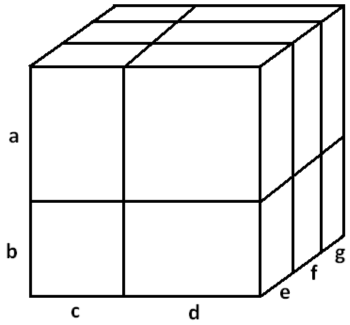
	b	1
a	ab	a
b	b^2	b
1	b	1

A good discussion to have with your students is whether it is important that the length of 1 in the width be the same as the length of 1 in the height. Continue the discussion as to whether it is important how you label the picture to represent the quantities.

Exercise 7

Draw a picture to represent the expression $(a+b) \times (c+d) \times (e+f+g)$.

Answer:



Closing (7 minutes)

The previous exercises demonstrate the distributive property of arithmetic, which we believe to hold for all real numbers, not just the positive whole numbers. The choice of the word "belief" is intended to reinforce the notion with students that we accept the property as true without the need for proof.

- We can see, geometrically, that the following are true for whole numbers (display on the board):
  $(a+b)^2 = a^2 + 2ab + b^2$
  $(a+b+1) \times (b+1) = ab + b^2 + a + 2b + 1$
  $(a+b) \times (c+d) \times (e+f+g) = ace + acf + acg + ade + adf + adg + bce + bcf + bcg + bde + bdf + bdg$
- Do we also believe these statements to be true for all real numbers: a, b, c, d, e, f, g?
- Which of these statements are extensions of the Distributive Property of arithmetic (stated in your student materials)?
  - *All of them*.

A Key Belief of Arithmetic:

The Distributive Property: **If $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ are real numbers, then $\boldsymbol{a(b+c) = ab + ac}$.**

Lesson Summary

The distributive property represents a key belief about the arithmetic of real numbers. This property can be applied to algebraic expressions using variables that represent real numbers.

Exit Ticket (3 minutes)

Name ____ Date____________________

Lesson 6: Algebraic Expressions—The Distributive Property

Exit Ticket

Consider the expression: $(x + y + 3) \times (y + 1)$.

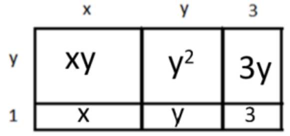
1. Draw a picture to represent the expression.

2. Write an equivalent expression by applying the distributive property.

Exit Ticket Sample Solutions

Consider the expression: $(x + y + 3) \times (y + 1)$.

1. Draw a picture to represent the expression.



2. Write an equivalent expression by applying the distributive property.

$(x + y + 3)(y + 1) = xy + \ x + y^2 + 4y + 3$

Problem Set Sample Solutions

1. Insert parentheses to make each statement true.

a.	$2 + 3 \times 4^2 + 1 = 81$	$(2 + 3) \times 4^2 + 1 = 81$
b.	$2 + 3 \times 4^2 + 1 = 85$	$(2 + 3) \times (4^2 + 1) = 85$
c.	$2 + 3 \times 4^2 + 1 = 51$	$2 + (3 \times 4^2) + 1 = 51$ *(or no parentheses at all is acceptable as well)*
d.	$2 + 3 \times 4^2 + 1 = 53$	$2 + 3 \times (4^2 + 1) = 53$

2. Using starting symbols of w, q, 2, and -2, which of the following expressions will NOT appear when following the rules of the game played in Exercise 3?

a. $7w + 3q + (-2)$

b. $q - 2$

c. $w - q$

d. $2w + 6$

e. $-2w + 2$

Expressions (c) and (e) cannot be obtained in this exercise.

Part (d) appears as $w + w + 2 + 2 + 2$, which is equivalent to $2w + 6$.

3. Luke wants to play the 4-number game with the numbers 1, 2, 3, and 4 and the operations of addition, multiplication, AND subtraction.

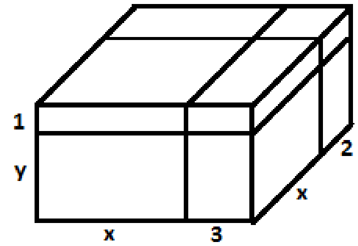
Leoni responds, "Or we just could play the 4-number game with just the operations of addition and multiplication, but now with the numbers -1, -2, -3, -4, 1, 2, 3, and 4 instead."

What observation is Leoni trying to point out to Luke?

Subtraction can be viewed as the addition of a negative (e.g., $x - 4 = x + (-4)$). By introducing negative integers, we need not consider subtraction as a new operation.

4. **Consider the expression: $(x+3)\cdot(y+1)\cdot(x+2)$.**

   a. **Draw a picture to represent the expression.**

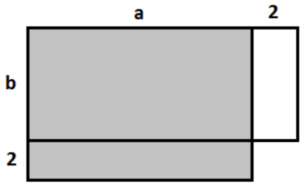


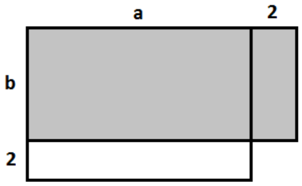
   b. **Write an equivalent expression by applying the distributive property.**

   $(y+1)(x+3)(x+2) = x^2y + 5xy + 6y + x^2 + 5x + 6$

5.

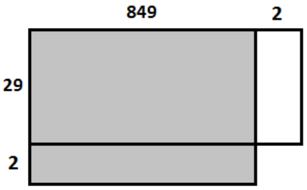
   a. **Given that $a > b$, which of the shaded regions is larger and why?**

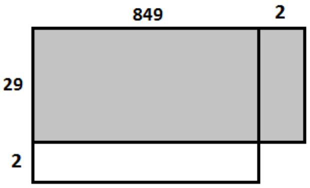




   The shaded region from the image on the left is larger than the shaded region from the image on the right. Both images are made up of the region of area $a \times b$ plus another region of either $2a$ (for the image on the left) or $2b$ (for the image on the right) since $a > b$, $2a > 2b$.

   b. **Consider the expressions 851×29 and 849×31. Which would result in a larger product? Use a diagram to demonstrate your result.**



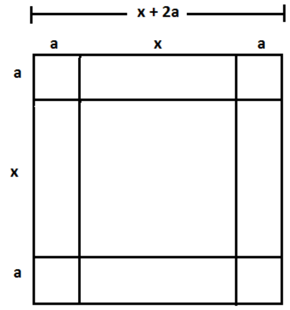


   $851 \cdot 29$ *can be written as:* $(849+2)29 = 849 \cdot 29 + 2 \cdot 29$ *and*

   $849 \cdot 31$ *can be written as:* $849(29+2) = 849 \cdot 29 + 2 \cdot 849$.

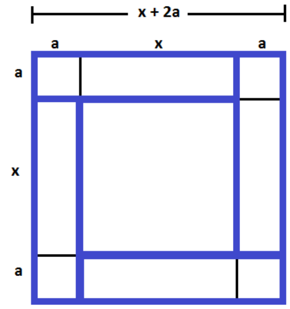
   Since $2 \cdot 29 < 2 \cdot 849$, *the product* $849 \cdot 31$ *is the larger product.*

6. **Consider the following diagram.**



Edna looked at the diagram and then highlighted the four small rectangles shown and concluded:

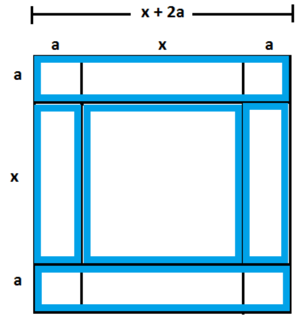
$(x+2a)^2 = x^2 + 4a(x+a)$.



a. **Michael, when he saw the picture, highlighted four rectangles and concluded:**

$(x+2a)^2 = x^2 + 2ax + 2a(x+2a)$.

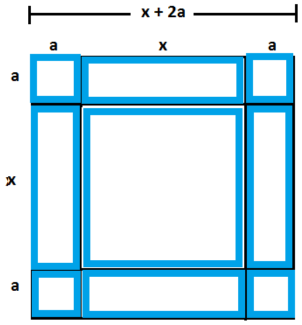
Which four rectangles and one square did he highlight?



b. **Jill, when she saw the picture, highlighted eight rectangles and squares (not including the square in the middle) to conclude:**

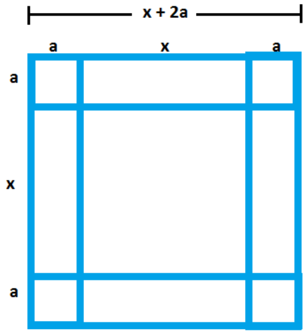
$(x + 2a)^2 = x^2 + 4ax + 4a^2$.

Which eight rectangles and squares did she highlight?



c. **When Fatima saw the picture, she exclaimed: $(x + 2a)^2 = x^2 + 4a(x + 2a) - 4a^2$. She claims she highlighted just four rectangles to conclude this. Identify the four rectangles she highlighted and explain how using them she arrived that the expression $x^2 + 4a(x + 2a) - 4a^2$.**

She highlighted each of the four rectangles that form a rim around the inner square. In doing so, she double counted each of the four $a \times a$ corners and, therefore, needed to subtract $4a^2$.



d. **Is each student's technique correct? Explain why or why not.**

Yes, all of the techniques are right. You can see how each one is correct using the diagrams. The students broke the overall area into parts and added up the parts. In Fatima's case, she ended up counting certain areas twice and had to compensate by subtracting those areas back out of her sum.

Lesson 7: Algebraic Expressions—The Commutative and Associative Properties

Student Outcomes

- Students use the commutative and associative properties to recognize structure within expressions and to prove equivalency of expressions.

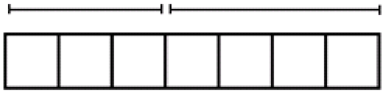
Classwork

Exercises 1–4 (15 minutes)

Have students discuss the following four exercises in pairs. Discuss the answers as a class.

Exercise 1

Suzy draws the following picture to represent the sum $3 + 4$:



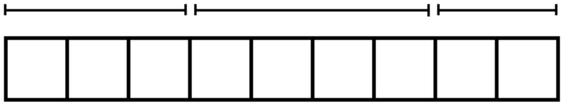
Ben looks at this picture from the opposite side of the table and says, "You drew $4 + 3$."

Explain why Ben might interpret the picture this way.

Ben read the picture from his left to his right on his side of the table.

Exercise 2

Suzy adds more to her picture and says, "The picture now represents $(3 + 4) + 2$."



How might Ben interpret this picture? Explain your reasoning.

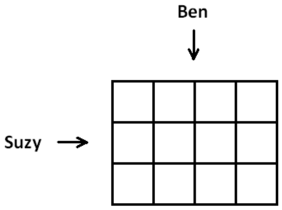
Reading from right to left, the solution would be $(2 + 4) + 3$. Make sure students have parentheses around $2 + 4$.

Exercise 3

Suzy then draws another picture of squares to represent the product 3×4. Ben moves to the end of the table and says, "From my new seat, your picture looks like the product 4×3."

What picture might Suzy have drawn? Why would Ben see it differently from his viewpoint?

Squares should be arranged in a grid. If a student responds that Suzy made 3 rows of 4, then Ben's viewpoint would be 4 rows of 3. Students should understand that Ben is seated to Suzy's left or right now, not across from her. Some students may need scaffolding here—have them physically move to see the different viewpoint.



Exercise 4

Draw a picture to represent the quantity $(3 \times 4) \times 5$ that also could represent the quantity $(4 \times 5) \times 3$ when seen from a different viewpoint.

Student solutions could vary here. Students may consider representing the problem as a 3 by 4 by 5 rectangular box. When viewed from different faces, the different expressions appear. With the 3 by 4 rectangle viewed as its base, the volume of the box might be computed as $(3 \times 4) \times 5$. But with the 4 by 5 rectangle viewed as its base, its volume would be computed as $(4 \times 5) \times 3$. Some students will likely repeat the 3×4 pattern 5 times in a row. This diagram viewed from the end of the table would be 4 dots repeated 5 times arranged in 3 columns.

Ask students a series of questions of the following type:

- Could the ideas developed in Exercises 1 and 2 be modified so as to explain why $2\frac{1}{2} + \frac{3}{19}$ should equal $\frac{3}{19} + 2\frac{1}{2}$?
- Or that $\left((-6562.65) + (-9980.77)\right) + 22$ should equal $\left(22 + (-9980.77)\right) + (-6562.65)$?
- How about that $\sqrt{2} + \frac{1}{\pi}$ should equal $\frac{1}{\pi} + \sqrt{2}$?
- Is it possible for a rectangle or a rectangular box to have a negative side length?
- Could the ideas developed in Exercises 3 and 4 be used to show that $(-3) \times \left(\frac{1}{\sqrt{7}}\right)$ should equal $\left(\frac{1}{\sqrt{7}}\right) \times (-3)$ or that $(\pi \times 17.2) \times \left(-\frac{16}{5}\right)$ should equal $\left(\left(-\frac{16}{5}\right) \times 17.2\right) \times \pi$?

Next, have students review the four properties of arithmetic provided in the student materials and ask the following:

Four Properties of Arithmetic:

The Commutative Property of Addition: If a and b are real numbers, then $a + b = b + a$.

The Associative Property of Addition: If a, b, and c are real numbers, then $(a + b) + c = a + (b + c)$.

The Commutative Property of Multiplication: If a and b are real numbers, then $a \times b = b \times a$.

The Associative Property of Multiplication: If a, b, and c are real numbers, then $(ab)c = a(bc)$.

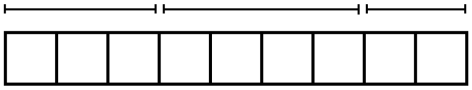
- Can you relate each of these properties to one of the previous exercises?
    - *Exercise 1 connects with the commutative property of addition.*
    - *Exercise 2 connects with the associative property of addition. (Students might mention that the commutative property of addition is also relevant to this exercise. This will be discussed fully in Exercise 5.)*
    - *Exercise 3 connects with the commutative property of multiplication.*
    - *Exercise 4 connects the associative property of multiplication. (Students might mention that the commutative property of multiplication is also relevant to this exercise. This will be discussed fully in Exercise 6.)*

Point out that the four opening exercises suggest that the commutative and associative properties of addition and multiplication are valid for whole numbers and probably apply to all real numbers as well. However, there is a weakness in the geometric models since negative side lengths and areas are not meaningful. We choose to believe these properties hold for *all* real numbers, including negative numbers.

Exercise 5 (10 minutes)

Exercise 5

Viewing the diagram below from two different perspectives illustrates that $(3 + 4) + 2$ equals $2 + (4 + 3)$.

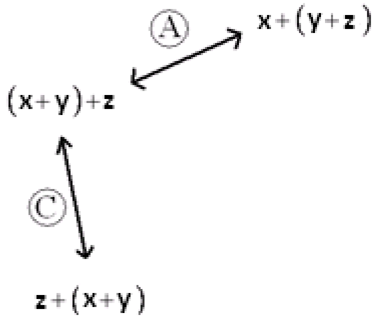


Is it true for all real numbers x, y, and z that $(x + y) + z$ should equal $(z + y) + x$?

(Note: The direct application of the associative property of addition only gives $(x + y) + z = x + (y + z)$.)

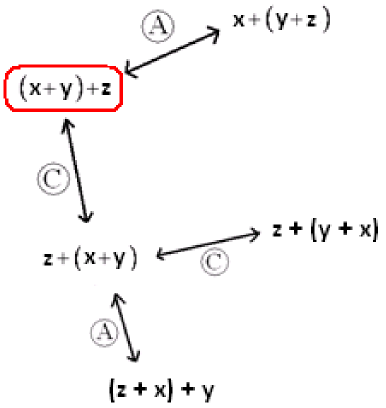
To answer this exercise with the class, create a “flow diagram” on the board as follows. This flow diagram will show how one can apply both the commutative and associative properties to prove the equivalence of these two expressions. Have students copy this work into their handouts.

Start by showing the application of each property on the expression $(x + y) + z$.

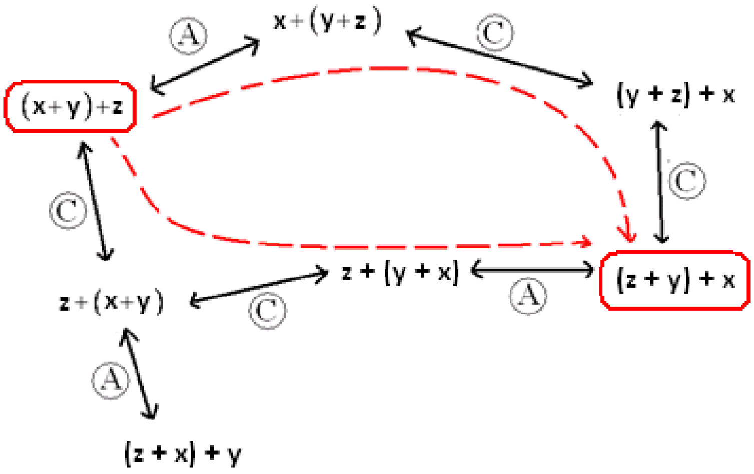


Here “A” represents an application of the associative property and “C” an application of the commutative property. Be sure students understand the application of the commutative property shown here.

Point out that we can extend this diagram by applying the commutative and associative properties to the new expressions in the diagram.



Note that there are multiple branches and options for extending this diagram. Direct the students to discover options that will chart a path from $(x+y)+z$ to $(z+y)+x$. Two possible paths are as follows:



Choose one of the paths in the flow diagram and show on the board how to write it as a mathematical proof of the statement that $(x+y)+z$ and $(z+y)+x$ are equivalent expressions. For example, for the lower of the two paths shown, write the following:

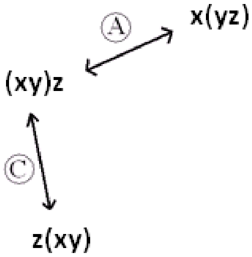
$$\begin{aligned}(x+y)+z &= z+(x+y) && \text{commutative property}\\ &= z+(y+x) && \text{commutative property}\\ &= (z+y)+x && \text{associative property}\end{aligned}$$

Exercise 6 (5 minutes)

Exercise 6

Draw a flow diagram and use it to prove that $(xy)z = (zy)x$ for all real numbers x, y, and z.

Here is the start of the diagram.



Students will likely realize the answer here is completely analogous to the solution to the previous exercise.

$$\begin{aligned}(xy)z &= z(xy) && \text{commutative property}\\ &= z(yx) && \text{commutative property}\\ &= (zy)x && \text{associative property}\end{aligned}$$

Have students complete Exercises 7 and 8.

Exercise 7 (5 minutes)

Exercise 7

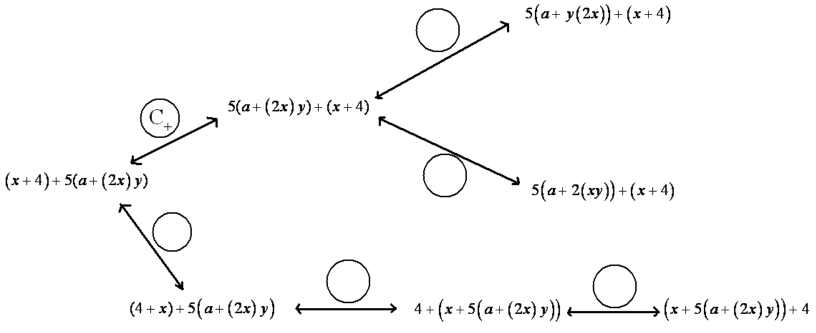
Use these abbreviations for the properties of real numbers and complete the flow diagram.

C_+ for the commutative property of addition

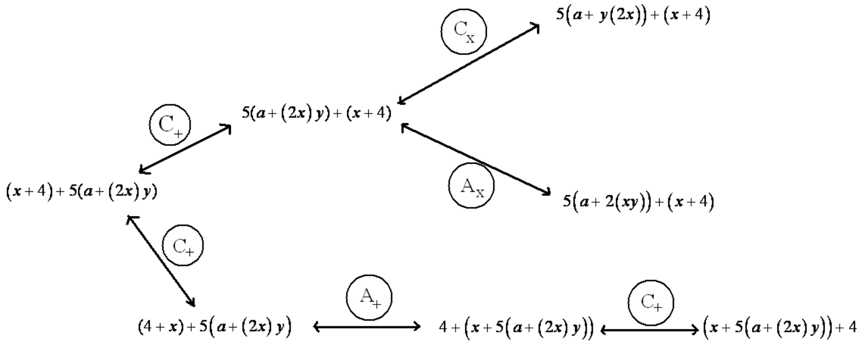
$C_\times$ for the commutative property of multiplication

A_+ for the associative property of addition

$A_\times$ for the associative property of multiplication



Answer:



Exercise 8 (2 minutes)

Exercise 8

Let a, b, c, and d be real numbers. Fill in the missing term of the following diagram to show that $((a+b)+c)+d$ is sure to equal $a+(b+(c+d))$.

$$\begin{tikzcd}((a+b)+c)+d \arrow[r, leftrightarrow, "\text{Ⓐ}"] & (a+(b+c))+d \arrow[r, leftrightarrow, "\text{Ⓐ}"] & \boxed{a+((b+c)+d)} \arrow[r, leftrightarrow, "\text{Ⓐ}"] & a+(b+(c+d))\end{tikzcd}$$

- This example illustrates that it is possible to prove, through repeated use of the associative property, that any two arrangements of parentheses around a given sum are equivalent expressions. For this reason it is deemed unnecessary to place parentheses among a sum of terms. (Present the following on the board.)

$$\begin{array}{lcl} ((a+b)+c)+d & & \\ a+(b+(c+d)) & \rightarrow & \text{“}a+b+c+d\text{”} \\ (a+b)+(c+d) & & \end{array}$$

MP.8

- From now on, we will accept this as common practice. In presenting a proof, writing the following:

$$\left(x+(y+z)+(w+6)\right)=\left((x+y)+(z+w)+6\right) \qquad \text{by the associative property}$$

OR

$$a+b+c+d=a+(b+c)+d \qquad \text{by the associative property}$$

for instance, is accepted.

The same holds for a product of symbols. Repeated application of the associative property of multiplication establishes the equivalency of $((xy)z)w$ and $x((yz)w)$, for example, and these can both be written simply as $xyzw$.

Closing (5 minutes)

- Throughout this lesson, we have used symbols for numbers and symbols for placeholders for numbers to create expressions. Let us now formalize these notions with definitions.

Numerical Symbol: A *numerical symbol* is a symbol that represents a specific number.

For example, $0, 1, 2, 3\frac{2}{3}, -3, -124.122, \pi, e$ are numerical symbols used to represent specific points on the real number line.

Variable Symbol: A *variable symbol* is a symbol that is a placeholder for a number.

It is possible that a question may restrict the type of number that a placeholder might permit; e.g., integers only or positive real numbers.

- The following is a general definition:

Algebraic Expression: An *algebraic expression* is either

1. **A numerical symbol or a variable symbol, or**
2. **The result of placing previously generated algebraic expressions into the two blanks of one of the four operators ((__) + (__), (__) − (__), (__) × (__), (__) ÷ (__)) or into the base blank of an exponentiation with exponent that is a rational number.**

- For example, x and 3 are algebraic expressions, and from that we can create the expression $x+3$ by placing each into the blanks of the addition operator. From there, we can create the expression $x(x+3)$ by placing x and $x+3$ into the blanks of the multiplication operator, and so on. According to this general definition, we can also create expressions of the form $\left(\frac{x(x+3)}{3x^2}\right)^{-2}$.

- Our notion of two expressions being "equivalent" has also been vague. We can now pinpoint what we mean:

Two algebraic expressions are *equivalent* if we can convert one expression into the other by repeatedly applying the commutative, associative, and distributive properties and the properties of rational exponents to components of the first expression.

- Some final jargon:

Numerical Expression: A *numerical expression* is an algebraic expression that contains only numerical symbols (no variable symbols), which evaluate to a single number.

The expression, $3 \div 0$, is not a numerical expression.

Equivalent Numerical Expressions: Two numerical expressions are *equivalent* if they evaluate to the same number.

- For example, although we formally proved $(3 + 4) + 2$ equals $(2 + 4) + 3$ through the commutative and associative properties, it is more reasonable to note that they both evaluate to 9.

Note that $1 + 2 + 3$ and $1 \times 2 \times 3$, for example, are equivalent numerical expressions (they are both 6) but $a + b + c$ and $a \times b \times c$ are not equivalent expressions.

Lesson Summary

The commutative and associative properties represent key beliefs about the arithmetic of real numbers. These properties can be applied to algebraic expressions using variables that represent real numbers.

Two algebraic expressions are *equivalent* if we can convert one expression into the other by repeatedly applying the commutative, associative, and distributive properties and the properties of rational exponents to components of the first expression.

Exit Ticket (3 minutes)

Name ____ Date____________________

Lesson 7: Algebraic Expressions—The Commutative and Associative Properties

Exit Ticket

Write a mathematical proof of the algebraic equivalence of $(pq)r$ and $(qr)p$.

Exit Ticket Sample Solution

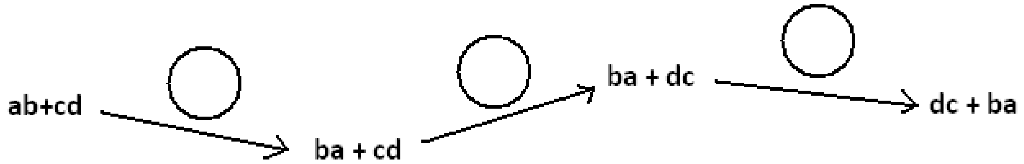
Write a mathematical proof of the algebraic equivalence of $(pq)r$ and $(qr)p$.

$(pq)r = p(qr)$ *associative property*

$= (qr)p$ *commutative property*

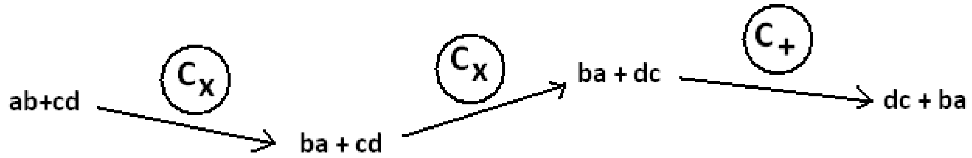
Problem Set Sample Solutions

1. **The following portion of a flow diagram shows that the expression $ab + cd$ is equivalent to the expression $dc + ba$.**



Fill in each circle with the appropriate symbol: Either C_+ (for the "commutative property of addition") or $C_\times$ (for the "commutative property of multiplication").

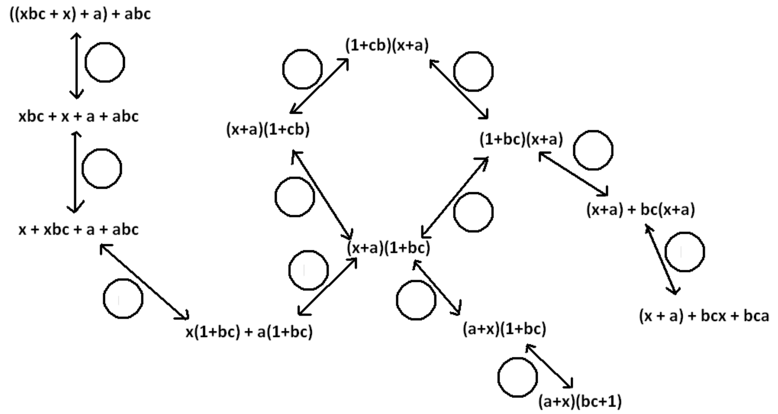
Answer:



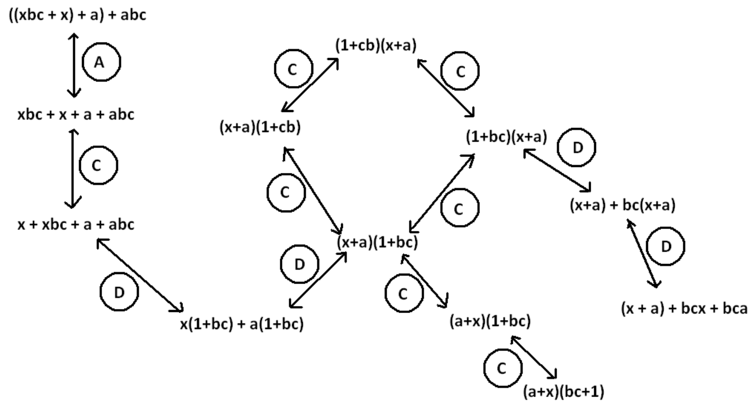
2. **Fill in the blanks of this proof showing that $(w+5)(w+2)$ is equivalent $w^2 + 7w + 10$. Write either "commutative property," "associative property," or "distributive property" in each blank.**

$(w+5)(w+2)$	$= (w+5)w + (w+5) \times 2$	*distributive property*
	$= w(w+5) + (w+5) \times 2$	*commutative property*
	$= w(w+5) + 2(w+5)$	*commutative property*
	$= w^2 + w \times 5 + 2(w+5)$	*distributive property*
	$= w^2 + 5w + 2(w+5)$	*commutative property*
	$= w^2 + 5w + 2w + 10$	*distributive property*
	$= w^2 + (5w + 2w) + 10$	*associative property*
	$= w^2 + 7w + 10$	

3. **Fill in each circle of the following flow diagram with one of the letters: C for commutative property (for either addition or multiplication), A for associative property (for either addition or multiplication), or D for distributive property.**



Answer:



4. **What is a quick way to see that the value of the sum $53 + 18 + 47 + 82$ is 200?**

$$53 + 18 + 47 + 82 = (53 + 47) + (18 + 82) = 100 + 100$$

5.

a. **If $ab = 37$ and $xy = \frac{1}{37}$, what is the value of the product $x \times b \times y \times a$?**

$$x \times b \times y \times a = (xy)(ab) = 1$$

b. **Give some indication as to how you used the commutative and associative properties of multiplication to evaluate $x \times b \times y \times a$ in part (a).**

$x \times b \times y \times a = x \times y \times a \times b$ *by two applications of the commutative property of multiplication and* $x \times y \times a \times b = (xy)(ab)$ *by the associative property of multiplication.*

c. **Did you use the associative and commutative properties of addition to answer Question 4?**

Yes, they were used in an analogous manner.

6. **The following is a proof of the algebraic equivalency of $(2x)^3$ and $8x^3$. Fill in each of the blanks with either the statement "commutative property" or "associative property."**

$(2x)^3 = 2x \cdot 2x \cdot 2x$

$= 2(x \times 2)(x \times 2)x$ *associative property*

$= 2(2x)(2x)x$ *commutative property*

$= 2 \cdot 2(x \times 2)x \cdot x$ *associative property*

$= 2 \cdot 2(2x)x \cdot x$ *commutative property*

$= (2 \cdot 2 \cdot 2)(x \cdot x \cdot x)$ *associative property*

$= 8x^3$

7. **Write a mathematical proof of the algebraic equivalency of $(ab)^2$ and a^2b^2.**

$(ab)^2 = (ab)(ab)$

$= a(ba)b$ *associative property*

$= a(ab)b$ *commutative property*

$= (aa)(bb)$ *associative property*

$= a^2b^2$

8.

a. **Suppose we are to play the 4-number game with the symbols a, b, c, and d to represent numbers, each used at most once, combined by the operation of addition ONLY. If we acknowledge that parentheses are unneeded, show there are essentially only 15 expressions one can write.**

By also making use of the commutative property of addition, we have the expressions:

$a,\ b,\ c,\ d,\ a+b,\ a+c,\ a+d,\ b+c,\ b+d,\ c+d,$

$a+b+c,\ a+b+d,\ a+c+d,\ b+c+d,\ a+b+c+d$

b. **How many answers are there for the multiplication ONLY version of this game?**

By analogous reasoning, there are only 15 expressions here too.

9. **Write a mathematical proof to show that $(x+a)(x+b)$ is equivalent to $x^2 + ax + bx + ab$.**

$(x+a)(x+b) = (x+a)x + (x+a)b$ (D)

$= x(x+a) + b(x+a)$ (C)

$= x^2 + xa + bx + ba$ (D)

$= x^2 + ax + bx + ab$ (C)

10. Recall the following rules of exponents:

$$x^a \cdot x^b = x^{a+b} \qquad \frac{x^a}{x^b} = x^{a-b} \qquad (x^a)^b = x^{ab}$$

$$(xy)^a = x^a y^a \qquad \left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}$$

Here x, y, a, and b are real numbers with x and y non-zero.

Replace each of the following expressions with an equivalent expression in which the variable of the expression appears only once with a positive number for its exponent. (For example, $\frac{7}{b^2} \cdot b^{-4}$ is equivalent to $\frac{7}{b^6}$.)

a. $(16x^2) \div (16x^5)$

$\frac{1}{x^3}$

b. $(2x)^4(2x)^3$

$128x^7$

c. $(9z^{-2})(3z^{-1})^{-3}$

$\frac{z}{3}$

d. $\left((25w^4) \div (5w^3)\right) \div (5w^{-7})$

w^8

e. $(25w^4) \div \left((5w^3) \div (5w^{-7})\right)$

$\frac{25}{w^6}$

Optional Challenge:

11. Grizelda has invented a new operation that she calls the "average operator." For any two real numbers a and b, she declares $a \oplus b$ to be the average of a and b:

$$a \oplus b = \frac{a+b}{2}$$

a. Does the average operator satisfy a commutative property? That is, does $a \oplus b = b \oplus a$ for all real numbers a and b?

Yes, use the fact that $\frac{x}{2} = \frac{1}{2} \cdot x$ for any real number x and the commutative property.

$$a \oplus b = \frac{a+b}{2} = \frac{1}{2} \cdot (a+b) = \frac{1}{2} \cdot (b+a) = \frac{b+a}{2} = b \oplus a$$

b. Does the average operator distribute over addition? That is, does $a \oplus (b+c) = (a \oplus b) + (a \oplus c)$ for all real numbers a, b, and c?

No. For instance, $2 \oplus (4+4) = 2 \oplus 8 = 5$, whereas $(2 \oplus 4) + (2 \oplus 4) = 3 + 3 = 6$.

Lesson 8: Adding and Subtracting Polynomials

Student Outcomes

- Students understand that the sum or difference of two polynomials produces another polynomial and relate polynomials to the system of integers; students add and subtract polynomials.

Classwork

Exercise 1 (7 minutes)

Have students complete Exercise 1(a) and use it for a brief discussion on the notion of base. Then have students continue with the remainder of the exercise.

Exercise 1

a. **How many quarters, nickels, and pennies are needed to make $\$1.13$?**

Answers will vary.

4 quarters, 2 nickels, 3 pennies

b. **Fill in the blanks:**

$\mathbf{8,943 = \underline{\ 8\ } \times 1000 + \underline{\ 9\ } \times 100 + \underline{\ 4\ } \times 10 + \underline{\ 3\ } \times 1}$

$\mathbf{= \underline{\ 8\ } \times 10^3 + \underline{\ 9\ } \times 10^2 + \underline{\ 4\ } \times 10 + \underline{\ 3\ } \times 1}$

c. **Fill in the blanks:**

$\mathbf{8,943 = \underline{\ 1\ } \times 20^3 + \underline{\ 2\ } \times 20^2 + \underline{\ 7\ } \times 20 + \underline{\ 3\ } \times 1}$

d. **Fill in the blanks:**

$\mathbf{113 = \underline{\ 4\ } \times 5^2 + \underline{\ 2\ } \times 5 + \underline{\ 3\ } \times 1}$

Next ask:

- Why do we use base 10? Why do we humans have a predilection for the number 10?
- Why do some cultures have base 20?
- How do you say “87” in French? How does the Gettysburg address begin?
    - *Quatre-vingt-sept: 4-20s and 7; Four score and seven years ago…*
- Computers use which base system?
    - *Base 2*

Extension:

- Mayan, Aztec, and Celtic all used base 20. The word “score” (which means 20) originated from the Celtic language.
- Students could be asked to research more on this and on the cultures who use or used base 5 and base 60.

Exercise 2 (5 minutes)

In Exercise 2, we are laying the foundation that polynomials written in standard form are simply base x "numbers." The practice of filling in specific values for x and finding the resulting values lays a foundation for connecting this algebra of polynomial expressions with the later lessons on polynomial functions (and other functions) and their inputs and outputs.

Work through Exercise 2 with the class.

Exercise 2

Now let's be as general as possible by not identifying which base we are in. Just call the base x.

Consider the expression $1 \times x^3 + 2 \times x^2 + 7 \times x + 3 \times 1$, or equivalently $x^3 + 2x^2 + 7x + 3$.

a. **What is the value of this expression if $x = 10$?**

$1,273$

b. **What is the value of this expression if $x = 20$?**

$8,943$

Point out that the expression we see here is just the generalized form of their answer from part (b) of Exercise 1. However, as we change x, we get a different number each time.

Exercise 3 (10 minutes)

Allow students time to complete Exercise 3 individually. Then elicit responses from the class.

Exercise 3

a. **When writing numbers in base 10, we only allow coefficients of 0 through 9. Why is that?**

Once you get ten of a given unit, you also have one of the unit to the left of that.

b. **What is the value of $22x + 3$ when $x = 5$? How much money is 22 nickels and 3 pennies?**

113

$\$1.13$

c. **What number is represented by $4x^2 + 17x + 2$ if $x = 10$?**

572

d. **What number is represented by $4x^2 + 17x + 2$ if $x = -2$ or if $x = \frac{2}{3}$?**

-16

$\frac{136}{9}$

e. **What number is represented by $-3x^2 + \sqrt{2}x + \frac{1}{2}$ when $x = \sqrt{2}$?**

$-\frac{7}{2}$

Point out, as highlighted by Exercises 1 and 3, that carrying is not necessary in this type of expression (polynomial expressions). For example, $4x^2 + 17x + 2$ is a valid expression. However, in base ten arithmetic, coefficients of value ten or greater are not conventional notation. Setting $x = 10$ in $4x^2 + 17x + 2$ yields 4 hundreds, 17 tens, and 2 ones, which is to be expressed as 5 hundreds, 7 tens, and 2 ones.

Discussion (11 minutes)

- The next item in your student materials is a definition for a polynomial expression. Read the definition carefully, and then create 3 polynomial expressions using the given definition.

Polynomial Expression: A *polynomial expression* is either

1. **A numerical expression or a variable symbol, or**
2. **The result of placing two previously generated polynomial expressions into the blanks of the addition operator (__+__) or the multiplication operator (__×__).**

- Compare your polynomial expressions with a neighbor's. Do your neighbor's expressions fall into the category of polynomial expressions?

Resolve any debates as to whether a given expression is indeed a polynomial expression by referring back to the definition and discussing as a class.

- Note that the definition of a polynomial expression includes subtraction (add the additive inverse instead), dividing by a non-zero number (multiply by the multiplicative inverse instead), and even exponentiation by a non-negative integer (use the multiplication operator repeatedly on the same numerical or variable symbol).

List several of the student-generated polynomials on the board. Include some that contain more than one variable.

Initiate the following discussion, presenting expressions on the board when relevant.

- Just as the expression $(3 + 4) \cdot 5$ is a numerical expression but not a number, $(x + 5) + (2x^2 - x)(3x + 1)$ is a polynomial expression but not technically a **polynomial.** We reserve the word *polynomial* for polynomial expressions that are written simply as a sum of monomial terms. This begs the question: What is a monomial?
- A **monomial** is a polynomial expression generated using only the multiplication operator (__×__). Thus, it does not contain + or − operators.
- Just as we would not typically write a number in factored form and still refer to it as a number (we might call it a number in factored form), similarly, we do not write a monomial in factored form and still refer to it as a monomial. We multiply any numerical factors together and condense multiple instances of a variable factor using (whole number) exponents.
- Try creating a monomial.
- Compare the monomial you created with your neighbor's. Is your neighbor's expression really a monomial? Is it written in the standard form we use for monomials?
- There are also such things as binomials and trinomials. Can anyone make a conjecture about what a binomial is and what a trinomial is and how they are the same or different from a polynomial?

Students may conjecture that a binomial has two of something and that a trinomial three of something. Further, they might conjecture that a polynomial has many of something. Allow for discussion and then state the following:

- A binomial is the sum (or difference) of two monomials. A trinomial is the sum (or difference) of three monomials. A polynomial, as stated earlier, is the sum of one or more monomials.
- The ***degree* of a monomial** is the sum of the exponents of the variable symbols that appear in the monomial.
- The ***degree* of a polynomial** is the degree of the monomial term with the highest degree.
- While polynomials can contain multiple variable symbols, most of our work with polynomials will be with **polynomials in one variable**.
- What do polynomial expressions in one variable look like? Create a polynomial expression in one variable and compare with your neighbor.

Post some of the student generated polynomials in one variable on the board.

- Let's relate polynomials to the work we did at the beginning of the lesson.
- Is this expression an integer in base 10? $10(100 + 22 - 2) + 3(10) + 8 - 2(2)$
- Is the expression equivalent to the integer $1{,}234$?
- How did we find out?
- We rewrote the first expression in our standard form, right?
- Polynomials in one variable have a standard form as well. Use your intuition of what standard form of a polynomial might be to write this polynomial expression as a polynomial in standard form: $2x(x^2 - 3x + 1) - (x^3 + 2)$ and compare your result with your neighbor.
  - *Students should arrive at the answer $x^3 - 6x^2 + 2x - 2$.*

Confirm that in standard form, we start with the highest degreed monomial, and continue in descending order.

- The **leading term** of a polynomial is the term of highest degree that would be written first if the polynomial is put into standard form. The **leading coefficient** is the coefficient of the leading term.
- What would you imagine we mean when we refer to the **constant term** of the polynomial?
  - *A constant term is any term with no variables. To find "the constant" term of a polynomial, be sure you have combined any and all constant terms into one single numerical term, written last if the polynomial is put into standard form. Note that a polynomial does not have to have a constant term (or could be said to have a constant term of 0).*

As an extension for advanced students, assign the task of writing of a formal definition for *standard form* of a polynomial. The formal definition is provided below for your reference:

> A polynomial expression with one variable symbol x is in **standard form** if it is expressed as, $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where n is a non-negative integer, and $a_0, a_1, a_2, \ldots, a_n$ are constant coefficients with $a_n \neq 0$. A polynomial expression in x that is in standard form is often called a *polynomial in x*.

Exercise 4 (5 minutes)

Exercise 4

Find each sum or difference by combining the parts that are alike.

a. $417 + 231 =$ __4__ **hundreds** $+$ __1__ **tens** $+$ __7__ **ones** $+$ __2__ **hundreds** $+$ __3__ **tens** $+$ __1__ **ones**
$=$ __6__ **hundreds** $+$ __4__ **tens** $+$ __8__ **ones**

b. $(4x^2 + x + 7) + (2x^2 + 3x + 1)$

$6x^2 + 4x + 8$

c. $(3x^3 - x^2 + 8) - (x^3 + 5x^2 + 4x - 7)$

$2x^3 - 6x^2 - 4x + 15$

d. $3(x^3 + 8x) - 2(x^3 + 12)$

$x^3 + 24x - 24$

e. $(5 - t - t^2) + (9t + t^2)$

$8t + 5$

f. $(3p + 1) + 6(p - 8) - (p + 2)$

$8p - 49$

Closing (3 minutes)

- How are polynomials analogous to integers?
  - *While integers are in base* 10*, polynomials are in base* x*.*
- If you add two polynomials together, is the result sure to be another polynomial? The difference of two polynomials?
  - *Students will likely reply, "yes," based on the few examples and their intuition.*
- Are you sure? Can you think of an example where adding or subtracting two polynomials does not result in a polynomial?
  - *Students thinking about* $x^2 - x^2 = 0$ *could suggest not. At this point, review the definition of a polynomial. Constant symbols are polynomials.*

Lesson Summary

A monomial is a polynomial expression generated using only the multiplication operator (__×__). Thus, it does not contain + or − operators. Monomials are written with numerical factors multiplied together and variable or other symbols each occurring one time (using exponents to condense multiple instances of the same variable).

A polynomial is the sum (or difference) of monomials.

The *degree* of a monomial is the sum of the exponents of the variable symbols that appear in the monomial.

The *degree* of a polynomial is the degree of the monomial term with the highest degree.

Exit Ticket (4 minutes)

Name ____ Date____________________

Lesson 8: Adding and Subtracting Polynomials

Exit Ticket

1. Must the sum of three polynomials again be a polynomial?

2. Find $(w^2 - w + 1) + (w^3 - 2w^2 + 99)$.

Exit Ticket Sample Solutions

1. **Must the sum of three polynomials again be a polynomial?**

   Yes.

2. **Find $(w^2 - w + 1) + (w^3 - 2w^2 + 99)$.**

   $w^3 - w^2 - w + 100$

Problem Set Sample Solutions

1. **Celina says that each of the following expressions is actually a binomial in disguise:**

   i. $5abc - 2a^2 + 6abc$

   ii. $5x^3 \cdot 2x^2 - 10x^4 + 3x^5 + 3x \cdot (-2)x^4$

   iii. $(t+2)^2 - 4t$

   iv. $5(a-1) - 10(a-1) + 100(a-1)$

   v. $(2\pi r - \pi r^2)r - (2\pi r - \pi r^2) \cdot 2r$

   For example, she sees that the expression in (i) is algebraically equivalent to $11abc - 2a^2$, which is indeed a binomial. (She is happy to write this as $11abc + (-2)a^2$, if you prefer.)

   Is she right about the remaining four expressions?

   She is right about the remaining four expressions. They all can be expressed as binomials.

2. **Janie writes a polynomial expression using only one variable, x, with degree 3. Max writes a polynomial expression using only one variable, x, with degree 7.**

   a. **What can you determine about the degree of the sum of Janie's and Max's polynomials?**

      The degree would be 7.

   b. **What can you determine about the degree of the difference of Janie's and Max's polynomials?**

      The degree would be 7.

3. **Suppose Janie writes a polynomial expression using only one variable, x, with degree of 5, and Max writes a polynomial expression using only one variable, x, with degree of 5.**

   a. **What can you determine about the degree of the sum of Janie's and Max's polynomials?**

      The maximum degree could be 5, but it could also be anything less than that. For example, if Janie's polynomial were $x^5 + 3x - 1$, and Max's were $-x^5 + 2x^2 + 1$, the degree of the sum is only 2.

   b. **What can you determine about the degree of the difference of Janie's and Max's polynomials?**

      The maximum degree could be 5, but it could also be anything less than that.

4. Find each sum or difference by combining the parts that are alike.

a. $(2p + 4) + 5(p - 1) - (p + 7)$

$6p - 8$

b. $(7x^4 + 9x) - 2(x^4 + 13)$

$5x^4 + 9x - 26$

c. $(6 - t - t^4) + (9t + t^4)$

$8t + 6$

d. $(5 - t^2) + 6(t^2 - 8) - (t^2 + 12)$

$4t^2 - 55$

e. $(8x^3 + 5x) - 3(x^3 + 2)$

$5x^3 + 5x - 6$

f. $(12x + 1) + 2(x - 4) - (x - 15)$

$13x + 8$

g. $(13x^2 + 5x) - 2(x^2 + 1)$

$11x^2 + 5x - 2$

h. $(9 - t - t^2) - \frac{3}{2}(8t + 2t^2)$

$-4t^2 - 13t + 9$

i. $(4m + 6) - 12(m - 3) + (m + 2)$

$-7m + 44$

j. $(15x^4 + 10x) - 12(x^4 + 4x)$

$3x^4 - 38x$

Lesson 9: Multiplying Polynomials

Student Outcomes

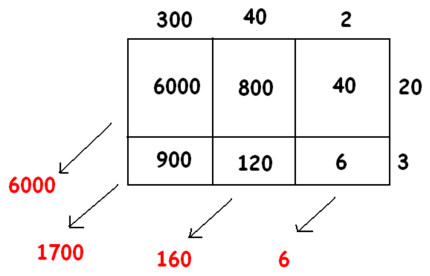
- Students understand that the product of two polynomials produces another polynomial; students multiply polynomials.

Classwork

Exercise 1 (15 minutes)

Exercise 1

a. Gisella computed 342×23 as follows:



Can you explain what she is doing? What is her final answer?

She is using an area model, finding the area of each rectangle and adding them together. Her final answer is $7,866$.

Use a geometric diagram to compute the following products:

b. $(3x^2 + 4x + 2) \times (2x + 3)$

$6x^3 + 17x^2 + 16x + 6$

c. $(2x^2 + 10x + 1)(x^2 + x + 1)$

$2x^4 + 12x^3 + 13x^2 + 11x + 1$

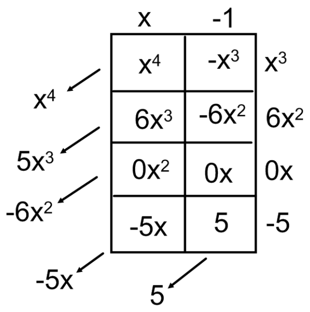
d. $(x - 1)(x^3 + 6x^2 - 5)$

$x^4 + 5x^3 - 6x^2 - 5x + 5$

Ask the students:

- What do you notice about the terms along the diagonals in the rectangles you drew?

Encourage students to recognize that in parts (b) and (c), the terms along the diagonals were all like terms; however, in part (d) one of the factors has no x-term. Allow students to develop a strategy for dealing with this, concluding with the suggestion of inserting the term $+\ 0x$, for a model that looks like the following:



x	-1	
x^4	$-x^3$	x^3
$6x^3$	$-6x^2$	$6x^2$
$0x^2$	$0x$	$0x$
$-5x$	5	-5

Students may naturally ask about the division of polynomials. This topic will be covered in Grade 11, Module 1. The extension challenge at the end of the lesson, however, could be of interest to students inquiring about this.

- Could we have found this product without the aid of a geometric model? What would that look like?

Go through the exercise applying the distributive property and collecting like terms. As a scaffold, remind students that variables are placeholders for numbers. If $x = 5$, for example, whatever the quantity on the right is (270), you have $5 - 1$ of "that quantity", or 5 of "that quantity" minus 1 of "that quantity". Similarly we have x of that quantity, minus 1 of that quantity:

$$(x-1)(x^3+6x^2-5)$$

$$x(x^3+6x^2-5)-1(x^3+6x^2-5)$$

$$x^4+6x^3-5x-\ x^3-6x^2+5$$

$$x^4+5x^3-6x^2-5x+5$$

Exercise 2 (5 minutes)

Have students work Exercise 2 independently and then compare answers with a neighbor. If needed, facilitate agreement on the correct answer by allowing students to discuss their thought processes and justify their solutions.

Exercise 2

Multiply the polynomials using the distributive property: $(3x^2+x-1)(x^4\ -\ 2x\ +\ 1)$.

$$3x^2(x^4-2x+1)+x(x^4-2x+1)-1(x^4-2x+1)$$

$$3x^6-6x^3+3x^2+x^5-2x^2+x-x^4+2x-1$$

$$3x^6+x^5-x^4-6x^3+x^2+3x-1$$

Exercise 3 (10 minutes)

Give students 10 minutes to complete Exercise 3 and compare their answers with a neighbor.

Exercise 3

The expression $10x^2 + 6x^3$ is the result of applying the distributive property to the expression $2x^2(5 + 3x)$. It is also the result of the applying the distributive property to $2(5x^2 + 3x^3)$ or to $x(10x + 6x^2)$, for example, or even to $1 \cdot (10x^2 + 6x^3)$!

For (i) to (x) below, write down an expression such that if you applied the distributive property to your expression it will give the result presented. Give interesting answers!

i. $6a + 14a^2$

ii. $2x^4 + 2x^5 + 2x^{10}$

iii. $6z^2 - 15z$

iv. $42w^3 - 14w + 77w^5$

v. $z^2(a + b) + z^3(a + b)$

vi. $\frac{3}{2}s^2 + \frac{1}{2}$

vii. $15p^3r^4 - 6p^2r^5 + 9p^4r^2 + 3\sqrt{2}p^3r^6$

viii. $0.4x^9 - 40x^8$

ix. $(4x + 3)(x^2 + x^3) - (2x + 2)(x^2 + x^3)$

x. $(2z + 5)(z - 2) - (13z - 26)(z - 3)$

Some possible answers:

i. $2a(3 + 7a)$ *or* $2(3a + 7a^2)$ *or* $a(6 + 14a)$

ii. $2x^4(1 + x + x^6)$ *or* $x(2x^3 + 2x^4 + 2x^9)$ *or* $2(x^4 + x^5 + x^{10})$

iii. $3z(2z - 5)$ *or* $3(2z^2 - 5z)$ *or* $z(6z - 15)$

iv. $7w(6w^2 - 2 + 11w^4)$ *or* $w(42w^2 - 14 + 77w^4)$

v. $z^2\big((a + b) + z(a + b)\big)$ *or* $z(z(a + b) + z^2(a + b))$

vi. $\frac{1}{2}(3s^2 + 1)$

vii. $3p^2r^2(5pr^2 - 2r^3 + 3p^2 + \sqrt{2}pr^4)$ *or* $p^2r^2(15pr^2 - 6r^3 + 9p^2 + 3\sqrt{2}pr^4)$

viii. $0.4x^8(x - 100)$ *or* $\frac{4}{10}x^8(x - 100)$

ix. $(x^2 + x^3)((4x + 3) - (2x + 2))$

x. $(z - 2)\big((2z + 5) - 13(z - 3)\big)$

Choose one (or more) to go through as a class, listing as many different re-writes as possible. Then remark:

- The process of making use of the distributive property "backwards" is factoring.

Exercise 4 (5 minutes)

Exercise 4

Sammy wrote a polynomial using only one variable, x, of degree 3. Myisha wrote a polynomial in the same variable of degree 5. What can you say about the degree of the product of Sammy's and Myisha's polynomials?

The degree of the product of the two polynomials would be 8.

Extension

Extension

Find a polynomial that, when multiplied by $2x^2 + 3x + 1$, gives the answer $2x^3 + x^2 - 2x - 1$.

$x - 1$

Closing (6 minutes)

- Is the product of two polynomials sure to be another polynomial?
    - *Yes, by the definition of polynomial expression given in Lesson 8, the product of any two polynomial expressions is again a polynomial expression, which can then be written in standard polynomial form through application of the distributive property.*
- Is a polynomial squared sure to be another polynomial (other integer powers)?
    - *Yes.*

Exit Ticket (4 minutes)

Name ____ Date____________________

Lesson 9: Multiplying Polynomials

Exit Ticket

1. Must the product of three polynomials again be a polynomial?

2. Find $(w^2 + 1)(w^3 - w + 1)$.

Exit Ticket Sample Solutions

1. **Must the product of three polynomials again be a polynomial?**

   Yes.

2. **Find $(w^2+1)(w^3-w+1)$.**

   w^5+w^2-w+1

Problem Set Sample Solutions

1. **Use the distributive property to write each of the following expressions as the sum of monomials.**

   a. $3a(4+a)$
   $3a^2+12a$

   b. $x(x+2)+1$
   x^2+2x+1

   c. $\frac{1}{3}(12z+18z^2)$
   $6z^2+4z$

   d. $4x(x^3-10)$
   $4x^4-40x$

   e. $(x-4)(x+5)$
   x^2+x-20

   f. $(2z-1)(3z^2+1)$
   $6z^3-3z^2+2z-1$

   g. $(10w-1)(10w+1)$
   $100w^2-1$

   h. $(-5w-3)w^2$
   $-5w^3-3w^2$

   i. $16s^{100}\left(\frac{1}{2}s^{200}+0.125s\right)$
   $8s^{300}+2s^{101}$

   j. $(2q+1)(2q^2+1)$
   $4q^3+2q^2+2q+1$

   k. $(x^2-x+1)(x-1)$
   x^3-2x^2+2x-1

   l. $3xz(9xy+z)-2yz(x+y-z)$
   $27x^2yz+3xz^2-2xyz-2y^2z+2yz^2$

   m. $(t-1)(t+1)(t^2+1)$
   t^4-1

   n. $(w+1)(w^4-w^3+w^2-w+1)$
   w^5+1

   o. $z(2z+1)(3z-2)$
   $6z^3-z^2-2z$

   p. $(x+y)(y+z)(z+x)$
   $2xyz+x^2y+x^2z+xy^2+xz^2+y^2z+yz^2$

   q. $\frac{x+y}{3}$
   $\frac{1}{3}x+\frac{1}{3}y$

   r. $(20f^{10}-10f^5)\div 5$
   $4f^{10}-2f^5$

   s. $-5y(y^2+y-2)-2(2-y^3)$
   $-3y^3-5y^2+10y-4$

   t. $\frac{(a+b-c)(a+b+c)}{17}$
   $\frac{1}{17}a^2+\frac{1}{17}b^2-\frac{1}{17}c^2+\frac{2}{17}ab$

   u. $(2x\div 9+(5x)\div 2)\div(-2)$
   $-\frac{49x}{36}$

   v. $(-2f^3-2f+1)(f^2-f+2)$
   $-2f^5+2f^4-6f^3+3f^2-5f+2$

2. **Use the distributive property (and your wits!) to write each of the following expressions as a sum of monomials. If the resulting polynomial is in one variable, write the polynomial in standard form.**

   a. $(a+b)^2$
   $a^2+2ab+b^2$

   b. $(a+1)^2$
   a^2+2a+1

   c. $(3+b)^2$
   b^2+6b+9

   d. $(3+1)^2$
   16

   e. $(x+y+z)^2$
   $x^2+y^2+z^2+2xy+2xz+2yz$

   f. $(x+1+z)^2$
   $x^2+z^2+2xz+2x+2z+1$

   g. $(3+z)^2$
   z^2+6z+9

   h. $(p+q)^3$
   $p^3+3p^2q+3pq^2+q^3$

   i. $(p-1)^3$
   p^3-3p^2+3p-1

   j. $(5+q)^3$
   $q^3+15q^2+75q+125$

3. **Use the distributive property (and your wits!) to write each of the following expressions as a polynomial in standard form.**

   a. $(s^2+4)(s-1)$
   s^3-s^2+4s-4

   b. $3(s^2+4)(s-1)$
   $3s^3-3s^2+12s-12$

   c. $s(s^2+4)(s-1)$
   $s^4-s^3+4s^2-4s$

   d. $(s+1)(s^2+4)(s-1)$
   s^4+3s^2-4

   e. $(u-1)(u^5+u^4+u^3+u^2+u+1)$
   u^6-1

   f. $\sqrt{5}(u-1)(u^5+u^4+u^3+u^2+u+1)$
   $\sqrt{5}u^6-\sqrt{5}$

   g. $(u^7+u^3+1)(u-1)(u^5+u^4+u^3+u^2+u+1)$
   $u^{13}+u^9-u^7+u^6-u^3-1$

4. **Beatrice writes down every expression that appears in this problem set, one after the other, linking them with "+" signs between them. She is left with one very large expression on her page. Is that expression a polynomial expression? That is, is it algebraically equivalent to a polynomial?**

   Yes.

   What if she wrote "−" signs between the expressions instead?

   Yes.

   What if she wrote "×" signs between the expressions instead?

   Yes.

Name ______________________________ Date ______________

1. Jacob lives on a street that runs east and west. The grocery store is to the east and the post office is to the west of his house. Both are on the same street as his house. Answer the questions below about the following story:

   At $1:00$ p.m., Jacob hops in his car and drives at a constant speed of 25 mph for 6 minutes to the post office. After 10 minutes at the post office, he realizes he is late and drives at a constant speed of 30 mph to the grocery store, arriving at $1:28$ p.m. He then spends 20 minutes buying groceries.

   a. Draw a graph that shows the distance Jacob's car is from his house with respect to time. Remember to label your axes with the units you chose and any important points (home, post office, grocery store).

b. On the way to the grocery store, Jacob looks down at his watch and notes the time as he passes his house. What time is it when he passes his house? Explain how you found your answer.

c. If he drives directly back to his house after the grocery story, what was the total distance he traveled to complete his errands? Show how you found your answer.

2. Jason is collecting data on the rate of water usage in the tallest skyscraper in the world during a typical day. The skyscraper contains both apartments and businesses. The electronic water meter for the building displays the total amount of water used in liters. At noon, Jason looks at the water meter and notes that the digit in the **ones** place on the water meter display changes too rapidly to read the digit and that the digit in the **tens** place changes every second or so.

   a. Estimate the total number of liters used in the building during one 24-hour day. Take into account the time of day when he made his observation. (Hint: Will water be used at the same rate at $2{:}00$ a.m. as at noon?) Explain how you arrived at your estimate.

   b. To what level of accuracy can Jason reasonably report a measurement if he takes it at precisely $12{:}00$ p.m.? Explain your answer.

   c. The meter will be checked at regular time spans (for example, every minute, every 10 minutes, and every hour). What is the minimum (or smallest) number of checks needed in a 24-hour period to create a reasonably accurate graph of the water usage **rate** with respect to time? (For example, 24 checks would mean checking the meter every hour; 48 checks would mean checking the meter every half hour.) Defend your choice by describing how the water usage rate might change during the day and how your choice could capture that change.

3. A publishing company orders black and blue ink in bulk for its two-color printing press. To keep things simple with its ink supplier, each time it places an order for blue ink, it buys B gallons, and each time it places an order for black ink, it buys K gallons. Over a one-month period, the company places m orders of blue ink and n orders of black ink.

   a. What quantities could the following expressions represent in terms of the problem context?

   $m + n$

   $mB + nK$

   $$\frac{mB + nK}{m + n}$$

   b. The company placed twice as many orders for black ink than for blue ink in January. Give interpretations for the following expressions in terms of the orders placed in January,

   $$\frac{m}{m+n} \quad \text{and} \quad \frac{n}{m+n},$$

   and explain which expression must be greater using those interpretations.

4. Sam says that he knows a clever set of steps to rewrite the expression

$$(x+3)(3x+8)-3x(x+3)$$

as a sum of two terms where the steps do not involve multiplying the linear factors first and then collecting like terms. Rewrite the expression as a sum of two terms (where one term is a number and the other is a product of a coefficient and variable) using Sam's steps if you can.

5. Using only the addition and multiplication operations with the numbers 1, 2, 3, and 4 each exactly once, it is possible to build a numeric expression (with parentheses to show the order used to build the expression) that evaluates to 21. For example, $1+((2+3)\cdot 4)$ is one such expression.

   a. Build two more numeric expressions that evaluate to 21 using the criteria above. Both must be different from the example given.

b. In both of your expressions, replace 1 with a, 2 with b, 3 with c, and 4 with d to get two algebraic expressions. For example, $a + \big((b + c) \cdot d\big)$ shows the replacements for the example given.

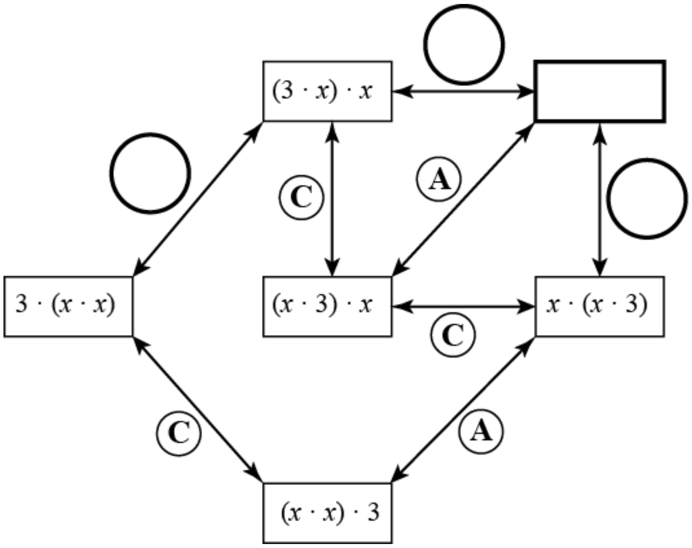
Are your algebraic expressions equivalent? Circle: Yes No

- If they are equivalent, prove that they are using the properties of operations.
- If not, provide **two** examples:

  (1) Find four different numbers (other than 0, 1, 2, 3, 4) that when substituted for a, b, c, and d into each expression, the expressions evaluate to **different numbers**, and

  (2) Find four different, non-zero numbers that when substituted into each expression, the expressions evaluate to the **same number**.

6. The diagram below, when completed, shows all possible ways to build equivalent expressions of $3x^2$ using multiplication. The equivalent expressions are connected by labeled segments stating which property of operations, **A** for associative property and **C** for commutative property, justifies why the two expressions are equivalent. Answer the following questions about $3x^2$ and the diagram.



a. Fill in the empty circles with **A** or **C** and the empty rectangle with the missing expression to complete the diagram.

b. Using the diagram above to help guide you, give *two different* proofs that $(x \cdot x) \cdot 3 = (3 \cdot x) \cdot x$.

7. Ahmed learned: "To multiply a whole number by ten, just place a zero at the end of the number." For example, 2813×10, he says, is 28,130. He doesn't understand why this "rule" is true.

a. What is the product of the polynomial, $2x^3 + 8x^2 + x + 3$, times the polynomial, x?

b. Use part (a) as a hint. Explain why the rule Ahmed learned is true.

8.

a. Find the following products:

i. $(x-1)(x+1)$

ii. $(x-1)(x^2 + x + 1)$

iii. $(x-1)(x^3 + x^2 + x + 1)$

iv. $(x-1)(x^4 + x^3 + x^2 + x + 1)$

v. $(x-1)(x^n + x^{n-1} + \cdots + x^3 + x^2 + x + 1)$

b. Substitute $x = 10$ into each of the products from parts (i) through (iv) and your answers to show how each of the products appears as a statement in arithmetic.

c. If we substituted $x = 10$ into the product $(x-2)(x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1)$ and computed the product, what number would result?

d. Multiply $(x-2)$ and $(x^7+x^6+x^5+x^4+x^3+x^2+x+1)$, and express your answer in standard form.

Substitute $x=10$ into your answer, and see if you obtain the same result that you obtained in part (c).

e. Francois says $(x-9)(x^7+x^6+x^5+x^4+x^3+x^2+x+1)$ must equal $x^7+x^6+x^5+x^4+x^3+x^2+x+1$ because when $x=10$, multiplying by "$x-9$" is the same as multiplying by 1.

i. Multiply $(x-9)(x^7+x^6+x^5+x^4+x^3+x^2+x+1)$.

ii. Put $x=10$ into your answer.

Is it the same as $x^7+x^6+x^5+x^4+x^3+x^2+x+1$ with $x=10$?

iii. Was Francois right?

A Progression Toward Mastery

Assessment Task Item		STEP 1 Missing or incorrect answer and little evidence of reasoning or application of mathematics to solve the problem.	STEP 2 Missing or incorrect answer but evidence of some reasoning or application of mathematics to solve the problem.	STEP 3 A correct answer with some evidence of reasoning or application of mathematics to solve the problem <u>or</u> an incorrect answer with substantial evidence of solid reasoning or application of mathematics to solve the problem.	STEP 4 A correct answer supported by substantial evidence of solid reasoning or application of mathematics to solve the problem.
1	**a** **N-Q.A.1** **N-Q.A.2**	Student was unable to respond to question. <u>OR</u> Student provided a minimal attempt to create an incorrect graph.	Student created a graph that reflects something related to the problem, but the axes did not depict the correct units of distance from the house on the y-axis and a measurement of time on the x-axis, or the graph indicated significant errors in calculations or reasoning.	Student created axes that depict distance from the house on the y-axis and some measurement of time on the x-axis, and the graph represents a reflection of what occurred but with errors in calculations, missing or erroneous axis labels, or choice of units that makes the graph difficult to obtain information from.	Student created and labeled the y-axis to represent distance from the house in miles and an x-axis to represent time (in minutes past 1:00 p.m.) and created a graph based on solid reasoning and correct calculations.
	b **N-Q.A.1**	Student answered incorrectly with no evidence of reasoning to support the answer. <u>OR</u> Student left item blank.	Student answered incorrectly but demonstrated some reasoning in explaining the answer.	Student answered 1:21 p.m. but did not either refer to a correct graph or provide sound reasoning to support the answer. <u>OR</u> Student answered incorrectly because either the graph in part (a) was incorrect	Student answered 1:21 p.m. and either referred to a correct graph from part (a) or provided reasoning and calculations to explain the answer.

				and the graph was referenced or because a minor calculation error was made but sound reasoning was used.	
	c **N-Q.A.1**	Student answered incorrectly with no evidence of reasoning to support the answer. OR Student left item blank.	Student answered incorrectly but demonstrated some reasoning in explaining the answer.	Student answered 12 miles but did not either refer to the work in part (a) or provide sound reasoning in support of the answer. OR Student answered incorrectly because either the work in part (a) was referenced, but the work was incorrect or because a minor calculation error was made but sound reasoning was used.	Student answered 12 miles and either referenced correct work from part (a) or provided reasoning and calculations to support the answer.
2	**a** **N-Q.A.3**	Student left the question blank. OR Student provided an answer that reflected no or very little reasoning.	Student either began with an assumption that was not based on the evidence of water being used at a rate of approximately 10 liters/second at noon. OR Student used poor reasoning in extending that reading to consider total use across 24 hours.	Student answered beginning with the idea that water was being used at a rate of approximately 10 liters/second at noon but made an error in the calculations to extend and combine that rate to consider usage across 24 hours. OR Student did not defend the choice by explaining water usage across the 24 hours and how it compares to the reading taken at noon.	Student answered beginning with the idea that water was being used at a rate of approximately 10 liters/second at noon and made correct calculations to extend and combine that rate to consider usage across 24 hours. AND Student defended the choice by explaining water usage across the 24 hours and how it compares to the reading taken at noon.

	b **N-Q.A.3**	Student left the question blank. OR Student provided an answer that reflected no or very little reasoning.	Student provided an answer that is outside of the range from "to the nearest ten" to "to the nearest hundred". OR Student provided an answer that is within the range but is not supported by an explanation.	Student answer ranged from "to the nearest ten" to "to the nearest hundred" but was not well supported by sound reasoning. OR Student answer contained an error in the way the explanation was written, even if it was clear what the student meant to say.	Student answer ranged from "to the nearest ten" to "to the nearest hundred" and was supported by correct reasoning that is expressed accurately.
	c **N-Q.A.3**	Student left the question blank. OR Student provided an answer that reflected no or very little reasoning.	Student answer was not in the range of 6 to 48 checks but provided some reasoning to justify the choice. OR Student answer was in that range, perhaps written in the form of "every x minutes" or "every x hours" but was not supported by an explanation with solid reasoning.	Student answer was in the range of 6 to 48 checks but was only given in the form of x checks per minute or x checks per hour; the answer was well supported by a written explanation. OR Student answer was given in terms of number of checks but was not well supported by a written explanation.	Student answer was in the range of 6 to 48 checks, and student provided solid reasoning to support the answer.
3	**a** **A-SSE.A.1a** **A-SSE.A.1b**	Student either did not answer. OR Student answered incorrectly for all three expressions.	Student answered one or two of the three correctly but left the other one blank or made a gross error in describing what it represented.	Student answered two of the three correctly and made a reasonable attempt at describing what the other one represented.	Student answered all three correctly.
	b **A-SSE.A.1a** **A-SSE.A.1b**	Student either did not answer. OR Student answered incorrectly for all three parts of the question.	Student understood that the expressions represented a portion of the orders for each color but mis-assigned the colors and/or incorrectly	Student understood that the expressions represented a portion of the orders for each color and correctly determined which one would be larger	Student understood that the expressions represented a portion of the orders for each color, correctly determined which one would be larger,

			determined which one would be larger.	but had errors in the way the answer was worded <u>OR</u> did not provide support for why $\frac{n}{m+n}$ would be larger.	and provided a well written explanation for why.
4	**A-SSE.A.1b** **A-SSE.A.2**	Student left the question blank. <u>OR</u> Student was unable to re-write the expression successfully, even by multiplying out the factors first.	Student got to the correct re-written expression of $8x + 24$ but did so by multiplying out the factors first <u>OR</u> did not show the work needed to demonstrate how $8x + 24$ was determined.	Student attempted to use structure to re-write the expression as described, showing the process, but student made errors in the process.	Student correctly used the process described to arrive at $8x + 24$ without multiplying out linear factors and demonstrated the steps for doing so.
5	**a–b** **A-SSE.A.2**	Student was unable to respond to many of the questions. <u>OR</u> Student left several items blank.	Student was only able to come up with one option for part (a) and, therefore, had only partial work for part (b). <u>OR</u> Student answered “Yes” for the question about equivalent expressions.	Student successfully answered part (a) and identified that the expressions created in part (b) were not equivalent, but there were minor errors in the answering of the remaining questions.	Student answered all four parts correctly and completely.
6	**a** **A-SSE.A.2**	Student left at least three items blank. <u>OR</u> Student answered at least three items incorrectly.	Student answered one or two items incorrectly or left one or more items blank.	Student completed circling task correctly and provided a correct ordering of symbols in the box, but the answer did not use parentheses or multiplication dots.	Student completed all four item correctly, including exact placement of parentheses and symbols for the box: $x \cdot (3 \cdot x)$.
	b **A-SSE.A.2**	Student did not complete either proof successfully.	Student attempted both proofs but made minor errors in both. <u>OR</u> Student only completed one proof, with or without errors.	Student attempted both proofs but made an error in one of them.	Student completed both proofs correctly, and the two proofs were different from one another.

7	**a** **A-APR.A.1**	Student left the question blank or demonstrated no understanding of multiplication of polynomials.	Student made more than one error in his multiplication but demonstrated some understanding of multiplication of polynomials.	Student made a minor error in the multiplication.	Student multiplied correctly and expressed the resulting polynomial as a sum of monomials.
	b **A-APR.A.1**	Student left the question blank or did not demonstrate a level of thinking that was higher than what was given in the problem's description of Ahmed's thinking.	Student used language that did not indicate an understanding of base x and/or the place value system. Student may have used language such as shifting or moving.	Student made only minor errors in the use of mathematically correct language to relate the new number to the old in terms of place value and/or the use of base x.	Student made no errors in the use of mathematically correct language to relate the new number to the old in terms of place value and/or the use of base x.
8	**a–c** **A-APR.A.1**	Student showed limited or no understanding of polynomial multiplication and of evaluating a polynomial for the given value of x.	Student made multiple errors but showed some understanding of polynomial multiplication. Student may not have combined like terms to present the product as the sum of monomials.	Student made one or two minor errors but demonstrated knowledge of polynomial multiplication and combining like terms to create the new polynomial. Student also showed understanding of evaluating a polynomial for the given value of x.	Student completed all products correctly, expressing each as a sum of monomials with like terms collected, and evaluated correctly when x is 10.
	d **A-APR.A.1**	Student showed limited or no understanding of polynomial multiplication and of evaluating a polynomial for the given value of x.	Student made multiple errors but showed some understanding of polynomial multiplication. Student may not have combined like terms to present the product as the sum of monomials. Student may have gotten an incorrect result when evaluating with $x = 10$.	Student made one or two minor errors but demonstrated knowledge of polynomial multiplication and combining like terms to create the new polynomial. Student also showed understanding of evaluating a polynomial for the given value of x.	Student correctly multiplied the polynomials and expressed the product as a polynomial in standard form. Student correctly evaluated with a value of 10 and answered "Yes".

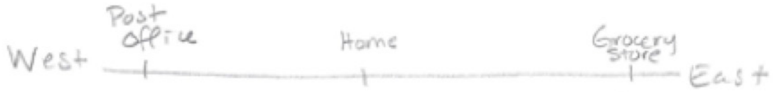
	e **A-APR.A.1**	Student was unable to demonstrate an understanding that part (iii) is "No" and/or demonstrated limited or no understanding of polynomial multiplication.	Student may have made some errors as he multiplied the polynomials and expressed the product as a sum of monomials. Student may have made some errors in the calculation of the value of the polynomial when x is 10. Student incorrectly answered part (iii) or applied incorrect reasoning.	Student may have made minor errors in multiplying the polynomials and expressing the product as a sum of monomials. Student may have made minor errors in calculating the value of the polynomial when x is 10. Student explained that the hypothesized equation being true when $x = 10$ does not make it true for all real x and/or explained that the two expressions are not algebraically equivalent.	Student correctly multiplied the polynomials and expressed the product as a sum of monomials with like terms collected. Student correctly calculated the value of the polynomial when x is 10. Student explained that the hypothesized equation being true when $x = 10$ does not make it true for all real x and/or explained that the two expressions are not algebraically equivalent.

Name ______________________________ Date ________________

1. Jacob lives on a street that runs east and west. The grocery store is to the east and the post office is to the west of his house. Both are on the same street as his house. Answer the questions below about the following story:

   At $1{:}00$ p.m., Jacob hops in his car and drives at a constant speed of 25 mph for 6 minutes to the post office. After 10 minutes at the post office, he realizes he is late, and drives at a constant speed of 30 mph to the grocery store, arriving at $1{:}28$ p.m. He then spends 20 minutes buying groceries.

   a. Draw a graph that shows the distance Jacob's car is from his house with respect to time. Remember to label your axes with the units you chose and any important points (home, post office, grocery store).



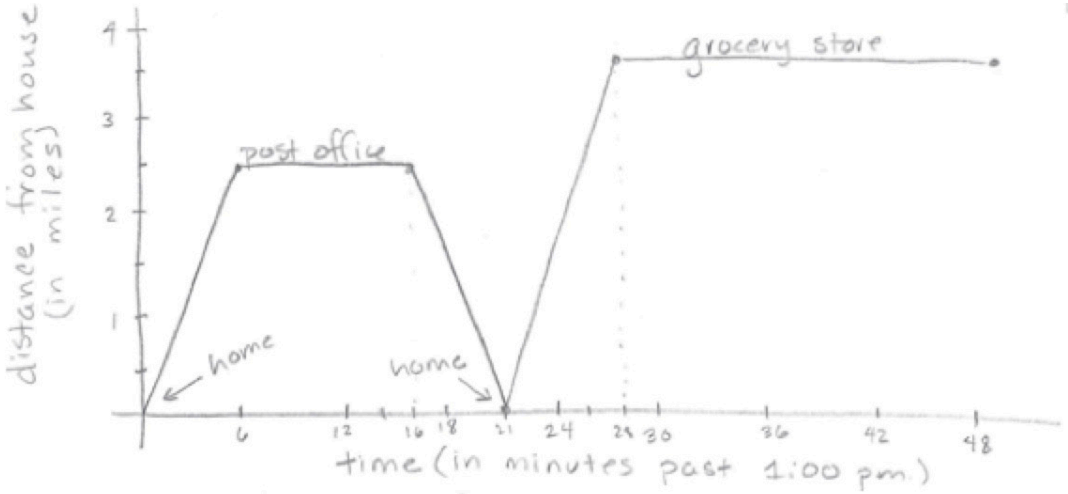
$$25\ \frac{\text{miles}}{\text{hour}} \times 6 \text{ minutes} \times 1\ \frac{\text{hour}}{60 \text{ minutes}} = 2.5 \text{ miles from house to post office}$$

$$30\ \frac{\text{miles}}{\text{hour}} \times 12 \text{ minutes} \times 1\ \frac{\text{hour}}{60 \text{ minutes}} = 6 \text{ miles from post office to store}$$

$$6 \text{ miles} - 2.5 \text{ miles} = 3.5 \text{ miles from home to store}$$

6 miles in 12 minutes is 1 mile in 2 minutes

so 2. 5 miles takes 5 minutes and 3.5 miles takes 7 minutes

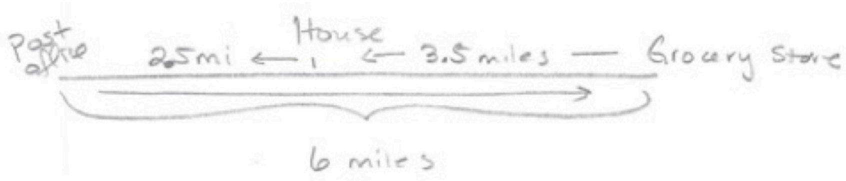


b. On the way to the grocery store, Jacob looks down at his watch and notes the time as he passes his house. What time is it when he passes his house? Explain how you found your answer.

It is 1:21. The graph shows the time as 21 minutes past 1:00 PM. He spent 6 minutes getting to the post office, 10 minutes at the post office, and 5 minutes getting from the post office to the point of passing by his house. You know it took 5 minutes for the last part because he traveled 30 miles per hour and went 2.5 miles. $2.5 \text{ miles} \times \frac{60 \text{ minutes}}{30 \text{ miles}} = 5 \text{ minutes}$

c. If he drives directly back to his house after the grocery story, what was the total distance he traveled to complete his errands? Show how you found your answer.

12 miles.



$$2.5 \text{ miles} + 6 \text{ miles} + 3.5 \text{ miles} = 12 \text{ miles}$$

You know it is 25 miles from the house to the post office because $25 \frac{\text{miles}}{\text{hour}} \times 6 \text{ minutes} \times 1 \frac{\text{hour}}{60 \text{ minutes}} = 2.5 \text{ miles.}$

You know it is 6 miles from the post office to the store because $30 \frac{\text{miles}}{\text{hour}} \times 12 \text{ minutes} \times 1 \frac{\text{hour}}{60 \text{ minutes}} = 6 \text{ miles.}$

2. Jason is collecting data on the rate of water usage in the tallest skyscraper in the world during a typical day. The skyscraper contains both apartments and businesses. The electronic water meter for the building displays the total amount of water used in liters. At noon, Jason looks at the water meter and notes that the digit in the **ones** place on the water meter display changes too rapidly to read the digit and that the digit in the **tens** place changes every second or so.

a. Estimate the total number of liters used in the building during one 24-hour day. Take into account the time of day when he made his observation. (Hint: Will water be used at the same rate at 2:00 a.m. as at noon?) Explain how you arrived at your estimate.

$$10 \frac{\text{liters}}{\text{second}} \times 60 \frac{\text{seconds}}{\text{minute}} \times 60 \frac{\text{minutes}}{\text{hour}} \times 18 \text{ hours} = 648{,}000 \text{ liters}$$

Since water is probably only used from about 5:00 AM to 11:00 PM, I did not multiply by 24 hours, but by 18 hours instead.

b. To what level of accuracy can Jason reasonably report a measurement if he takes it at precisely $12:00$ p.m.? Explain your answer.

It can be reported within ±10 liters, since he can read the 10's place, but it is changing by a 10 during the second he reads it.

c. The meter will be checked at regular time spans (for example, every minute, every 10 minutes, and every hour). What is the minimum (or smallest) number of checks needed in a 24-hour period to create a reasonably accurate graph of the water usage **rate** with respect to time? (For example, 24 checks would mean checking the meter every hour; 48 checks would mean checking the meter every half hour.) Defend your choice by describing how the water usage rate might change during the day and how your choice could capture that change.

24 checks. Every hour would be good to show the peaks in usage during morning and evening hours from those in the apartments. And it might also show that businesses stop using it after business hours. It would depend on what portion of the building is business vs. apartments.

3. A publishing company orders black and blue ink in bulk for its two-color printing press. To keep things simple with its ink supplier, each time it places an order for blue ink, it buys B gallons, and each time it places an order for black ink, it buys K gallons. Over a one-month period, the company places m orders of blue ink and n orders of black ink.

a. What quantities could the following expressions represent in terms of the problem context?

$m+n$ – Total number of ink orders over a one-month period.

$mB+nK$ – Total gallons of ink ordered over a one-month period.

$\frac{mB+nK}{m+n}$ – Average number of gallons of ink per order.

b. The company placed twice as many orders for black ink than for blue ink in January. Give interpretations for the following expressions in terms of the orders placed in January,

$$\frac{m}{m+n} \quad \text{and} \quad \frac{n}{m+n},$$

and explain which expression must be greater using those interpretations.

$\frac{m}{m+n}$ is the fraction of orders that are for blue ink.

$\frac{n}{m+n}$ is the fraction of orders that are for black ink.

$\frac{n}{m+n}$ would be bigger, 2 times as big as $\frac{m}{m+n}$ because they ordered twice as many orders for black ink than for blue ink.

4. Sam says that he knows a clever set of steps to rewrite the expression

$$(x+3)(3x+8)-3x(x+3)$$

as a sum of two terms where the steps do not involve multiplying the linear factors first and then collecting like terms. Rewrite the expression as a sum of two terms (where one term is a number and the other is a product of a coefficient and variable) using Sam's steps if you can.

$$\big((3x+8)-3x\big)\cdot(x+3)$$

$$8(x+3)$$

$$8x+24$$

5. Using only the addition and multiplication operations with the numbers 1, 2, 3, and 4 each exactly once, it is possible to build a numeric expression (with parentheses to show the order used to build the expression) that evaluates to 21. For example, $1+\big((2+3)\cdot 4\big)$ is one such expression.

   a. Build two more numeric expressions that evaluate to 21 using the criteria above. Both must be different from the example given.

$$(1+2)\cdot(3+4)=21$$

$$\big((2+4)+1\big)\cdot 3=21$$

b. In both of your expressions, replace 1 with a, 2 with b, 3 with c, and 4 with d to get two algebraic expressions. For example, $a + \big((b + c) \cdot d\big)$ shows the replacements for the example given.

$$(a + b) \cdot (c + d) = ac + ad + bc + bd$$

$$\big((b + d) + a\big) \cdot c = ac + bc + dc$$

Are your algebraic expressions equivalent? Circle: Yes [✓ No]

- If they are equivalent, prove that they are using the properties of operations.
- If not, provide **two** examples:

  (1) Find four different numbers (other than 0, 1, 2, 3, 4) that when substituted for a, b, c, and d into each expression, the expressions evaluate to **different numbers**, and

$a = 5 \qquad b = 10 \qquad c = 20 \qquad d = 30$

$$(5 + 10) \cdot (20 + 30) = 750$$

$$\big((10 + 30) + 5\big) \cdot 20 = 900$$

  (2) Find four different, non-zero numbers that when substituted into each expression, the expressions evaluate to the **same number**.

$5, 6, 11, 7$

$(5 + 6) \cdot (11 + 7) = 11 \cdot 18 = 198$

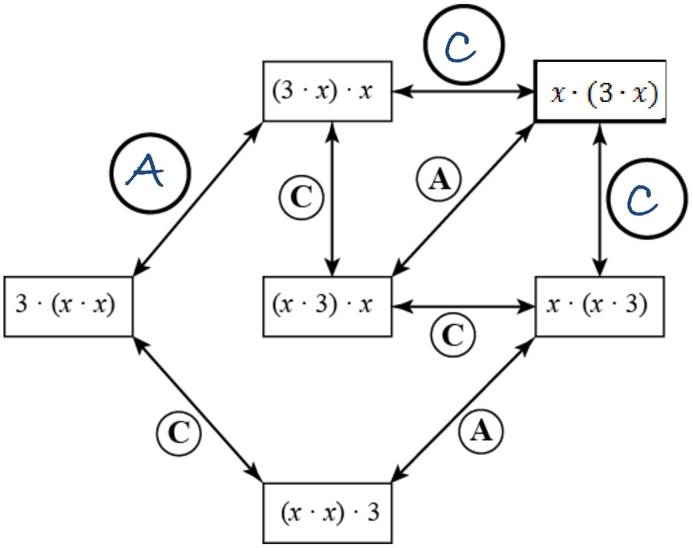
$\big((6 + 7) + 5\big) \cdot 11 = 18 \cdot 11 = 198$

$(ac + ad + bc + bd)$ needs to equal $(ac + bc + dc)$;

so, $(ad + bd)$ needs to equal (dc);

so, $(a + b)$ needs to equal c.

6. The diagram below, when completed, shows all possible ways to build equivalent expressions of $3x^2$ using multiplication. The equivalent expressions are connected by labeled segments stating which property of operations, **A** for associative property and **C** for commutative property, justifies why the two expressions are equivalent. Answer the following questions about $3x^2$ and the diagram.



a. Fill in the empty circles with **A** or **C** and the empty rectangle with the missing expression to complete the diagram.

b. Using the diagram above to help guide you, give *two different* proofs that $(x \cdot x) \cdot 3 = (3 \cdot x) \cdot x$.

1. $(x \cdot x) \cdot 3 = x \cdot (x \cdot 3)$ by Associate Property
   $x \cdot (x \cdot 3) = x \cdot (3 \cdot x)$ by Commutative Property
   $x \cdot (3 \cdot x) = (3 \cdot x) \cdot x$ by Commutative Property
2. $(x \cdot x) \cdot 3 = 3 \cdot (x \cdot x)$ by Commutative Property
   $3 \cdot (x \cdot x) = (3 \cdot x) \cdot x$ by Associate Property

7. Ahmed learned: "To multiply a whole number by ten, just place a zero at the end of the number." For example, 2813×10, he says, is 28,130. He doesn't understand why this "rule" is true.

a. What is the product of the polynomial, $2x^3 + 8x^2 + x + 3$, times the polynomial, x?

$$2x^4 + 8x^3 + x^2 + 3x$$

b. Use part (a) as a hint. Explain why the rule Ahmed learned is true.

When you multiply by the same number as the base, it creates a new number where each digit in the original number is now one place-value higher so that there is nothing left (no numbers) to represent the ones' digit, which leads to a trailing "0" in the ones' digit.

8.

a. Find the following products:

i. $(x-1)(x+1)$

$x^2+x-x-1$
x^2-1

ii. $(x-1)\left(x^2+x+1\right)$

$x^3+x^2+x-x^2-x-1$
x^3-1

iii. $(x-1)(x^3+x^2+x+1)$

$x^4+x^3+x^2+x-x^3-x^2-x-1$
x^4-1

iv. $(x-1)(x^4+x^3+x^2+x+1)$

$x^5+x^4+x^3+x^2+x-x^4-x^3-x^2-x-1$
x^5-1

v. $(x-1)(x^n+x^{n-1}+\cdots+x^3+x^2+x+1)$

$x^{n+1}-1$

b. Substitute $x=10$ into each of the products from parts (i) through (iv) and your answers to show how each of the products appears as a statement in arithmetic.

i. $(10-1)\cdot(10+1)=(100-1)$
$9\cdot(11)=99$

ii. $(10-1)\cdot(100+10+1)=(1000-1)$
$9\cdot(111)=999$

iii. $(10-1)\cdot(1{,}000+100+10+1)=(10{,}000-1)$
$9\cdot(1{,}111)=9{,}999$

iv. $(10-1)\cdot(10{,}000+1{,}000+100+10+1)=(100{,}000-1)$
$9\cdot(11{,}111)=99{,}999$

c. If we substituted $x=10$ into the product $(x-2)(x^7+x^6+x^5+x^4+x^3+x^2+x+1)$ and computed the product, what number would result?

$8\cdot(11{,}111{,}111)=88{,}888{,}888$

d. Multiply $(x - 2)$ and $(x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1)$ and express your answer in standard form.

$$x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x - 2x^7 - 2x^6 - 2x^5 - 2x^4 - 2x^3 - 2x^2 - 2x - 2$$
$$x^8 - x^7 - x^6 - x^5 - x^4 - x^3 - x^2 - x - 2$$

Substitute $x = 10$ into your answer and see if you obtain the same result as you obtained in part (c).

$10^8 - 10^7 - 10^6 - 10^5 - 10^4 - 10^3 - 10^2 - 10 - 2 = 88{,}888{,}888$. Yes, I get the same answer.

e. Francois says $(x - 9)(x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1)$ must equal $x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$ because when $x = 10$, multiplying by "$x - 9$" is the same as multiplying by 1.

i. Multiply $(x - 9)(x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1)$.

$$x^8 - 8x^7 - 8x^6 - 8x^5 - 8x^4 - 8x^3 - 8x^2 - 8x - 9$$

ii. Put $x = 10$ into your answer.

$$100{,}000{,}000 - 80{,}000{,}000 - 8{,}000{,}000 - 800{,}000 - 80{,}000 - 8{,}000 - 800 - 80 - 9$$

$$100{,}000{,}000 - 88{,}888{,}889 = 11{,}111{,}111$$

Is it the same as $x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$ with $x = 10$?

Yes.

iii. Was Francois right?

No, just because it is true when x is 10, doesn't make it true for all real x. The two expressions are not algebraically equivalent.

New York State Common Core

Mathematics Curriculum

Topic C:

Solving Equations and Inequalities

A-CED.A.3, A-CED.A.4, A-REI.A.1, A-REI.B.3, A-REI.C.5, A-REI.C.6, A-REI.D.10, A-REI.D.12

Focus Standard:	A-CED.A.3	Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.**
	A-CED.A.4	Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. *For example, rearrange Ohm's law* $V = IR$ *to highlight resistance* R*.*
	A-REI.A.1	Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method.
	A-REI.B.3	Solve linear equations and inequalities in one variable, including equations with coefficients represented by letters.
	A-REI.C.5	Prove that, given a system of two equations in two variables, replacing one equation by the sum of that equation and a multiple of the other produces a system with the same solutions.
	A-REI.C.6	Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables.
	A-REI.D.10	Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line).
	A-REI.D.12	Graph the solutions to a linear inequality in two variables as a half-plane (excluding the boundary in the case of a strict inequality), and graph the solution set to a system of linear inequalities in two variables as the intersection of the corresponding half-planes.

Instructional Days:	15
Lesson 10:	True and False Equations (P)[1]
Lesson 11:	Solution Sets for Equations and Inequalities (P)
Lesson 12:	Solving Equations (P)
Lesson 13:	Some Potential Dangers when Solving Equations (P)
Lesson 14:	Solving Inequalities (P)
Lesson 15:	Solution Sets of Two or More Equations (or Inequalities) Joined by "And" or "Or" (E)
Lesson 16:	Solving and Graphing Inequalities Joined by "And" or "Or" (P)
Lesson 17:	Equations Involving Factored Expressions (S)
Lesson 18:	Equations Involving a Variable Expression in the Denominator (P)
Lesson 19:	Rearranging Formulas (P)
Lessons 20:	Solution Sets to Equations with Two Variables (P)
Lessons 21:	Solution Sets to Inequalities with Two Variables (P)
Lessons 22–23:	Solution Sets to Simultaneous Equations (P, E)
Lesson 24:	Applications of Systems of Equations and Inequalities (E)

Teaching the process of how to solve an equation is fraught with well-meaning models and procedures suggested by textbook curricula (balance scales, algebra tiles, equivalent equations, etc.) that are often incompatible with what it actually means "to solve." An equation with variables can be viewed as a question asking for which values of the variables (the solution set) will result in true number sentences when those values are substituted into the equation. Equations are manifestly about *numbers* and understanding true *and false* number sentences. In Algebra I, the application of this idea expands to include solutions to compound statements such as equations or inequalities joined by "and" or "or," including simultaneous systems of equations or inequalities.

The Common Core Learning Standards rightfully downplay the notion of equivalent equations and instead place a heavy emphasis on students studying the solution sets to equations. In Lessons 12–14 of this topic, students formalize descriptions of what they learned before (true/false equations, solution sets, identities, properties of equality, etc.) and learn how to explain the steps of solving equations to construct viable arguments to justify their solution methods. They then learn methods for solving inequalities, again by focusing on ways to preserve the (now infinite) solution sets. With these methods now on firm footing, students investigate in Lessons 15–18 solution sets of equations joined by "and" or "or" and investigate ways to change an equation such as squaring both sides, which changes the solution set in a controlled (and often useful) way. In Lesson 19, students learn to use these same skills as they rearrange formulas to define one quantity in terms of another. Finally, in Lessons 20–24, students apply all of these new skills and understandings as they work through solving equations and inequalities with two variables including systems of such equations and inequalities.

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[1] Lesson Structure Key: **P**-Problem Set Lesson, **M**-Modeling Cycle Lesson, **E**-Exploration Lesson, **S**-Socratic Lesson

Lesson 10: True and False Equations

Student Outcomes

- Students understand that an equation is a statement of equality between two expressions. When values are substituted for the variables in an equation, the equation is either true or false. Students find values to assign to the variables in equations that make the equations true statements.

Classwork

Exercise 1 (5 minutes)

Give students a few minutes to reflect on Exercise 1. Then ask students to share their initial reactions and thoughts in answering the questions.

Exercise 1

a. Consider the statement: "The President of the United States is a United States citizen."
Is the statement a grammatically correct sentence?
What is the subject of the sentence? What is the verb in the sentence?
Is the sentence true?

b. Consider the statement: "The President of France is a United States citizen."
Is the statement a grammatically correct sentence?
What is the subject of the sentence? What is the verb in the sentence?
Is the sentence true?

c. Consider the statement: "$2 + 3 = 1 + 4$."
This is a sentence. What is the verb of the sentence? What is the subject of the sentence?
Is the sentence true?

d. Consider the statement: "$2 + 3 = 9 + 4$."
Is this statement a sentence? And if so, is the sentence true or false?

Hold a general class discussion about parts (c) and (d) of the exercise. Be sure to raise the following points:

- One often hears the chime that "Mathematics is a language." And indeed it is. For us reading this text, that language is English. (And if this text were written in French, that language would be French, or if this text were written in Korean, that language would be Korean.)
- A mathematical statement, such as $2 + 3 = 1 + 4$, is a grammatically correct sentence. The subject of the sentence is the numerical expression "$2 + 3$", and its verb is "equals" or "is equal to." The numerical expression "$1 + 4$" renames the subject ($2 + 3$). We say that the statement is TRUE because these two numerical expressions evaluate to the same numerical value (namely, five).

- The mathematical statement $2 + 3 = 9 + 4$ is also a grammatically correct sentence, but we say it is FALSE because the numerical expression to the left (the subject of the sentence) and the numerical expression to the right do not evaluate to the same numerical value.

  (Perhaps remind students of parts (a) and (b) of the exercise: grammatically correct sentences can be false.)

- Recall the definition:

A number sentence is a statement of equality between two numerical expressions.

A *number sentence* is said to be *true* if both numerical expressions are equivalent (that is, both evaluate to the same number). It is said to be *false* otherwise. True and false are called *truth values*.

Exercise 2 (7 minutes)

Have students complete this exercise independently, and then review the answers as a class.

Exercise 2

Determine whether the following number sentences are TRUE or FALSE.

a. $4 + 8 = 10 + 5$

FALSE

b. $\frac{1}{2} + \frac{5}{8} = 1.2 - 0.075$

TRUE

c. $(71 \cdot 603) \cdot 5876 = 603 \cdot (5876 \cdot 71)$

TRUE. The commutative and associative properties of multiplication demand these numerical expressions match.

d. $13 \times 175 = 13 \times 90 + 85 \times 13$

TRUE. Notice the right side equals $13 \times (90 + 85)$*.*

e. $(7 + 9)^2 = 7^2 + 9^2$

FALSE

f. $\pi = 3.141$

FALSE (The value of π *is not exactly* 3.141*.)*

g. $\sqrt{(4+9)} = \sqrt{4} + \sqrt{9}$

FALSE

h. $\frac{1}{2} + \frac{1}{3} = \frac{2}{5}$

FALSE

i. $\frac{1}{2}+\frac{1}{3}=\frac{2}{6}$

FALSE

j. $\frac{1}{2}+\frac{1}{3}=\frac{5}{6}$

TRUE

k. $3^2+4^2=7^2$

FALSE

l. $3^2\times 4^2=12^2$

TRUE

m. $3^2\times 4^3=12^6$

FALSE

n. $3^2\times 3^3=3^5$

TRUE

Exercise 3 (3 minutes)

Allow students to answer the questions in their student materials, and then discuss with the class.

Exercise 3

a. **Could a number sentence be both TRUE and FALSE?**

b. **Could a number sentence be neither TRUE nor FALSE?**

A number sentence has a left-hand numerical expression that evaluates to a single number and has a right-hand numerical expression that also evaluates to a single numerical value. Either these two single values match or they do not. A numerical sentence is thus either TRUE or FALSE (and not both).

An algebraic equation is a statement of equality between two expressions.

***Algebraic equations* can be number sentences (when both expressions are numerical), but often they contain symbols whose values have not been determined.**

Exercise 4 (6 minutes)

Exercise 4

a. **Which of the following are algebraic equations?**

   i. $3.1x - 11.2 = 2.5x + 2.3$

   ii. $10\pi^4 + 3 = 99\pi^2$

   iii. $\pi + \pi = 2\pi$

   iv. $\frac{1}{2} + \frac{1}{2} = \frac{2}{4}$

   v. $79\pi^3 + 70\pi^2 - 56\pi + 87 = (60\pi + 29{,}928)/\pi^2$

   All of them.

b. **Which of them are also number sentences?**

   Numbers (ii), (iii), (iv), and (v). (Note that the symbol π has a value that has is already stated or known.)

c. **For each number sentence, state whether the number sentence is true or false.**

   (ii) False, (iii) True, (iv) False, (v) False. Note that (ii) and (v) are both very close to evaluating to true. Some calculators may not be able to discern the difference. Wolfram Alpha's web-based application can be used to reveal the differences.

Exercise 5 (9 minutes)

Discuss the three cases for algebraic equations given in the student materials, and based on the preparedness of your students, complete the exercise as a whole class, in small groups, in pairs, or individually.

Exercise 5

When algebraic equations contain a symbol whose value has not yet been determined, we use analysis to determine whether:

1. **The equation is true for all the possible values of the variable(s), or**
2. **The equation is true for a certain set of the possible value(s) of the variable(s), or**
3. **The equation is never true for any of the possible values of the variable(s).**

For each of the three cases, write an algebraic equation that would be correctly described by that case. Use only the variable, x, where x represents a real number.

1. $2(x + 3) = 2x + 6$*; By the distributive property, the two expressions on each side of the equal sign are algebraically equivalent; therefore, the equation is true for all possible real numbers,* x*.*
2. $x + 5 = 11$*; This equation is only a true number sentence if* $x = 6$*. Any other real number would make the equation a false number sentence.*
3. $x^2 = -1$*; There is no real number* x *that could make this equation a true number sentence.*

Share and discuss some possible answers for each.

Example 1 (4 minutes)

Example 1

Consider the following scenario.

Julie is 300 feet away from her friend's front porch and observes, "Someone is sitting on the porch."

Given that she didn't specify otherwise, we would assume that the "someone" Julie thinks she sees is a human. We can't guarantee that Julie's observatory statement is true. It could be that Julie's friend has something on the porch that merely looks like a human from far away. Julie assumes she is correct and moves closer to see if she can figure out who it is. As she nears the porch she declares, "Ah, it is our friend, John Berry."

- Often in mathematics, we observe a situation and make a statement we believe to true. Just as Julie used the word "someone", in mathematics we use variables in our statements to represent quantities not yet known. Then, just as Julie did, we "get closer" to study the situation more carefully and find out if our "someone" exists and if so "who" it is.
- Notice that we are comfortable assuming that the "someone" Julie referred to is a human, even though she didn't say so. In mathematics we have a similar assumption. If it is not stated otherwise, we assume that variable symbols represent a real number. But in some cases, we might say the variable represents an integer or an even integer or a positive integer, for example.
- Stating what type of number the variable symbol represents is called stating its **domain**.

Exercise 6 (6 minutes)

- In the sentence $w^2 = 4$, w can represent any real number we care to choose (its domain). If we choose to let w be 5, then the number sentence is false. If we let $w = 2$, then the sentence is true. Is there another value for w that would also make the sentence true?
    - $w = -2$

Exercise 6

Name a value of the variable that would make each equation a true number sentence.

Here are several examples of how we can name the value of the variable:

Let $w = -2$. Then $w^2 = 4$ is true.

or

$w^2 = 4$ is true when $w = -2$

or

$w^2 = 4$ is true if $w = -2$

or

$w^2 = 4$ is true for $w = -2$ and $w = 2$.

There might be more than one option for what numerical values to write. (And feel free to write more than one possibility.)

Warning: Some of these are tricky. Keep your wits about you!

a. Let ______________. Then $7 + x = 12$ is true.

$x = 5$

b. Let ______________. Then $3r + 0.5 = \frac{37}{2}$ is true.

$r = 6$

c. $m^3 = -125$ is true for ________________.

$m = -5$

d. A number x and its square, x^2, have the same value when __________________.

$x = 1$ or when $x = 0$.

e. The average of 7 and n is -8 if __________________.

$n = -23$.

f. Let ________________________. Then $2a = a + a$ is true.

$a =$ any real number

g. $q + 67 = q + 68$ is true for __________________.

There is no value one can assign to q to turn this equation into a true statement.

Exit Ticket (5 minutes)

Name ____ Date____________________

Lesson 10: True and False Equations

Exit Ticket

1. Consider the following equation, where a represents a real number: $\sqrt{a+1} = \sqrt{a} + 1$.

   Is this statement a number sentence? If so, is the sentence TRUE or FALSE?

2. Suppose we are told that b has the value 4. Can we determine whether the equation below is TRUE or FALSE? If so, say which it is; if not, state that it cannot be determined. Justify your answer.

$$\sqrt{b+1} = \sqrt{b} + 1$$

3. For what value of c is the following equation TRUE?

$$\sqrt{c+1} = \sqrt{c} + 1$$

Exit Ticket Sample Solutions

1. Consider the following equation, where a represents a real number: $\sqrt{a+1} = \sqrt{a} + 1$.

   Is this statement a number sentence? If so, is the sentence TRUE or FALSE?

   No, it is not a number sentence because no value has been assigned to a. Thus, it is neither TRUE nor FALSE.

2. Suppose we are told that b has the value 4. Can we determine whether the equation below is TRUE or FALSE? If so, say which it is; if not, state that it cannot be determined. Justify your answer.

   $$\sqrt{b+1} = \sqrt{b} + 1$$

   FALSE, the left-hand expression has value $\sqrt{4+1} = \sqrt{5}$ and the right-hand expression has value $2 + 1 = 3$. These are not the same value.

3. For what value of c is the following equation TRUE?

   $$\sqrt{c+1} = \sqrt{c} + 1$$

   $\sqrt{c+1} = \sqrt{c} + 1$, if we let $c = 0$.

Problem Set Sample Solutions

Determine whether the following number sentences are true or false.

1. $18 + 7 = \frac{50}{2}$ *TRUE*

2. $3.123 = 9.369 \cdot \frac{1}{3}$ *TRUE*

3. $(123 + 54) \cdot 4 = 123 + (54 \cdot 4)$ *FALSE*

4. $5^2 + 12^2 = 13^2$ *TRUE*

5. $(2 \times 2)^2 = \sqrt{256}$ *TRUE*

6. $\frac{4}{3} = 1.333$ *FALSE*

In the following equations, let $x = -3$ and $y = \frac{2}{3}$. Determine whether the following equations are true, false, or neither true nor false.

7. $xy = -2$ *TRUE*

8. $x + 3y = -1$ *TRUE*

9. $x + z = 4$ *Neither TRUE nor FALSE*

10. $9y = -2x$ *TRUE*

11. $\frac{y}{x} = -2$ *FALSE*

12. $\frac{-\frac{2}{x}}{y} = -1$ *FALSE*

For each of the following, assign a value to the variable, x, to make the equation a true statement.

13. $(x^2 + 5)(3 + x^4)(100x^2 - 10)(100x^2 + 10) = 0$ for __________________.

    $x = \frac{1}{\sqrt{10}}$ *or* $x = -\frac{1}{\sqrt{10}}$

14. $\sqrt{(x+1)(x+2)} = \sqrt{20}$ for __________________.

    $x = 3$ *or* $x = -6$

15. $(d+5)^2 = 36$ for ________________.

$d = 1$ or $d = -11$

16. $(2z+2)(z^5-3)+6 = 0$ for ________________.

$z = 0$ seems the easiest answer.

17. $\frac{1+x}{1+x^2} = \frac{3}{5}$ for ________________.

$x = 2$ works.

18. $\frac{1+x}{1+x^2} = \frac{2}{5}$ for ________________.

$x = 3$ works. So does $x = -\frac{1}{2}$

19. The diagonal of a square of side length L is 2 inches long when ________________.

$L = \sqrt{2}$ inches

20. $\left(T-\sqrt{3}\right)^2 = T^2 + 3$ for ________________.

$T = 0$

21. $\frac{1}{x} = \frac{x}{1}$ if ________________.

$x = 1$ and also if $x = -1$

22. $\left(2+\left(2-\left(2+\left(2-(2+r)\right)\right)\right)\right) = 1$ for ______________.

$r = -1$

23. $x + 2 = 9$

for $x = 7$

24. $x + 2^2 = -9$

for $x = -13$

25. $-12t = 12$

for $t = -1$

26. $12t = 24$

for $t = 2$

27. $\frac{1}{b-2} = \frac{1}{4}$

for $b = 6$

28. $\frac{1}{2b-2} = -\frac{1}{4}$

for $b = -1$

29. $\sqrt{x} + \sqrt{5} = \sqrt{x+5}$

for $x = 0$

30. $(x-3)^2 = x^2 + (-3)^2$

for $x = 0$

31. $x^2 = -49$

No real number will make the equation true.

32. $\frac{2}{3} + \frac{1}{5} = \frac{3}{x}$

for $x = \frac{45}{13}$

Fill in the blank with a variable term so that the given value of the variable will make the equation true.

33. $\underline{\ x\ } + 4 = 12;\ x = 8$

34. $\underline{\ 2x\ } + 4 = 12;\ x = 4$

Fill in the blank with a constant term so that the given value of the variable will make the equation true.

35. $4y - \underline{\ 0\ } = 100;\ y = 25$

36. $4y - \underline{\ 24\ } = 0;\ y = 6$

37. $r + \underline{\ 0\ } = r;$ r *is any real number*

38. $r \times \underline{\ 1\ } = r;$ r *is any real number*

Generate the following:

39. An equation that is always true

40. An equation that is true when $x = 0$

41. An equation that is never true

42. An equation that is true when $t = 1$ or $t = -1$

43. An equation that is true when $y = -0.5$

44. An equation that is true when $z = \pi$

Problems 39–44: Answers vary

Lesson 11: Solution Sets for Equations and Inequalities

Student Outcomes

- Students understand that an equation with variables is often viewed as a question asking for the set of values one can assign to the variables of the equation to make the equation a true statement. They see the equation as a "filter" that sifts through all numbers in the domain of the variables, sorting those numbers into two disjoint sets: the solution set and the set of numbers for which the equation is false.
- Students understand the commutative, associate, and distributive properties as identities; e.g., equations whose solution sets are the set of all values in the domain of the variables.

Classwork

Example 1 (2 minutes)

- Consider the equation shown in Example 1 of your student materials, $x^2 = 3x + 4$, where x represents a real number.
- Since we have not stated the value of x, this is not a number sentence.

Example 1

Consider the equation, $x^2 = 3x + 4$, where x represents a real number.

a. **Are the expressions x^2 and $3x + 4$ algebraically equivalent?**

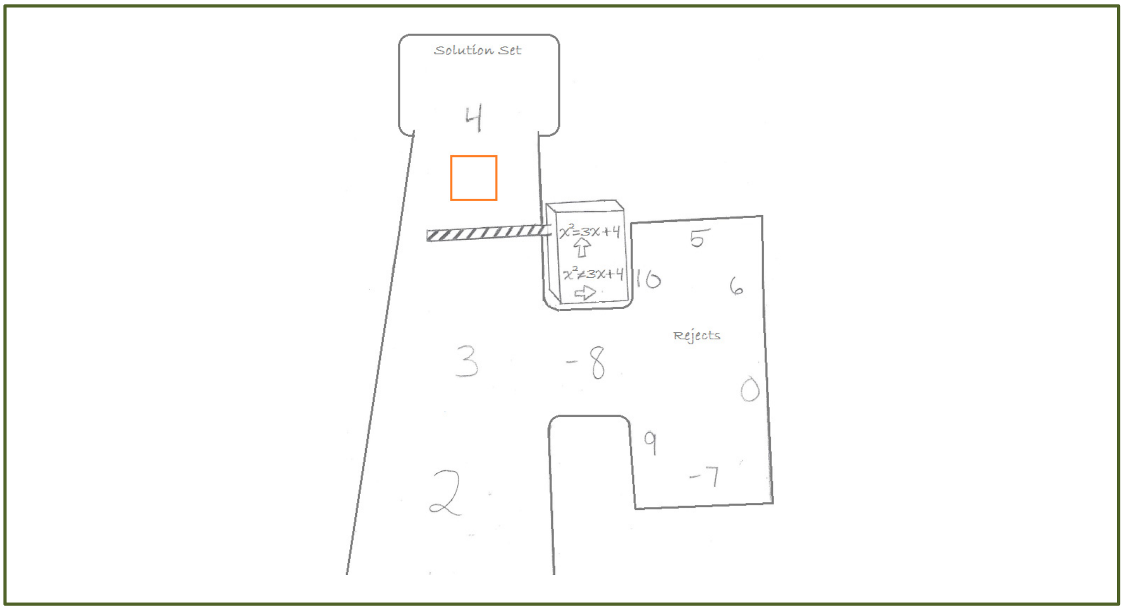
No.

- Then we cannot guarantee there will be any real value of x that will make the equation true.

b. **The following table shows how we might "sift" through various values to assign to the variable symbol x in the hunt for values that would make the equation true.**

x-VALUE	THE EQUATION	TRUTH VALUE
Let $x = 0$	$0^2 = 3(0) + 4$	FALSE
Let $x = 5$	$5^2 = 3(5) + 4$	FALSE
Let $x = 6$	$6^2 = 3(6) + 4$	FALSE
Let $x = -7$	$(-7)^2 = 3(-7) + 4$	FALSE
Let $x = 4$	$4^2 = 3(4) + 4$	TRUE
Let $x = 9$	$9^2 = 3(9) + 4$	FALSE
Let $x = 10$	$10^2 = 3(10) + 4$	FALSE
Let $x = -8$	$(-8)^2 = 3(-8) + 4$	FALSE

- Of course, we should sift through ALL the real numbers if we are seeking all values that make the equation $x^2 = 3x + 4$ true. (This makes for quite a large table!) So far we have found that setting x equal to 4 yields a true statement.
- Look at the image in your student materials. Can you see what is happening here and how it relates to what we have been discussing?
  - *The numbers are going down the road and being accepted into the solution set or rejected based on whether or not the equation is true.*



- There happens to be just one more value we can assign to x that makes $x^2 = 3x + 4$ a true statement. Would you like to continue the search to find it?
  - $x = -1$

Example 2 (1 minute)

Example 2

Consider the equation $7 + p = 12$.

	THE NUMBER SENTENCE	**TRUTH VALUE**
Let $p = 0$	$7 + 0 = 12$	**FALSE**
Let $p = 4$		
Let $p = 1 + \sqrt{2}$		
Let $p = \frac{1}{\pi}$		
Let $p = 5$		

- Here's a table that could be used to hunt for the value(s) of p that make the equation true:

p-VALUE	THE EQUATION	TRUTH VALUE
Let $p = 0$	$7 + 0 = 12$	FALSE
Let $p = 4$	$7 + 4 = 12$	FALSE
Let $p = 1 + \sqrt{2}$	$7 + (1 + \sqrt{2}) = 12$	FALSE
Let $p = \frac{1}{\pi}$	$7 + \frac{1}{\pi} = 12$	FALSE
Let $p = 5$	$7 + 5 = 12$	TRUE

- But is a table necessary for this question? Is it obvious what value(s) we could assign to p to make the equation true?

Discussion (2 minutes)

The solution set of an equation written with only one variable is the set of all values one can assign to that variable to make the equation a true statement. Any one of those values is said to be a *solution to the equation.*

To *solve an equation* means to *find the solution set* for that equation.

- Recall that it is usually assumed that one is sifting through all the real numbers to find the solutions to an equation, but a question or a situation might restrict the domain of values we should sift through. We might be required to sift only through integer values, the positive real numbers, or the non-zero real numbers, for example. The context of the question should make this clear.

Example 3 (1 minute)

Give students 1 minute or less to complete the exercise and then discuss the answer.

Example 3

Solve for a: $a^2 = 25$.

We know that setting $a = 5$ or setting $a = -5$ makes $a^2 = 25$ a true statement. And a little thought shows that these are the only two values we can assign to make this so. The solution set is just the set containing the numbers 5 and -5. (And since the question made no mention of restricting the domain of values we should consider, we shall assume both these values are admissible solutions for this question.)

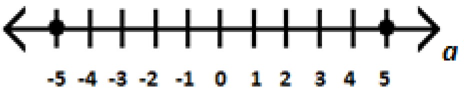
Discussion (6 minutes)

One can describe a solution set in any of the following ways:

IN WORDS: $a^2 = 25$ has solutions 5 and -5. (That is, $a^2 = 25$ is true when $a = 5$ or $a = -5$.)

IN SET NOTATION: The solution set of $a^2 = 25$ is $\{-5, 5\}$.

IN A GRAPHICAL REPRESENTATION ON A NUMBER LINE: The solution set of $a^2 = 25$ is



In this graphical representation, a solid dot is used to indicate a point on the number line that is to be included in the solution set. (WARNING: The dot one physically draws is larger than the point it represents. One hopes that it is clear from the context of the diagram which point each dot refers to.)

How set notation works.

- **The curly brackets { } indicate we are denoting a set. A set is essentially a collection of things, e.g., letters, numbers, cars, people. In this case, the things are numbers.**
- **From this example, the numbers -5 and 5 are called elements of the set. No other elements belong in this particular set because no other numbers make the equation $a^2 = 25$ true.**
- **When elements are listed, they are listed in increasing order.**
- **Sometimes, a set is empty; it has no elements. In which case, the set looks like { }. We often denote this with the symbol, $\emptyset$. We refer to this *as the empty set* or *the null set*.**

Exercise 1 (3 minutes)

Allow students to work independently, making sense of the problem and persevering in solving it. Have the students discuss the problem and its solution. As much as possible, let the students find the way to the solution and articulate how they know on their own, interjecting questions as needed to spawn more conversation.

MP.1

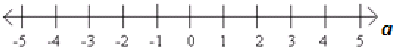
Exercise 1

Solve for a: $a^2 = -25$. Present the solution set in words, in set notation, and graphically.

IN WORDS: The solution set to this equation is the empty set. There are no real values to assign to a to make the equation true.

IN SET NOTATION: The solution set is { } (the empty set).

IN A GRAPHICAL REPRESENTATION: The solution set is



Exercise 2

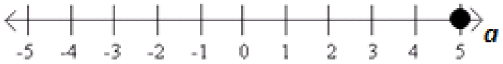
Exercise 2

Depict the solution set of $7 + p = 12$ in words, in set notation, and graphically.

IN WORDS: $7 + p = 12$ has the solution $p = 5$.

IN SET NOTATION: The solution set is $\{5\}$.

IN A GRAPHICAL REPRESENTATION:



Example 4 (4 minutes)

Example 4

Solve $\frac{x}{x} = 1$ for x, over the set of positive real numbers. Depict the solution set in words, in set notation, and graphically.

- The question statement indicates that we are to consider assigning values to x only from the set of positive real numbers. Let's create a table to get a feel for the problem. (It might actually be helpful this time.)

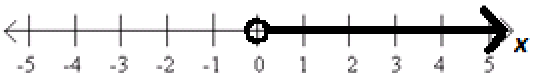
x-VALUE	THE EQUATION	TRUTH VALUE
Let $x = 2$	$\frac{2}{2} = 1$	TRUE
Let $x = 7$	$\frac{7}{7} = 1$	TRUE
Let $x = 0.01$	$\frac{0.01}{0.01} = 1$	TRUE
Let $x = 562\frac{2}{3}$	$\frac{562\frac{2}{3}}{562\frac{2}{3}} = 1$	TRUE
Let $x = 10^{100}$	$\frac{10^{100}}{10^{100}}$	TRUE
Let $x = \pi$	$\frac{\pi}{\pi} = 1$	TRUE

It seems that each and every positive real number is a solution to this equation.

IN WORDS: The solution set is the set of all positive real numbers.

IN SET NOTATION: This is $\{x \text{ real} \mid x > 0\}$.

IN A GRAPHICAL REPRESENTATION:



Discussion (4 minutes)

Some comments on set notation:

- If it is possible to list the elements in a set, then one might do so, for example:

  $\{-3, 5, \sqrt{40}\}$ is the set containing the three real numbers -3, 5, and $\sqrt{40}$.

  $\{1, 2, 3, \ldots, 10\}$ is the set containing the ten integers 1 through 10. (The ellipsis is used to state that the pattern suggested continues.)

- If it is not possible or not easy to list the elements in a set, then use the notation:

  { variable symbol number type | a description }

- For example:

  $\{x$ real $\mid x > 0\}$ reads as, "The set of all real numbers that are greater than zero."

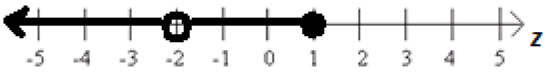
  $\{p$ integer $\mid -3 \leq p < 100\}$ reads as, "The set of all integers that are greater than or equal to -3 and smaller than 100."

  $\{y$ real $\mid y \neq 0\}$ reads as, "The set of all real numbers that are not equal to zero."

- The vertical bar "|" in this notation is often read as "that" or "such that."

Some comments on graphical representations are as follows:

- One uses solid dots to denote points on a number line, real numbers, to be included in the solution set, and open dots to indicate points to be excluded.
- Solid lines (possibly with arrows to indicate "extend indefinitely to the right" or "extend indefinitely to the left") are used to indicate intervals of points on the number line (intervals of real numbers) all to be included in the set.
- For example,



represents the set of real numbers: $\{z$ real $\mid z \leq 1$ and $z \neq -2\}$

Exercise 3 (3 minutes)

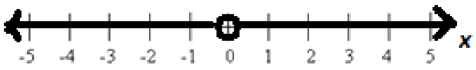
Exercise 3

Solve $\frac{x}{x} = 1$ for x, over the set of all non-zero real numbers. Describe the solution set in words, in set notation, and graphically.

IN WORDS: The solution set is the set of all non-zero real numbers.

IN SET NOTATION: $\{x$ real $\mid x \neq 0\}$

IN A GRAPHICAL REPRESENTATION:



Example 5 (4 minutes)

Note that the following example is important to discuss with care.

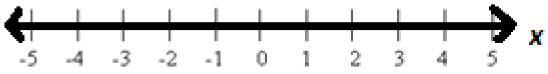
Example 5

Solve for x: $x(3+x) = 3x + x^2$.

- Since it is not specified otherwise, we should again assume that we are considering solutions from the set of all real numbers.
- In drawing a table to sift for possible solutions, students might come to suspect that every real value for x is a solution to this equation.
- The distributive property states that, for all real numbers a, b, and c, the expressions $a(b+c)$ and $ab+ac$ are sure to have the same numerical value. The commutative property for multiplication states for all real numbers d and e, the expressions de and ed have the same numerical value.
- Consequently, we can say, for any value we assign to x:

$$x(3+x) = x \cdot 3 + x^2,$$

that is, $x(3+x) = 3x + x^2$ is sure to be a true numerical statement. This proves that the solution set to this equation is the set of all real numbers.



- It is awkward to express the set of all real numbers in set notation. We simply write the “blackboard script” ℝ for the set of all real numbers. (By hand, one usually just draws a double vertical bar in the capital letter: ℝ.)

Exercise 4 (2 minutes)

MP.3

Exercise 4

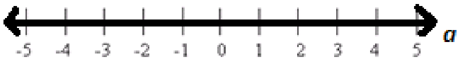
Solve for α: $\alpha + \alpha^2 = \alpha(\alpha + 1)$. Describe carefully the reasoning that justifies your solution. Describe the solution set in words, in set notation, and graphically.

IN WORDS: By the distributive property, we have $\alpha + \alpha^2 = \alpha(1+\alpha)$. This is a true numerical statement no matter what value we assign to α. By the commutative property of addition, we have $\alpha + \alpha^2 = \alpha(\alpha+1)$, which is a true numerical statement no matter what real value we assign to α.

The solution set is the set of all real numbers.

IN SET NOTATION: The solution set is ℝ.

IN GRAPHICAL REPRESENTATION:



Discussion (2 minutes)

- Recall, what does it mean for two expressions to be algebraically equivalent?
  - *One expression can be converted to the other by repeatedly applying the commutative, associative, and distributive properties or the properties of rational exponents to either expression.*
- When the left side of an equation is algebraically equivalent to the right side of an equation, what will the solution set be?
  - *All real numbers.*

An identity is an equation that is always true.

Exercise 5 (1 minute)

Exercise 5

Identify the properties of arithmetic that justify why each of the following equations has a solution set of all real numbers:

a. $2x^2 + 4x = 2(x^2 + 2x)$

b. $2x^2 + 4x = 4x + 2x^2$

c. $2x^2 + 4x = 2x(2 + x)$

(a) Distributive property; (b) commutative property of addition; (c) multiple properties: distributive property, commutative property of addition, and maybe even associative property of multiplication if we analyze the interpretation of $2x^2$ ***with extreme care.***

Exercise 6 (2 minutes)

Exercise 6

Create an expression for the right side of each equation such that the solution set for the equation will be all real numbers. (There is more than one possibility for each expression. Feel free to write several answers for each one.)

a. $2x - 5 =$ __________

b. $x^2 + x =$ __________

c. $4 \cdot x \cdot y \cdot z =$ __________

d. $(x + 2)^2 =$ __________

Sample Answers: ***(a)*** $-5 + 2x$ ***or*** $2\left(x - \frac{5}{2}\right)$ ***(b)*** $x + x^2$ ***or*** $x(x + 1)$ ***or*** $x(1 + x)$

(c) any rearranging of the factors ***(d)*** $(2 + x)^2$ ***or*** $x^2 + 4x + 4$

Closing/Example 6/Exercise 7 (5 minutes)

- We can extend the notion of a solution set of an equation to that of a solution set of an inequality (that is, a statement of inequality between two expressions).

- For instance, we can make sense of the following example.

Example 6

Solve for w: $w + 2 > 4$.

In discussing this problem, have students realize:

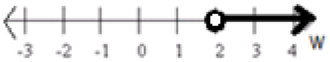
- An inequality between two numerical expressions also has a well-defined truth value: true or false.
- Just as for equations, one can "sift" through real values of the variable in an inequality to find those values that make the inequality a true statement. The solution set of an inequality is set of all real values that make the inequality true.

Have students describe the solution set to $w + 2 > 4$ in words, in set notation, and in graphical representation.

IN WORDS: w must be greater than 2.

IN SET NOTATION: $\{w \text{ real} \mid w > 2\}$

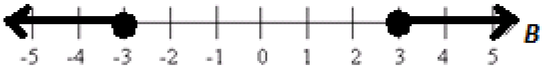
IN GRAPHICAL REPRESENTATION:



Review their answers, and then have them complete Exercise 7.

Exercise 7

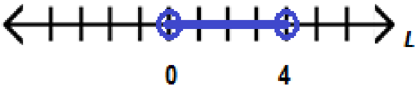
a. **Solve for B: $B^2 \geq 9$. Describe the solution set using a number line.**



MP.2

b. **What is the solution set to the statement: "Sticks of lengths 2 yards, 2 yards, and L yards make an isosceles triangle"? Describe the solution set in words and on a number line.**

L must be greater than 0 yards and less than 4 yards.



Lesson Summary

The solution set of an equation written with only one variable symbol is the set of all values one can assign to that variable to make the equation a true number sentence. Any one of those values is said to be a *solution to the equation.*

To *solve an equation* means to *find the solution set* for that equation.

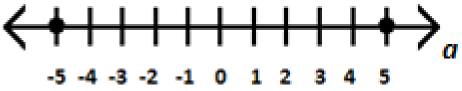
One can describe a solution set in any of the following ways:

IN WORDS: $a^2 = 25$ has solutions 5 and -5. (That is, $a^2 = 25$ is true when $a = 5$ or $a = -5$.)

IN SET NOTATION: The solution set of $a^2 = 25$ is $\{-5, 5\}$.

It is awkward to express the set of infinitely many numbers in set notation. In these cases we can use the notation: {variable symbol number type | a description}. For example $\{x \text{ real } \mid x > 0\}$ reads, "x is a real number where x is greater than zero." The symbol $\mathbb{R}$ can be used to indicate all real numbers.

IN A GRAPHICAL REPRESENTAION ON A NUMBER LINE: The solution set of $a^2 = 25$ is as follows:



In this graphical representation, a solid dot is used to indicate a point on the number line that is to be included in the solution set. (WARNING: The dot one physically draws is larger than the point it represents! One hopes that it is clear from the context of the diagram which point each dot refers to.)

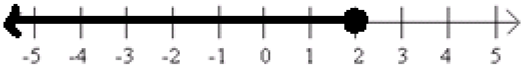
Exit Ticket (3 minutes)

Name ____ Date____________________

Lesson 11: Solution Sets for Equations and Inequalities

Exit Ticket

1. Here is the graphical representation of a set of real numbers:



   a. Describe this set of real numbers in words.

   b. Describe this set of real numbers in set notation.

   c. Write an equation or an inequality that has the set above as its solution set.

2. Indicate whether each of the following equations is sure to have a solution set of all real numbers. Explain your answers for each.

   a. $3(x+1) = 3x+1$

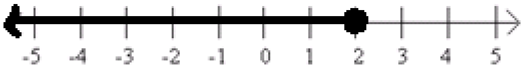
   b. $x+2 = 2+x$

   c. $4x(x+1) = 4x+4x^2$

   d. $3x(4x)(2x) = 72x^3$

Exit Ticket Sample Solutions

1. **Here is the graphical representation of a set of real numbers:**



   a. **Describe this set of real numbers in words.**

   The set of all real numbers less than or equal to two.

   b. **Describe this set of real numbers in set notation.**

   $\{r \text{ real} \mid r \leq 2\}$ (Students might use any variable.)

   c. **Write an equation or an inequality that has the set above as its solution set.**

   $w - 7 \leq -5$ (Answers will vary. Students might use any variable.)

2. **Indicate whether each of the following equations is sure to have a solution set of all real numbers. Explain your answers for each.**

   a. $3(x+1) = 3x + 1$

   No, the two algebraic expressions are not equivalent.

   b. $x + 2 = 2 + x$

   Yes, the two expressions are algebraically equivalent by application of the commutative property.

   c. $4x(x+1) = 4x + 4x^2$

   Yes, the two expressions are algebraically equivalent by application of the distributive property and the commutative property.

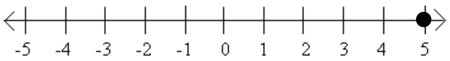
   d. $3x(4x)(2x) = 72x^3$

   No, the two algebraic expressions are not equivalent.

Problem Set Sample Solutions

For each solution set graphed below, (a) describe the solution set in words, (b) describe the solution set in set notation, and (c) write an equation or an inequality that has the given solution set.

1.

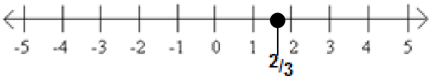


a. the set of all real numbers equal to 5

b. $\{5\}$

c. answers vary

2.

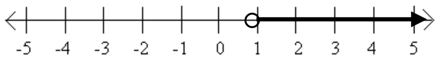


a. the set of all real numbers equal to $\frac{2}{3}$

b. $\left\{\frac{2}{3}\right\}$

c. answers vary

3.

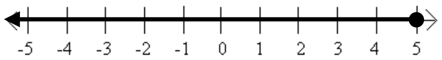


a. the set of all real numbers greater than 1

b. $\{x \text{ real} \mid x > 1\}$

c. answers vary

4.

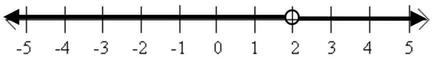


a. the set of all real numbers less or equal to 5

b. $\{x \text{ real} \mid x \leq 5\}$

c. answers vary

5.

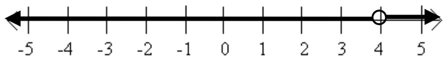


a. the set of all real numbers not equal to 2

b. $\{x \text{ real} \mid x \neq 2\}$

c. answers vary

6.

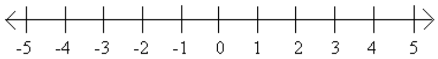


a. the set of all real numbers not equal to 4

b. $\{x \text{ real} \mid x \neq 4\}$

c. answers vary

7.

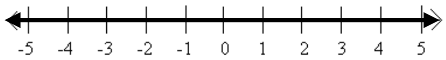


a. the null set

b. { } or ∅

c. answers vary

8.

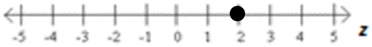
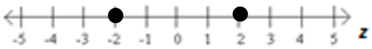
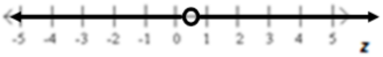
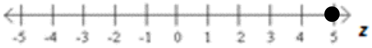
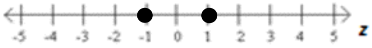
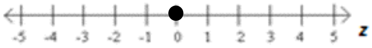
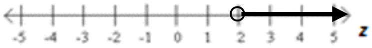
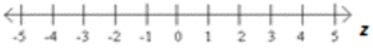
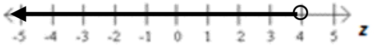
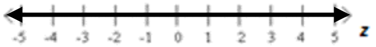


a. the set of all real numbers

b. $\{\mathbb{R}.\}$

c. answers vary

Fill in the chart below.

	SOLUTION SET IN WORDS	SOLUTION SET IN SET NOTATION	GRAPH
9. $z = 2$	*The set of real numbers equal to* 2	$\{2\}$	
10. $z^2 = 4$	*The set of real numbers equal to* 2 *or* -2	$\{-2, 2\}$	
11. $4z \neq 2$	*The set of real numbers not equal to* $\frac{1}{2}$.	$\{z \text{ real} \mid z \neq \frac{1}{2}\}$	
12. $z - 3 = 2$	*The set of real numbers equal to* 5	$\{5\}$	
13. $z^2 + 1 = 2$	*The set of real numbers equal to* 1 *or* -1	$\{-1, 1\}$	
14. $z = 2z$	*The set of real numbers equal to* 0	$\{0\}$	
15. $z > 2$	*The set of real numbers greater than* 2	$\{z \text{ real} \mid z > 2\}$	
16. $z - 6 = z - 2$	*The null set*	{ }	
17. $z - 6 < -2$	*The set of real numbers less than* 4	$\{z \text{ real} \mid z < 4\}$	
18. $4(z - 1) \geq 4z - 4$	*The set of all real numbers*	$\mathbb{R}$	

For Problems 19–24, answer the following: Are the two expressions algebraically equivalent? If so, state the property (or properties) displayed. If not, state why (the solution set may suffice as a reason) and change the equation, ever so slightly, e.g., touch it up, to create an equation whose solution set is all real numbers.

19. $x(4 - x^2) = (-x^2 + 4)x$

Yes, commutative

20. $\frac{2x}{2x} = 1$

No, the solution set is $\{x \text{ real} \mid x \neq 0\}$*. If we changed it to* $\frac{2x}{1} = 2x$*, it would have a solution set of all real numbers.*

21. $(x - 1)(x + 2) + (x - 1)(x - 5) = (x - 1)(2x - 3)$

Yes, distributive

22. $\frac{x}{5} + \frac{x}{3} = \frac{2x}{8}$

No, the solution set is $\{0\}$*. If we changed it to* $\frac{x}{5} + \frac{x}{3} = \frac{8x}{15}$*, it would have a solution set of all real numbers.*

23. $x^2 + 2x^3 + 3x^4 = 6x^9$

No, neither the coefficients nor the exponents are added correctly. One way it could have a solution set of all real numbers would be $x^2 + 2x^3 + 3x^4 = x^2(1 + 2x + 3x^2)$*.*

24. $x^3 + 4x^2 + 4x = x(x + 2)^2$

Yes, distributive

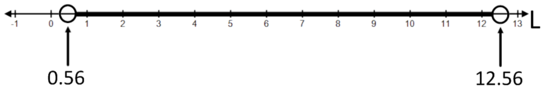
25. **Solve for w:** $\frac{6w+1}{5} \neq 2$**. Describe the solution set in set notation.**

$\left\{w \text{ real} \mid w \neq 1\frac{1}{2}\right\}$

26. **Edwina has two sticks, one 2 yards long and the other 2 meters long. She is going to use them, with a third stick of some positive length, to make a triangle. She has decided to measure the length of the third stick in units of feet.**

a. **What is the solution set to the statement: "Sticks of lengths 2 yards, 2 meters, and L feet make a triangle"? Describe the solution set in words and through a graphical representation.**

One meter is equivalent, to two decimal places, to 3.28 *feet. We have that* L *must be a positive length greater than* 0.56 *feet and less than* 12.56 *feet. Within these values, the sum of any two sides will be greater than the third side.*



b. **What is the solution set to the statement: "Sticks of lengths 2 yards, 2 meters, and L feet make an isosceles triangle"? Describe the solution set in words and through a graphical representation.**

$L = 6$ *feet or* $L = 6.56$ *feet.*

c. **What is the solution set to the statement: "Sticks of lengths 2 yards, 2 meters, and L feet make an equilateral triangle"? Describe the solution set in words and through a graphical representation.**

The solution set is the empty set.

Lesson 12: Solving Equations

Student Outcomes

- Students are introduced to the formal process of solving an equation: starting from the assumption that the original equation has a solution. Students explain each step as following from the properties of equality. Students identify equations that have the same solution set.

Classwork

Opening Exercise (4 minutes)

Opening Exercise

Answer the following questions.

a. **Why should the equations $(x-1)(x+3)=17+x$ and $(x-1)(x+3)=x+17$ have the same solution set?**

   The commutative property.

b. **Why should the equations $(x-1)(x+3)=17+x$ and $(x+3)(x-1)=17+x$ have the same solution set?**

   The commutative property.

c. **Do you think the equations $(x-1)(x+3)=17+x$ and $(x-1)(x+3)+500=517+x$ should have the same solution set? Why?**

   Yes, 500 was added to both sides.

d. **Do you think the equations $(x-1)(x+3)=17+x$ and $3(x-1)(x+3)=51+3x$ should have the same solution set? Explain why.**

   Yes, both sides were multiplied by 3.

Discussion (4 minutes)

Allow students to attempt to justify their answers for (a) and (b) above. Then summarize with the following:

- We know that the commutative and associative properties hold for all real numbers. We also know that variables are placeholders for real numbers, and the value(s) assigned to a variable that make an equation true is the "solution." If we apply the commutative and associative properties of real numbers to an expression, we obtain an equivalent expression. Therefore, equations created this way (by applying the commutative and associative properties to one or both expressions) consist of expressions equivalent to those in the original equation.
- In other words, if x is a solution to an equation, then it will also be a solution to any new equation we make by applying the commutative and associative properties to the expression in that equation.

Exercise 1 (3 minutes)

Exercise 1

a. **Use the commutative property to write an equation that has the same solution set as**

$\mathbf{x^2 - 3x + 4 = (x+7)(x-12)(5).}$

$\mathbf{-3x + x^2 + 4 = (x+7)(5)(x-12)}$

b. **Use the associative property to write an equation that has the same solution set as**

$\mathbf{x^2 - 3x + 4 = (x+7)(x-12)(5).}$

$(\mathrm{x}^2 - 3\mathrm{x}) + 4 = \big((\mathrm{x}+7)(\mathrm{x}-12)\big)(5)$

c. **Does this reasoning apply to the distributive property as well?**

Yes, it does apply to the distributive property.

Discussion (3 minutes)

- Parts (c) and (d) of the Opening Exercise rely on key properties of equality. What are they?

Call on students to articulate and compare their thoughts as a class discussion. In middle school, these properties are simply referred to as the *if-then moves*. Introduce their formal names to the class, the *additive and multiplicative properties of equality*. Summarize with the following as you write it on the board:

- So, whenever $a = b$ is true, then $a + c = b + c$ will also be true for all real numbers c.
- What if $a = b$ is false?
  - *Then $a + c = b + c$ will also be false.*
- Is it also ok, to subtract a number from both sides of the equation?
  - *Yes, this is the same operation as adding the opposite of that number.*
- Whenever $a = b$ is true, then $ac = bc$ will also be true, and whenever $a = b$ is false, $ac = bc$ will also be false for all non-zero real numbers c.
- So, we have said earlier that applying the distributive, associative, and commutative properties does not change the solution set, and now we see that applying the additive and multiplicative properties of equality also preserves the solution set (does not change it).
- Suppose I see the equation $|x| + 5 = 2$. (Write the equation on the board.)
- Is it true, then, that $|x| + 5 - 5 = 2 - 5$? (Write the equation on the board.)

MP.1

Allow students to verbalize their answer and challenge each other if they disagree. If they all say, "yes", prompt students with, "Are you sure?" until one or more students articulate that we would get the false statement: $|x| = -3$. Then summarize with the following points. Give these points great emphasis, perhaps adding grand gestures or voice inflection to recognize the importance of this moment, as it addresses a major part of **A-REI.A.1**:

- So our idea that adding the same number to both sides gives us another true statement depends on the idea that the first equation has a value of x that makes it true to begin with.

MP.1

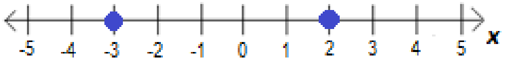
- It is a big assumption that we make when we start to solve equations using properties of equality. We are assuming there is some value for the variable that makes the equation true. **IF** there is, then it makes sense that applying the properties of equality will give another true statement. But we must be cognizant of that big "IF".
- What if there was no value of x to make the equation true? What is the effect of adding a number to both sides of the equation or multiplying both sides by a non-zero number?
    - *There still will be no value of x that makes the equation true. The solution set is still preserved; it will be the empty set.*
- Create another equation that initially seems like a reasonable equation to solve but in fact has no possible solution.

Exercise 2 (7 minutes)

Exercise 2

Consider the equation $x^2 + 1 = 7 - x$.

a. **Verify that this has the solution set $\{-3, 2\}$. Draw this solution set as a graph on the number line. *We will later learn how to show that these happen to be the ONLY solutions to this equation.***



$2^2 + 1 = 7 - 2$ *True.* $(-3)^2 + 1 = 7 - -3$ *True.*

b. **Let's add four to both sides of the equation and consider the new equation $x^2 + 5 = 11 - x$. Verify 2 and -3 are still solutions.**

$2^2 + 5 = 11 - 2$ *True.* $(-3)^2 + 5 = 11 - (-3)$ *True. They are still solutions.*

c. **Let's now add x to both sides of the equation and consider the new equation $x^2 + 5 + x = 11$. Are 2 and -3 still solutions?**

$2^2 + 5 + 2 = 11$ *True.* $(-3)^2 + 5 + -3 = 11$ *True.* *2 and -3 are still solutions.*

d. **Let's add -5 to both sides of the equation and consider the new equation $x^2 + x = 6$. Are 2 and -3 still solutions?**

$2^2 + 2 = 6$ *True.* $(-3)^2 + -3 = 6$ *True.* *2 and -3 are still solutions.*

e. **Let's multiply both sides by $\frac{1}{6}$ to get $\frac{x^2+x}{6} = 1$. Are 2 and -3 still solutions?**

$\frac{2^2+2}{6} = 1$ *True.* $\frac{(-3)^2+(-3)}{6} = 1$ *True.* *2 and -3 are still solutions.*

f. **Let's go back to part (d) and add $3x^3$ to both sides of the equation and consider the new equation $x^2 + x + 3x^3 = 6 + 3x^3$. Are 2 and -3 still solutions?**

$2^2 + 2 + 3(2)^3 = 6 + 3(2)^3$ $(-3)^2 + -3 + 3(-3)^3 = 6 + 3(-3)^3$

$4 + 2 + 24 = 6 + 24$ *True.* $9 - 3 - 81 = 6 - 81$ *True.*

2 and -3 are still solutions.

Discussion (4 minutes)

- In addition to applying the commutative, associative, and distributive properties to equations, according to the exercises above, what else can be done to equations that does not change the solution set?
  - *Adding a number to or subtracting a number from both sides.*
  - *Multiplying or dividing by a non-zero number.*
- What we discussed in Example 1 can be rewritten slightly to reflect what we have just seen: If x is a solution to an equation, it will also be a solution to the new equation formed when the same number is added to (or subtracted from) each side of the original equation or when the two sides of the original equation are multiplied by the same number or divided by the same non-zero number. These are referred to as the Properties of Equality. This now gives us a strategy for finding solution sets.
- Is $x = 5$ an equation? If, so what is its solution set?
  - *Yes, its solution set is* 5.
- This example is so simple that it is hard to wrap your brain around, but it points out that if ever we have an equation that is this simple, we know its solution set.
- We also know the solution sets to some other simple equations, such as

  (a) $w^2 = 64$ (b) $7 + P = 5$ (c) $3\beta = 10$

- Here's the strategy:

  If we are faced with the task of solving an equation, that is, finding the solution set of the equation:

  Use the commutative, associative, distributive properties

  AND

  Use the properties of equality (adding, subtracting, multiplying, dividing by non-zeros)

  to keep rewriting the equation into one whose solution set you easily recognize. (We observed that the solution set will not change under these operations.)

- This usually means rewriting the equation so that all the terms with the variable appear on one side of the equation.

Exercise 3 (5 minutes)

Exercise 3

a. **Solve for r: $\frac{3}{2r} = \frac{1}{4}$**

$r = 6$

b. **Solve for s: $s^2 + 5 = 30$**

$s = 5, s = -5$

c. **Solve for y: $4y - 3 = 5y - 8$**

$y = 5$

Exercise 4 (5 minutes)

- Does it matter which step happens first? Let's see what happens with the following example.

Do a quick count-off or separate the class into quadrants. Give groups their starting points. Have each group designate a presenter, so the whole class can see the results.

MP.3

Exercise 4

Consider the equation $3x + 4 = 8x - 16$. Solve for x using the given starting point.

Group 1	Group 2	Group 3	Group 4
Subtract $3x$ from both sides	*Subtract 4 from both sides*	*Subtract $8x$ from both sides*	*Add 16 to both sides*
$3x + 4 - 3x = 8x - 16 - 3x$	$3x + 4 - 4 = 8x - 16 - 4$	$3x + 4 - 8x = 8x - 16 - 8x$	$3x + 4 + 16 = 8x - 16 + 16$
$4 = 5x - 16$	$3x = 8x - 20$	$-5x + 4 = -16$	$3x + 20 = 8x$
$4 + 16 = 5x - 16 + 16$	$3x - 8x = 8x - 20 - 8x$	$-5x + 4 - 4 = -16 - 4$	$3x + 20 - 3x = 8x - 3x$
$20 = 5x$	$-5x = -20$	$-5x = -20$	$20 = 5x$
$\frac{20}{5} = \frac{5x}{5}$	$\frac{-5x}{-5} = \frac{-20}{-5}$	$\frac{-5x}{-5} = \frac{-20}{-5}$	$\frac{20}{5} = \frac{5x}{5}$
$4 = x$	$x = 4$	$x = 4$	$4 = x$
$\{4\}$	$\{4\}$	$\{4\}$	$\{4\}$

- Therefore, according to this exercise, does it matter which step happens first? Explain why or why not.
  - *No, because the properties of equality produce equivalent expressions, no matter the order in which they happen.*
- How does one know "how much" to add/subtract/multiply/divide? What's the goal of using the properties and how do they allow equations to be solved?

Encourage students to verbalize their strategies to the class and to question each other's reasoning and question the precision of each other's description of their reasoning. From middle school, students recall that the goal is to isolate the variable by making 0s and 1s. Add/subtract numbers to make the zeros, and multiply/divide numbers to make the 1s. The properties say any numbers will work, which is true, but with the 0s and 1s goal in mind, equations can be solved very efficiently.

- The ability to pick the most efficient solution method comes with practice.

Closing Exercise (5 minutes)

Answers will vary. As time permits, share several examples of student responses.

MP.2

Closing Exercise

Consider the equation $3x^2 + x = (x - 2)(x + 5)x$.

a. Use the commutative property to create an equation with the same solution set.

$x + 3x^2 = (x + 5)(x - 2)x$

b. Using the result from (a), use the associative property to create an equation with the same solution set.

$(x + 3x^2) = ((x + 5)(x - 2))x$

c. Using the result from (b), use the distributive property to create an equation with the same solution set.

$x + 3x^2 = x^3 + 3x^2 - 10x$

d. **Using the result from (c), add a number to both sides of the equation.**

$x + 3x^2 + 5 = x^3 + 3x^3 - 10x + 5$

e. **Using the result from (d), subtract a number from both sides of the equation.**

$x + 3x^2 + 5 - 3 = x^3 + 3x^2 - 10x + 5 - 3$

MP.2

f. **Using the result from (e), multiply both sides of the equation by a number.**

$4(x + 3x^2 + 2) = 4(x^3 + 3x^2 - 10x + 2)$

g. **Using the result from (f), divide both sides of the equation by a number.**

$x + 3x^2 + 2 = x^3 + 3x^2 - 10x + 2$

h. **What do all seven equations have in common? Justify your answer.**

They will all have the same solution set.

Lesson Summary

If x is a solution to an equation, it will also be a solution to the new equation formed when the same number is added to (or subtracted from) each side of the original equation or when the two sides of the original equation are multiplied by (or divided by) the same non-zero number. These are referred to as the *Properties of Equality*.

If one is faced with the task of solving an equation, that is, finding the solution set of the equation:

> **Use the *commutative*, *associative*, and *distributive properties*, AND use the *properties of equality* (adding, subtracting, multiplying by non-zeros, dividing by non-zeros) to keep rewriting the equation into one whose solution set you easily recognize. (We believe that the solution set will not change under these operations.)**

Exit Ticket (5 minutes)

Name ____ Date_____________________

Lesson 12: Solving Equations

Exit Ticket

Determine which of the following equations have the same solution set by recognizing properties, rather than solving.

a. $2x + 3 = 13 - 5x$

b. $6 + 4x = -10x + 26$

c. $6x + 9 = \frac{13}{5} - x$

d. $0.6 + 0.4x = -x + 2.6$

e. $3(2x + 3) = \frac{13}{5} - x$

f. $4x = -10x + 20$

g. $15(2x + 3) = 13 - 5x$

h. $15(2x + 3) + 97 = 110 - 5x$

Exit Ticket Sample Solutions

Determine which of the following equations have the same solution set by recognizing properties, rather than solving.

a. $2x+3=13-5x$ **b.** $6+4x=-10x+26$ **c.** $6x+9=\frac{13}{5}-x$

d. $0.6+0.4x=-x+2.6$ **e.** $3(2x+3)=\frac{13}{5}-x$ **f.** $4x=-10x+20$

g. $15(2x+3)=13-5x$ **h.** $15(2x+3)+97=110-5x$

(a), (b), (d), and (f) have the same solution set. (c), (e), (g), and (h) have the same solution set.

Problem Set Sample Solutions

1. **Which of the following equations have the same solution set? Give reasons for your answers that do not depend on solving the equations.**

I. $x-5=3x+7$ **II.** $3x-6=7x+8$ **III.** $15x-9=6x+24$

IV. $6x-16=14x+12$ **V.** $9x+21=3x-15$ **VI.** $-0.05+\frac{x}{100}=\frac{3x}{100}+0.07$

I, V, and VI all have the same solution set; V is the same as I after multiplying both sides by 3 and switching the left side with the right side; VI is the same as 1 after dividing both sides by 100 and using the commutative property to rearrange the terms on the left side of the equation.

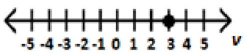
II and IV have the same solution set. IV is the same as II after multiplying both sides by 2 and subtracting 4 from both sides.

III does not have the same solution set as any of the others.

Solve the following equations, check your solutions, and then graph the solution sets.

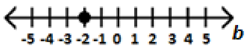
2. $-16-6v=-2(8v-7)$

$\{3\}$



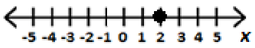
3. $2(6b+8)=4+6b$

$\{-2\}$



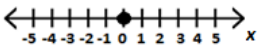
4. $x^2-4x+4=0$

$\{2\}$



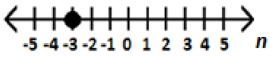
5. $7-8x=7(1+7x)$

$\{0\}$



6. $39-8n=-8(3+4n)+3n$

$\{-3\}$



7. $(x-1)(x+5)=x^2+4x-2$

no solution

8. $x^2 - 7 = x^2 - 6x - 7$

$\{0\}$

9. $-7 - 6a + 5a = 3a - 5a$

$\{7\}$

10. $7 - 2x = 1 - 5x + 2x$

$\{-6\}$

11. $4(x - 2) = 8(x - 3) - 12$

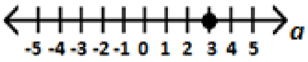
$\{7\}$

12. $-3(1 - n) = -6 - 6n$

$\left\{-\frac{1}{3}\right\}$

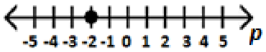
13. $-21 - 8a = -5(a + 6)$

$\{3\}$



14. $-11 - 2p = 6p + 5(p + 3)$

$\{-2\}$



15. $\frac{x}{x+2} = 4$

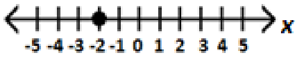
$\left\{-\frac{8}{3}\right\}$

16. $2 + \frac{x}{9} = \frac{x}{3} - 3$

$\left\{\frac{45}{2}\right\}$

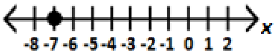
17. $-5(-5x - 6) = -22 - x$

$\{-2\}$



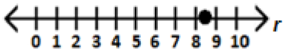
18. $\frac{x+4}{3} = \frac{x+2}{5}$

$\{-7\}$



19. $-5(2r - 0.3) + 0.5(4r + 3) = -64$

$\left\{\frac{67}{8}\right\}$



Lesson 13: Some Potential Dangers when Solving Equations

Student Outcomes

- Students learn "if-then" moves using the properties of equality to solve equations. Students also explore moves that may result in an equation having more solutions than the original equation.

In previous lessons we have looked at techniques for solving equations, a common theme throughout algebra. In this lesson, we will examine some potential dangers where our intuition about algebra may need to be examined.

Exercise 1 (4 minutes)

Give students a few minutes to answer the questions individually. Then, elicit responses from students.

Exercise 1

a. **Describe the property used to convert the equation from one line to the next:**

$x(1-x)+2x-4=8x-24-x^2$	
$x-x^2+2x-4=8x-24-x^2$	*distributive property*
$x+2x-4=8x-24$	*added x^2 to both sides of the equation*
$3x-4=8x-24$	*collected like terms*
$3x+20=8x$	*added 24 to both sides of the equation*
$20=5x$	*subtracted $3x$ from both sides of the equation*

In each of the steps above, we applied a property of real numbers and/or equations to create a new equation.

b. **Why are we sure that the initial equation $x(1-x)+2x-4=8x-24-x^2$ and the final equation $20=5x$ have the same solution set?**

We established last class that making use of the commutative, associative, and distributive properties, and properties of equality to "rewrite" an equation does not change the solution set of the equation.

c. **What is the common solution set to all these equations?**

$x=4$

- Do we know for certain that $x=4$ is the solution to every equation shown? Explain why.
  - *Have students verify this by testing the solution in a couple of the equations.*

Exercise 2 (4 minutes)

Work through the exercise as a class. Perhaps have one student writing the problem on the board and one student writing the operation used in each step as the class provides responses.

Emphasize that the solution obtained in the last step is the same as the solution to each of the preceding equations. The moves made in each step did not change the solution set.

Consider:

- Students may point out that they solved the equation in a different way but got the same answer. Consider allowing them to show their approach and discuss whether or not it was algebraically sound.

Exercise 2

Solve the equation for x. For each step, describe the operation used to convert the equation.

$$3x - [8 - 3(x-1)] = x + 19$$

$3x - [8 - 3(x-1)] = x + 19$	
$3x - (8 - 3x + 3) = x + 19$	*distributive property*
$3x - (11 - 3x) = x + 19$	*commutative property/collected like terms*
$3x - 11 + 3x = x + 19$	*distributive property*
$6x - 11 = x + 19$	*commutative property/collected like terms*
$5x - 11 = 19$	*subtracted x from both sides*
$5x = 30$	*added 11 to both sides*
$x = 6$	*divided both sides by 5*

Exercise 3 (8 minutes)

Exercise 3

Solve each equation for x. For each step, describe the operation used to convert the equation.

a. $7x - [4x - 3(x-1)] = x + 12$

$\{3\}$

b. $2[2(3 - 5x) + 4] = 5[2(3 - 3x) + 2]$

$\{2\}$

c. $\frac{1}{2}(18 - 5x) = \frac{1}{3}(6 - 4x)$

$\{6\}$

Note with the class that students may have different approaches that arrived at the same answer.

Ask students how they handled the fraction in part (c).

- Was it easier to use the distributive property first or multiply both sides by 6 first?

Discussion (10 minutes)

Use the following sample dialog to inspire a similar exchange between you and your students where you play the part of Mike, suggesting ideas of actions you could perform on both sides of an equation that would not predictably preserve the solution set of the original equation. Start by asking students to summarize what they have been studying over the last two lessons and then make Mike's first suggestion. Be sure to provide more than one idea for things that could be done to both sides of an equation that might result in solutions that are not part of the solution set for the original equation, and conclude with an affirmation that you can try anything, but you will have to check to see if your solutions work with the original equation.

MP.3

- Fergus says, "Basically, what I've heard over the last two lessons is that whatever you do to the left side of the equation, do the same thing to the right side. Then solutions will be good."
- Lulu says, "Well, we've only said that for the properties of equality – adding quantities and multiplying by non-zero quantities. (And associative, commutative, and distributive properties too.) Who knows if it is true in general?"
- Mike says, "Okay … Here's an equation:

$$\frac{x}{12} = \frac{1}{3}$$

- If I follow the idea, "Whatever you do to the left, do to the right as well," then I am in trouble. What if I decide to remove the denominator on the left and also remove the denominator on the right. I get $x = 1$. Is that a solution?"
- Fergus replies, "Well, that is silly. We all know that is a wrong thing to do. You should multiply both sides of that equation by 12. That gives $x = 4$, and that does give the correct solution."
- Lulu says, "Okay Fergus, you have just acknowledged that there are some things we can't do! Even if you don't like Mike's example, he's got a point."
- Mike or another student says, "What if I take your equation and choose to square each side. This gives

$$\frac{x^2}{144} = \frac{1}{9}$$

- Multiplying through by 144 gives $x^2 = \frac{144}{9} = 16$, which has solutions $x = 4$ AND $x = -4$."
- Fergus responds, "Hmmm. Okay I do see the solution $x = 4$, but the appearance of $x = -4$ as well is weird."
- Mike says, "Lulu is right. Over the past two days we have learned that using the commutative, associative, and distributive properties, along with the properties of equality (adding and multiplying equations throughout) definitely DOES NOT change solution sets. BUT if we do anything different from this we might be in trouble."

MP.3

- Lulu continues, "Yeah! Basically when we start doing unusual operations on an equation, we are really saying that **IF** we have a solution to an equation, then it should be a solution to the next equation as well. BUT remember, it could be that there was no solution to the first equation anyway!"
- Mike says, "So feel free to start doing weird things to both sides of an equation if you want (though you might want to do sensible weird things!), but all you will be getting are possible CANDIDATES for solutions. You are going to have to check at the end if they really are solutions."

Exercises 4–7 (12 minutes)

Allow students to work through Exercises 4–7 either individually or in pairs. Point out that they are trying to determine what impact certain moves have on the solution set of an equation.

Exercise 4

Consider the equations $x+1=4$ and $(x+1)^2=16$.

a. **Verify that $x=3$ is a solution to both equations.**

$3+1=4$
$(3+1)^2=16$

b. **Find a second solution to the second equation.**

$x=-5$

c. **Based on your results, what effect does squaring both sides of an equation appear to have on the solution set?**

Answers will vary. The new equation seems to retain the original solution and add a second solution.

Exercise 5

Consider the equations $x-2=6-x$ and $(x-2)^2=(6-x)^2$.

a. **Did squaring both sides of the equation affect the solution sets?**

No, $x=4$ is the only solution to both equations.

b. **Based on your results, does your answer to part (c) of the previous question need to be modified?**

The new equation retains the original solution and may add a second solution.

MP.3

Exercise 6

Consider the equation $x^3+2=2x^2+x$.

a. **Verify that $x=1$, $x=-1$, and $x=2$ are each solutions to this equation.**

$(1)^3+2=2(1)^2+1$ True

$(-1)^3+2=2(-1)^2+(-1)$ *True*

$(2)^3+2=2(2)^2+2$ *True*

MP.3

b. **Bonzo decides to apply the action "Ignore the exponents" on each side of the equation. He gets $x + 2 = 2x + x$. In solving this equation, what does he obtain? What seems to be the problem with his technique?**

$x = 1$; *The problem is that he only finds one of the three solutions to the equation.*

c. **What would Bonzo obtain if he applied his "method" to the equation $x^2 + 4x + 2 = x^4$? Is it a solution to the original equation?**

$x = -\frac{1}{2}$; *No, it is not a solution to the original equation.*

Exercise 7

Consider the equation $x - 3 = 5$.

a. **Multiply both sides of the equation by a constant, and show that the solution set did not change.**

$$7(x-3) = 7(5)$$
$$7(8-3) = 7(5)$$
$$7(5) = 7(5)$$

Now, multiply both sides by x.

$$x(x-3) = 5x$$

b. **Show that $x = 8$ is still a solution to the new equation.**

$$8(8-3) = 5(8)$$
$$8(5) = 5(8)$$

c. **Show that $x = 0$ is also a solution to the new equation.**

$$0(0-3) = 5(0)$$

Now, multiply both sides by the factor $x - 1$.

$$(x-1)x(x-3) = 5x(x-1)$$

d. **Show that $x = 8$ is still a solution to the new equation.**

$$(8-1)(8)(8-3) = 5(8)(8-1)$$
$$(7)(8)(5) = 5(8)(7)$$

e. **Show that $x = 1$ is also a solution to the new equation.**

$$(1-1)(1)(1-3) = 5(1)(1-1)$$
$$0(1)(-2) = 5(1)(0)$$
$$0 = 0$$

f. **Based on your results, what effect does multiplying both sides of an equation by a constant have on the solution set of the new equation?**

Multiplying by a constant does not change the solution set.

g. **Based on your results, what effect does multiplying both sides of an equation by a variable factor have on the solution set of the new equation?**

Multiplying by a variable factor could produce additional solution(s) to the solution set.

Scaffold:

Have early finishers explore the idea of cubing both sides of an equation. If $x = 2$, then $x^3 = 8$. If $x^3 = 8$, can x equal any real number besides 2?

Review answers and discuss the following points:

- Does squaring both sides of an equation change the solution set?
  - *Sometimes but not always!*
- For Exercise 6, was it just luck that Bonzo got one out of the three correct answers?
  - *Yes, in part (c), the answer obtained is not a solution to the original equation.*

Consider having students make up another problem to verify.

- What effect did multiplying both sides by a variable factor have on the solution set?
  - *In our case, it added another solution to the solution set.*
- Can we predict what the second solution will be?

Have students make up another problem to test the prediction.

Closing (2 minutes)

- What moves have we seen that do not change the solution set of an equation?
- What moves did change the solution set?
- What limitations are there to the principle "whatever you do to one side of the equation, you must do to the other side?"

Lesson Summary

Assuming that there is a solution to an equation, applying the distributive, commutative, and associative properties and the properties of equality to equations will not change the solution set.

Feel free to try doing other operations to both sides of an equation, but be aware that the new solution set you get contains possible <u>candidates</u> for solutions. You have to plug each one into the original equation to see if it really is a solution to your original equation.

Exit Ticket (5 minutes)

Name ____ Date____________________

Lesson 13: Some Potential Dangers when Solving Equations

Exit Ticket

1. Solve the equation for x. For each step, describe the operation and/or properties used to convert the equation.

$$5(2x-4)-11=4+3x$$

2. Consider the equation $x+4=3x+2$.

   a. Show that adding $x+2$ to both sides of the equation does not change the solution set.

   b. Show that multiplying both sides of the equation by $x+2$ adds a second solution of $x=-2$ to the solution set.

Exit Ticket Sample Responses

1. **Solve the equation for x. For each step, describe the operation and/or properties used to convert the equation.**

$$5(2x - 4) - 11 = 4 + 3x$$

Solution set is $\{5\}$*.*

2. **Consider the equation $x + 4 = 3x + 2$.**

   a. **Show that adding $x + 2$ to both sides of the equation does not change the solution set.**

$x + 4 = 3x + 2$	$x + 4 + x + 2 = 3x + 2 + x + 2$
$4 = 2x + 2$	$2x + 6 = 4x + 4$
$2 = 2x$	$2 = 2x$
$1 = x$	$1 = x$

   b. **Show that multiplying both sides of the equation by $x + 2$ adds a second solution of $x = -2$ to the solution set.**

$$(x + 2)(x + 4) = (x + 2)(3x + 2)$$
$$(-2 + 2)(-2 + 4) = (-2 + 2)(3(-2) + 2)$$
$$(0)(2) = (0)(-4)$$
$$0 = 0$$

Problem Set Sample Responses

1. **Solve each equation for x. For each step, describe the operation used to convert the equation. How do you know that the initial equation and the final equation have the same solution set?**

Steps will vary as in the exit ticket and exercises.

   a. $\frac{1}{5}[10 - 5(x - 2)] = \frac{1}{10}(x + 1)$

   Solution set is $\left\{\frac{39}{11}\right\}$*.*

   b. $x(5 + x) = x^2 + 3x + 1$

   Solution set is $\left\{\frac{1}{2}\right\}$*.*

   c. $2x(x^2 - 2) + 7x = 9x + 2x^3$

   Solution set is $\{0\}$*.*

2. **Consider the equation $x + 1 = 2$.**

Students should write the new equations and the solution sets:

   a. **Find the solution set.**

   Solution set is $\{1\}$*.*

b. **Multiply both sides by $x+1$ and find the solution set of the new equation.**

New solution set is $\{\pm 1\}$.

c. **Multiply both sides of the original equation by x and find the solution set of the new equation.**

New solution set is $\{0, 1\}$.

3. **Solve the equation $x+1=2x$ for x. Square both sides of the equation and verify that your solution satisfies this new equation. Show that $-\frac{1}{3}$ satisfies the new equation but not the original equation.**

The solution of $x+1=2x$ *is* $x=1$. *The equation obtained by squaring is* $(x+1)^2=4x^2$.

Let $x=1$ *in the new equation.* $(1+1)^2=4(1)^2$ *is true, so* $x=1$ *is still as solution.*

Let $x=-\frac{1}{3}$ *in the new equation.* $\left(-\frac{1}{3}+1\right)^2=4\left(-\frac{1}{3}\right)^2$ *is true, so x =* $-\frac{1}{3}$ *is also a solution to the new equation.*

4. **Consider the equation $x^3=27$.**

a. **What is the solution set?**

Solution set is $\{3\}$.

b. **Does multiplying both sides by x change the solution set?**

Yes.

c. **Does multiplying both sides by x^2 change the solution set?**

Yes.

5. **Consider the equation $x^4=16$.**

a. **What is the solution set?**

Solution set is $\{-2, 2\}$.

b. **Does multiplying both sides by x change the solution set?**

Yes.

c. **Does multiplying both sides by x^2 change the solution set?**

Yes.

Lesson 14: Solving Inequalities

Student Outcomes

- Students learn *if-then* moves using the addition and multiplication properties of inequality to solve inequalities and graph the solution sets on the number line.

Classwork

Exercise 1 (5 minutes)

Allow students time to work through the warm up individually. Then, discuss results. Let several students share values that work. For each part of the exercise, demonstrate that these values still work for each new inequality.

Exercise 1

Consider the inequality $x^2 + 4x \geq 5$.

a. **Sift through some possible values to assign to x that make this inequality a true statement. Find at least two positive values that work and at least two negative values that work.**

Any value such that $x \leq -5$ or $x \geq 1$.

b. **Should your four values also be solutions to the inequality $x(x+4) \geq 5$? Explain why or why not. Are they?**

Yes, the inequality can be returned to its original form by using the distributive property, so the two are equivalent.

c. **Should your four values also be solutions to the inequality $4x + x^2 \geq 5$? Explain why or why not. Are they?**

Yes, the terms on the left-hand side can be rearranged using the commutative property.

d. **Should your four values also be solutions to the inequality $4x + x^2 - 6 \geq -1$? Explain why or why not. Are they?**

Yes, 6 was subtracted from both sides of the inequality. If the same principle of equality holds for inequalities, then the original solutions should still work. (Make sure students confirm this with their answers from (a).

e. **Should your four values also be solutions to the inequality $12x + 3x^2 \geq 15$? Explain why or why not. Are they?**

Yes, both sides were multiplied by 3. Using the same multiplication principle of equality, the solution set should not change. (Again, make sure students test this.)

Scaffold:

- Remind students about the significance of the open circle or solid circle on the endpoint of the solution set.

While discussing parts (b) and (c), guide the class discussion to lead to the following conclusion:

- Just like all our previous work on equations, rewriting an inequality via the commutative, associative, and distributive properties of the real numbers does not change the solution set of that inequality.

Make the point that we're talking about $>$, $<$, $\geq$, and $\leq$.

While discussing (d), point out that it appears we are choosing to accept the *addition (and subtraction) property of inequality* (If $A > B$, then $A + c > B + c$). Students have previously referred to this property as an "if-then" move.

Ask students to articulate the property as formally as they can; for example, "adding a value to each side of an inequality does not change the solution set of that inequality."

Conduct a similar discussion to those conducted in Lesson 12. Include the following:

- Could the solution set to an inequality be changed by applying the commutative, associative, or distributive properties to either side of an inequality?
    - *No, the solution set would not change*.
- Could the solution set to an inequality be changed by applying the additive property of inequality?
    - *No, the solution set would not change.*

While discussing (e), make the same argument for multiplying both sides of an inequality by a *positive value*:

If $A > B$, then $kA > kB$ provided k is positive.

This action also does not change the solution set of an inequality.

- So if x is a solution to an inequality, it will also be a solution to the new inequality formed when the same number is added to or subtracted from each side of the original inequality or when the two sides of the original inequality are multiplied by the same positive number. This gives us a strategy for finding solution sets.

Example 1 (2 minutes)

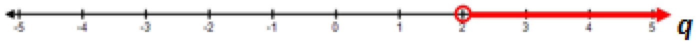
Work through the example as a class using the addition and multiplication properties of inequality.

Example 1

What is the solution set to the inequality $5q + 10 > 20$? Express the solution set in words, in set notation, and graphically on the number line.

q can be any value that is more than 2.

$q > 2$



- Point out to students that these are the only actions that we *know* do not affect the solution set.
- Dividing both sides by 5 is still applying the multiplication principle (multiplying both sides by $\frac{1}{5}$).

Exercise 2 (6 minutes)

Allow students time to complete the problems individually and then pair up to compare responses.

Exercise 2

Find the solution set to each inequality. Express the solution in set notation and graphically on the number line.

a. $x + 4 \le 7$

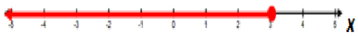
$x \le 3$

b. $\frac{m}{3} + 8 \ne 9$

$m \ne 3$

c. $8y + 4 < 7y - 2$

$y < -6$



d. $6(x - 5) \ge 30$

$x \ge 10$

e. $4(x - 3) > 2(x - 2)$

$x > 4$

Exercise 3 (10 minutes)

Have students work in pairs on the exploration. Then, discuss results as a class. Make sure that students can provide a reason or proof with their responses. Allow several students to share their reasoning for each one.

When discussing (a), ask the following:

- What is the smallest value of B that still works in the second inequality?

For part (b), examine how the solution sets differ.

Exercise 3

Recall the discussion on all the strange ideas for what could be done to both sides of an equation. Let's explore some of the same issues here but with inequalities. Recall, in this lesson we have established that adding (or subtracting) and multiplying through by positive quantities does not change the solution set of an inequality. We've made no comment about other operations.

a. ***Squaring:*** **Do $B \le 6$ and $B^2 \le 36$ have the same solution set? If not, give an example of a number that is in one solution set but not the other.**

No. In the first inequality, B can equal 6 or any number smaller. The second inequality can only equal numbers from 6 down to -6. For example, $B = -7$ is not in the solution set.

b. ***Multiplying through by a negative number*****: Do $5 - C > 2$ and $-5 + C > -2$ have the same solution set? If not, give an example of a number that is in one solution set but not the other.**

No. The first inequality has a solution of $C < 3$, and the second has a solution of $C > 3$. The number 4, for example, is a solution to the second but not the first inequality.

c. **Bonzo's ignoring exponents: Do $y^2 < 5^2$ and $y < 5$ have the same solution set?**

No. When both sides are squared, we end up introducing the possibility that y can be negative but not less than -5. For example, $y = -5$ is a solution for the second inequality but not the first.

- Recall that we have established that making use of the properties of inequalities:

  If $A > B$, then $A + c > B + c$ for any real number c.

  If $A > B$, then $kA > kB$ for any positive real number k.

  Along with all the usual commutative, associative, and distributive properties, to rewrite an inequality does not change the solution set of that inequality.
- Any action different from these offers no guarantee that your work is yielding valid solutions. These may be candidates you can use to check for solutions, but they must be checked (as we saw in the previous exercise).
- The next two examples illustrate some more dangers.

Example 2 (2 minutes)

Work through the responses as a class reminding students of dangers that were seen in Lesson 13.

MP.3

Example 2

Jojo was asked to solve $6x + 12 < 3x + 6$, for x. She answered as follows:

$6x + 12 < 3x + 6$	
$6(x + 2) < 3(x + 2)$	**Apply the distributive property.**
$6 < 3$	**Multiply through by $\frac{1}{x+2}$.**

a. **Since the final line is a false statement, she deduced that there is no solution to this inequality (that the solution set is empty).**

What is the solution set to $6x + 12 < 3x + 6$?

$x < -2$

b. **Explain why Jojo came to an erroneous conclusion.**

Multiplying through by $\frac{1}{x+2}$ is dangerous, as we do not know whether or not $\frac{1}{x+2}$ is positive or negative (or even well defined). Its value depends on the value one chooses to assign to the variable. We could, potentially, multiply through by a negative value, making the final step invalid. (The property of inequalities allows for only multiplication though by positives.) We do not know then if this step is actually valid. Things can, and clearly have, gone wrong.

Example 3 (3 minutes)

- During the last exercise, we saw that when both sides were multiplied by -1 the solution set of the inequality changed.
- So, we can't multiply through by -1 as this does not match the allowable operations given by the properties of inequality. However, we can use the property: If $A > B$, then $A + c > B + c$ for any real number c. Can you figure out how to use this property in a way that is helpful?

MP.1

Allow students time to work the problem, individually or in pairs, and then discuss the solution.

Example 3

Solve $-q \geq -7$, for q.

$-q \geq -7$

$0 \geq -7 + q$ *Add q to both sides*

$7 \geq q$ *Add 7 to both sides*

Exercises 4–7 (10 minutes)

Allow students to work individually or in pairs for each of the following exercises, pausing to review or compare answers with a neighbor or the class after each exercise.

Exercise 4

Find the solution set to each inequality. Express the solution in set notation and graphically on the number line.

a. $-2f < -16$

$f > 8$

b. $-\frac{x}{12} \leq \frac{1}{4}$

$x \geq -3$

c. $6 - a \geq 15$

$a \leq -9$

d. $-3(2x + 4) > 0$

$x < -2$

Recall the Properties of Inequality:

- **Addition Property of Inequality:**
  If $A > B$, then $A + c > B + c$ for any real number c.
- **Multiplication Property of Inequality:**
  If $A > B$, then $kA > kB$ for any <u>positive</u> real number k.

Exercise 5

Use the properties of inequality to show that each of the following are true for any real numbers p and q.

MP.2

a. If $p \geq q$, then $-p \leq -q$.

$$p \geq q$$
$$p - q \geq q - q$$
$$p - q \geq 0$$
$$p - p - q \geq 0 - p$$
$$-q \geq -p$$

b. If $p < q$, then $-5p > -5q$.

$$p < q$$
$$5p < 5q$$
$$5p - 5p - 5q < 5q - 5p - 5q$$
$$-5q < -5p$$
$$-5p > -5q$$
$$-p \leq -q$$

MP.2

c. **If $p \le q$, then $-0.03\,p \ge -0.03\,q$.**

$$p \le q$$
$$0.03p \le 0.03q$$
$$0.03p - 0.03p - 0.03q \le 0.03q - 0.03p - 0.03q$$
$$-0.03q \le -0.03p$$
$$-0.03p \ge -0.03q$$

d. **Based on the results from (a) through (c), how might we expand the multiplication property of inequality?**

If $A > B$, then $kA < kB$ for any negative real number k.

Exercise 6

Solve $-4 + 2t - 14 - 18t > -6\ - 100t$, for t in two different ways: first without ever multiplying through by a negative number and then by first multiplying through by $-\frac{1}{2}$.

$$-18 - 16t > -6 - 100t$$
$$-16t + 100t > -6 + 18$$
$$84t > 12$$
$$t > \frac{1}{7}$$

$$-\frac{1}{2}(-4 + 2t - 14 - 18t) < -\frac{1}{2}(-6 - 100t)$$
$$2 - t + 7 + 9t < 3 + 50t$$
$$9 + 8t < 3 + 50t$$
$$6 < 42t$$
$$\frac{1}{7} < t$$

Exercise 7

Solve $-\frac{x}{4} + 8 < \frac{1}{2}$, for x in two different ways: first without ever multiplying through by a negative number and then by first multiplying through by -4.

$$-\frac{x}{4} + 8 < \frac{1}{2}$$
$$8 - \frac{1}{2} < \frac{x}{4}$$
$$4 \cdot \frac{15}{2} < 4 \cdot \frac{x}{4}$$
$$30 < x$$

$$-4(-\frac{x}{4} + 8) > -4\left(\frac{1}{2}\right)$$
$$x - 32 > -2$$
$$x > 30$$

Closing (2 minutes)

- What moves do we know do not change the solution set of an inequality?
- What moves did we see today that did change the solution set?

Exit Ticket (5 minutes)

Name ____ Date____________________

Lesson 14: Solving Inequalities

Exit Ticket

1. Find the solution set to each inequality. Express the solution in set notation and graphically on the number line.

   a. $6x - 5 < 7x + 4$

   b. $x^2 + 3(x - 1) \geq x^2 + 5$

2. Fergus was absent for today's lesson and asked Mike to explain why the solution to $-5x > 30$ is $x < -6$. Mike said, "Oh! That's easy. When you multiply by a negative, just flip the inequality." Provide a better explanation to Fergus about why the direction of the inequality is reversed.

Exit Ticket Sample Solutions

1. **Find the solution set to each inequality. Express the solution in set notation and graphically on the number line.**

a. $6x - 5 < 7x + 4$

$-5 < x + 4$

$-9 < x$

b. $x^2 + 3(x-1) \geq x^2 + 5$

$3x - 3 \geq 5$

$3x \geq 8$

$x \geq \frac{8}{3}$

2. **Fergus was absent for today's lesson and asked Mike to explain why the solution to $-5x > 30$ is $x < -6$. Mike said, "Oh! That's easy. When you multiply by a negative, just flip the inequality." Provide a better explanation to Fergus about why the direction of the inequality is reversed.**

The multiplication property of inequality only applies when multiplying by a positive value. Otherwise, the addition property must be used.

$$0 > 5x + 30$$
$$-30 > 5x$$
$$-6 > x$$

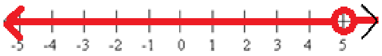
Flipping the direction of the inequality is simply a shortcut for using the addition property.

Problem Set Sample Solutions

1. **Find the solution set to each inequality. Express the solution in set notation and graphically on the number line.**

a. $2x < 10$

$\{x \text{ real} \mid x < 5\}$



b. $-15x \geq -45$

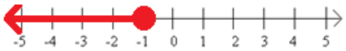
$\{x \text{ real} \mid x \leq 3\}$

c. $\frac{2}{3}x \neq \frac{1}{2} + 2$

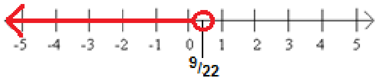
$\{x \text{ real} \mid x \neq 15/4\}$

d. $-5(x-1) \geq 10$

$\{x \text{ real} \mid x \leq -1\}$



e. $13x < 9(1 - x)$



$\{x \text{ real} \mid x < 9/22\}$

2. **Find the mistake in the following set of steps in a student's attempt to solve $5x + 2 \geq x + \frac{2}{5}$, for x. What is the correct solution set?**

$$5x + 2 \geq x + \frac{2}{5}$$

$$5\left(x + \frac{2}{5}\right) \geq x + \frac{2}{5} \quad \text{(factoring out 5 on the left side)}$$

$$5 \geq 1 \quad \text{(dividing by } \left(x + \tfrac{2}{5}\right)\text{)}$$

So, the solution set is the empty set.

The third line of the solution is incorrect. Since x is a variable, we do not know whether $x + \frac{2}{5}$ is positive, negative, or zero. The correct solution to the given problem is $x \geq -\frac{2}{5}$.

3. **Solve $-\frac{x}{16} + 1 \geq -\frac{5x}{2}$, for x without multiplying by a negative number. Then solve by multiplying through by -16.**

$$-\frac{x}{16} + 1 \geq -\frac{40x}{16}$$

$$\frac{40x}{16} - \frac{x}{16} \geq -1$$

$$\frac{39x}{16} \geq -1$$

$$x \geq -\frac{16}{39}$$

$$-16\left(-\frac{x}{16} + 1\right) \leq -16\left(-\frac{5x}{2}\right)$$

$$x - 16 \leq 40x$$

$$-16 \leq 39x$$

$$-\frac{16}{39} \leq x$$

4. **Lisa brought half of her savings to the bakery and bought 12 croissants for $\$14.20$. The amount of money she brings home with her is more than $\$2$. Use an inequality to find how much money she had in her savings before going to the bakery. (Write the inequalities that represents the situation and solve it.)**

The inequality is $2 < \frac{x}{2} - 14.2$. The original savings amount must have been more than $\$32.40$.

Lesson 15: Solution Sets of Two or More Equations (or Inequalities) Joined by “And” or “Or”

Student Outcomes

- Students describe the solution set of two equations (or inequalities) joined by either “and” or “or” and graph the solution set on the number line.

Classwork

Exercise 1 (6 minutes)

It may be helpful to some students to review some of the vocabulary used here, such as compound sentence (a sentence that contains at least two clauses) or declarative sentence (a sentence in the form of a statement).

Give students a few minutes to work on the exploration independently and then 1 minute to compare answers with a partner. Discuss results as a class, particularly the difference between separating the declarations by “and” and by “or.”

Exercise 1

Determine whether each claim given below is true or false.

a. Right now, I am in math class and English class.

False

b. Right now, I am in math class or English class.

True (assuming they are answering this in class)

c. $3 + 5 = 8$ **and** $5 < 7 - 1$.

True

d. $10 + 2 \neq 12$ **and** $8 - 3 > 0$

False

e. $3 < 5 + 4$ **or** $6 + 4 = 9$.

True

f. $16 - 20 > 1$ **or** $5.5 + 4.5 = 11$

False

These are all examples of declarative compound sentences.

g. When the two declarations in the sentences above were separated by “and,” what had to be true to make the statement true?

Both declarations had to be true.

h. When the two declarations in the sentences above were separated by “or,” what had to be true to make the statement true?

At least one declaration had to be true.

Discuss the following points with students:

- The word "and" means the same thing in a compound mathematical sentence as it does in an English sentence.
- If two clauses are separated by "and," both clauses must be true for the entire compound statement to be deemed true.
- The word "or" also means a similar thing in a compound mathematical sentence as it does in an English sentence. However, there is an important distinction: In English the word "or" is commonly interpreted as the exclusive or, one condition or the other is true, but not both. In mathematics, either or both could be true.
- If two clauses are separated by "or," one or both of the clauses must be true for the entire compound statement to be deemed true.

Example 1 (5 minutes)

Work through the four examples as a class.

Example 1

Solve each system of equations and inequalities.

a. $x + 8 = 3$ **or** $x - 6 = 2$

$x = -5$ *or* $x = 8$

$\{-5, 8\}$

b. $4x - 9 = 0$ **or** $3x + 5 = 2$

$x = \frac{9}{4}$ *or* $x = -1$

$\left\{-1, \frac{9}{4}\right\}$

c. $x - 6 = 1$ **and** $x + 2 = 9$

$x = 7$ *and* $x = 7$

$\{7\}$

d. $2w - 8 = 10$ **and** $w > 9$.

The empty set.

$\emptyset$

Exploratory Challenge/Exercise 2 (10 minutes)

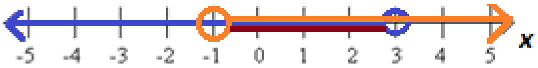
Provide students with colored pencils and allow them a couple of minutes to complete (a) through (c). Then, stop and discuss the results.

Exercise 2

a. **Using a colored pencil, graph the inequality $x < 3$ on the number line below.**

b. **Using a different colored pencil, graph the inequality $x > -1$ on the number line below.**

c. **Using a third colored pencil, darken the section of the number line where $x < 3$ and $x > -1$.**



- In order for the compound sentence $x > -1$ and $x < 3$ to be true, what has to be true about x?
  - *x has to be both greater than -1 and less than 3. (Students might also verbalize that it must be between -1 and 3, not including the points -1 and 3.)*
- On the graph, where do the solutions lie?
  - *Between -1 and 3, not including the points -1 and 3*

Have students list some of the solutions to the compound inequality. Make sure to include examples of integer and non-integer solutions.

- How many solutions are there to this compound inequality?
  - *An infinite number*

Introduce the abbreviated way of writing this sentence:

- Sometimes this is written as $-1 < x < 3$.

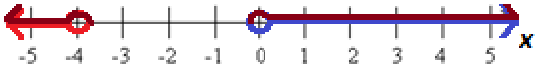
Use this notation to further illustrate the idea of x representing all numbers ***strictly between*** -1 and 3.

Allow students a couple of minutes to complete (d) *through* (f). Then, stop and discuss the results.

d. Using a colored pencil, graph the inequality $x < -4$ on the number line below.

e. Using a different colored pencil, graph the inequality $x > 0$ on the number line below.

f. Using a third colored pencil, darken the section of the number line where $x < -4$ or $x > 0$.



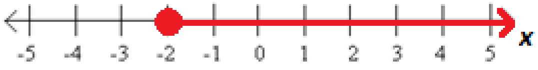
- In order for the compound sentence $x < -4$ or $x > 0$ to be true, what has to be true about x?
  - *It could either be less than -4, or it could be greater than 0.*
- On the graph, where do the solutions lie?
  - *To the left of -4 and to the right of 0*

Have students list solutions to the compound inequality. Make sure to include examples of integer and non-integer solutions.

- How many solutions are there to this compound inequality?
  - *Infinitely many*
- Would it be acceptable to abbreviate this compound sentence as follows: $0 < x < -4$?
  - *No.*
- Explain why not.
  - *Those symbols suggest that x must be greater than zero and less than -4 at the same time, but the solution is calling for x to be either less than -4 or greater than zero.*

Allow students a couple of minutes to complete (g) *through* (i) and discuss answers.

g. **Graph the compound sentence $x > -2$ or $x = -2$ on the number line below.**



h. **How could we abbreviate the sentence $x > -2$ or $x = -2$?**

$x \geq -2$

i. **Rewrite $x \leq 4$ as a compound sentence and graph the solutions to the sentence on the number line below.**

$x < 4$ or $x = 4$

Example 2 (3 minutes)

Work through Example 2 as a class.

Example 2

Graph each compound sentence on a number line.

a. $x = 2$ or $x > 6$

b. $x \leq -5$ or $x \geq 2$

Rewrite as a compound sentence and graph the sentence on a number line.

c. $1 \leq x \leq 3$

$x \geq 1$ and $x \leq 3$

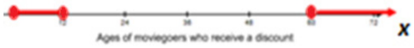
Exercise 3 (5 minutes)

Give students a couple of minutes to read through Exercise 3 and try it independently before comparing answers with a neighbor or discussing as a class.

Exercise 3

Consider the following two scenarios. For each, specify the variable and say, "W is the width of the rectangle," for example, and write a compound inequality that represents the scenario given. Draw its solution set on a number line.

Scenario	Variable	Inequality	Graph
a. Students are to present a persuasive speech in English class. The guidelines state that the speech must be at least 7 minutes but not exceed 12 minutes.	*Let x = length of time of the speech.*	*$x \geq 7$ and $x \leq 12$* *$7 \leq x \leq 12$*	

b. Children and senior citizens receive a discount on tickets at the movie theater. To receive a discount, a person must be between the ages of 2 and 12, including 2 and 12, or 60 years of age or older.	*Let x = age of moviegoer who receives a discount.*	$2 \le x \le 12$ *or* $x \ge 60$	

Scaffolding:

- Ask students to think of other scenarios that could be modeled using a compound inequality.

Exercise 4 (10 minutes)

Give students time to work on the problems, and then allow for sharing of answers, possibly with a neighbor or with the class.

Exercise 4

Determine if each sentence is true or false. Explain your reasoning.

a. $8 + 6 \le 14$ **and** $\frac{1}{3} < \frac{1}{2}$.

True

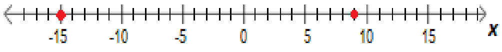
b. $5 - 8 < 0$ **or** $10 + 13 \neq 23$

True

Solve each system and graph the solution on a number line.

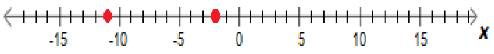
c. $x - 9 = 0$ **or** $x + 15 = 0$

$\{9, -15\}$



d. $5x - 8 = -23$ **or** $x + 1 = -10$

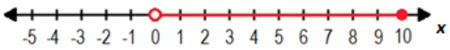
$\{-3, -11\}$



Graph the solution set to each compound inequality on a number line.

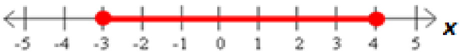
e. $x < -8$ **or** $x > -8$

f. $0 < x \le 10$



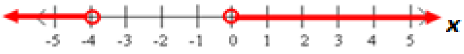
Write a compound inequality for each graph.

g.



$-3 \le x \le 4$

h.



$x < -4$ *or* $x > 0$

i. **A poll shows that a candidate is projected to receive 57% of the votes. If the margin for error is plus or minus 3%, write a compound inequality for the percentage of votes the candidate can expect to get.**

let x = percentage of votes $54 \le x \le 60$

j. **Mercury is one of only two elements that is liquid at room temperature. Mercury is non-liquid for temperatures less than -38.0°F or greater than 673.8°F. Write a compound inequality for the temperatures at which mercury is non-liquid.**

Let x = temperatures (in degrees F) for which mercury is non-liquid $x < -38$ or $x > 673.8$

As an extension, students can come up with ways to alter parts (a) and (b) to make them false compound statements. Share several responses.

Ask the following:

- What would be a more concise way of writing the sentence for part (e)?
  - $x \neq 8$
- For part (f), list some numbers that are solutions to the inequality.
- What is the largest possible value of x?
  - 10
- What is the smallest possible value of x?
  - This is tougher to answer. x can be infinitely close to 0 but cannot equal zero. Therefore, there is no absolute smallest value for x in this case.

For parts (i) and (j), make sure students specify what the variable they choose represents.

MP.2

Scaffolding:

- The other element that is liquid at room temperature is bromine. Students could be asked to look up the temperatures at which bromine is non-liquid and write a similar compound inequality.

Closing (2 minutes)

MP.6

Lead a conversation on the idea that in math, as in English, it is important that we are precise in our use of language and that we are able to read (and comprehend) and write mathematical sentences. Ask students to give examples to justify why the precision is important in math, and why it is important in English.

Reinforce that, in mathematical sentences, like in English sentences, a compound sentence separated by

AND is true if <u>both clauses are true</u>.

OR is true if <u>at least one of the clauses is true</u>.

Lesson Summary

In mathematical sentences, like in English sentences, a compound sentence separated by

AND is true if ____.

OR is true if ____.

Exit Ticket (5 minutes)

Name ____ Date____________________

Lesson 15: Solution Sets of Two or More Equations (or Inequalities) Joined by “And” or “Or”

Exit Ticket

1.

a. Solve the system and graph the solution set on a number line.

$x - 15 = 5$ or $2x + 5 = 1$

b. Write a different system of equations that would have the same solution set.

2. Swimming pools must have a certain amount of chlorine content. The United States standard for safe levels of chlorine in swimming pools is at least 1 part per million and no greater than 3 parts per million. Write a compound inequality for the acceptable range of chlorine levels.

3. Consider each of the following compound sentences:

$x < 1$ and $x > -1$ $x < 1$ or $x > -1$

Does the change of word from “and” to “or” change the solution set?

Use number-line graphs to support your answer.

Exit Ticket Sample Solutions

1.

a. **Solve the system and graph the solution set on a number line.**

$x - 15 = 5$ or $2x + 5 = 1$

$x = 20$ or $x = -2$ $\{-2, 20\}$

b. **Write a different system of equations that would have the same solution set.**

sample answer: $x - 20 = 0$ or $x + 2 = 0$

2. **Swimming pools must have a certain amount of chlorine content. The United States standard for safe levels of chlorine in swimming pools is at least 1 part per million and no greater than 3 parts per million. Write a compound inequality for the acceptable range of chlorine levels.**

Let $x =$ chlorine level in a swimming pool (in parts per million) $1 \le x \le 3$

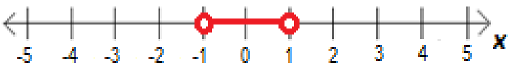
3. **Consider each of the following compound sentences:**

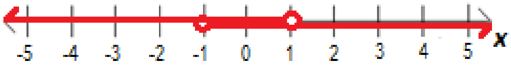
$x < 1$ and $x > -1$ $x < 1$ or $x > -1$

Does the change of word from "and" to "or" change the solution set?

Use number-line graphs to support your answer.

For the first sentence, both statements must be true, so x can only equal values that are both greater than -1 and less than 1. For the second sentence, only one statement must be true, so x must be greater than -1 or less than 1. This means x can equal any number on the number line.





Problem Set Sample Solutions

1. **Consider the inequality $0 < x < 3$.**

a. **Rewrite the inequality as a compound sentence.**

$x > 0$ and $x < 3$

b. **Graph the inequality on a number line.**

c. **How many solutions are there to the inequality? Explain.**

There are an infinite number of solutions. x can be any value between 0 and 3, which includes the integer values of 1 and 2 as well as non-integer values. The set of numbers between 0 and 3 is infinite.

d. What are the largest and smallest possible values for x? Explain.

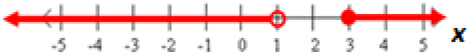
There is no absolute largest or absolute smallest value for x. x can be infinitely close to 0 or to 3 but cannot equal either value.

e. If the inequality is changed to $0 \leq x \leq 3$, then what are the largest and smallest possible values for x?

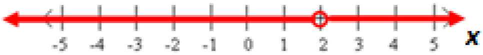
In this case, we can define the absolute maximum value to be 3 and the absolute minimum value to be 0.

Write a compound inequality for each graph.

2. $x < 1$ *or* $x \geq 3$



3. $x < 2$ *or* $x > 2$, which can be written as $x \neq 2$



Write a single or compound inequality for each scenario.

4. The scores on the last test ranged from 65% to 100%.

$x =$ *scores on last test* $65 \leq x \leq 100$

5. To ride the roller coaster, one must be at least 4 feet tall.

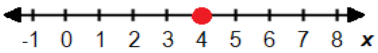
$x =$ *height (in feet) to ride the roller coaster* $x \geq 4$

6. Unsafe body temperatures are those lower than 96°F or above 104°F.

$x =$ *body temperature (in degrees F) that are unsafe* $x < 96$ *or* $x > 104$

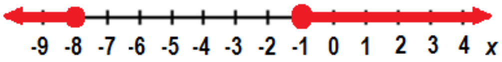
Graph the solution(s) to each of the following on a number line.

7. $x - 4 = 0$ and $3x + 6 = 18$

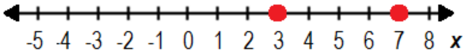


8. $x < 5$ and $x \neq 0$

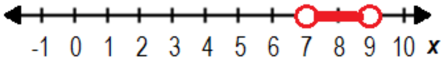
9. $x \leq -8$ or $x \geq -1$



10. $3(x - 6) = 3$ or $5 - x = 2$



11. $x < 9$ and $x > 7$



12. $x + 5 < 7$ or $x = 2$

Lesson 16: Solving and Graphing Inequalities Joined by “And” or “Or”

Student Outcomes

- Students solve two inequalities joined by “and” or “or” and then graph the solution set on the number line.

Classwork

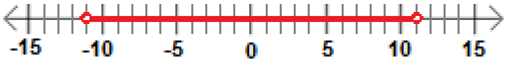
Exercise 1 (5 minutes)

Do parts (a)–(c) and as much of (d)–(e) as time permits depending on the level of your students. You can also present the challenge problem given below if time allows.

Exercise 1

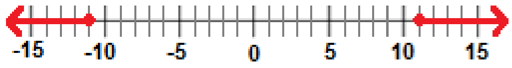
a. **Solve $w^2 = 121$, for w. Graph the solution on a number line.**

b. **Solve $w^2 < 121$, for w. Graph the solution on a number line and write the solution set as a compound inequality.**



$-11 < x < 11$

c. **Solve $w^2 \geq 121$ for w. Graph the solution on a number line and write the solution set as a compound inequality.**

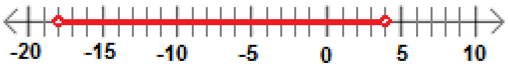


$x \leq -11$ *or* $x \geq 11$

d. **Quickly solve $(x+7)^2 = 121$, for x. Graph the solution on a number line.**

e. **Use your work from part (d) to quickly graph the solution on a number line to each inequality below.**

i. $(x+7)^2 < 121$



ii. $(x+7)^2 \geq 121$

Extension

Use the following to challenge students who finish early.

a. Poindexter says that $(a + b)^2$ equals $a^2 + 2ab + b^2$. Is he correct?

b. Solve $x^2 + 14x + 49 < 121$, for x. Present the solution graphically on a number line.

Exercises 2–3 (6 minutes)

Give students 4 minutes to work on Exercises 2 and 3. Then, discuss the results as a class. Students are applying their knowledge from the previous lesson to solve an unfamiliar type of problem.

Scaffolding:

- Remind students of their experience from the previous lesson. For a statement separated by "and" to be true, BOTH statements must be true. If it is separated by "or," at least one statement must be true.

Exercise 2

Consider the compound inequality $-5 < x < 4$

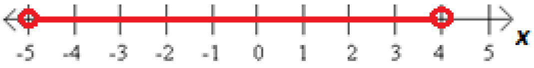
a. **Rewrite the inequality as a compound statement of inequality.**

$x > -5$ and $x < 4$

b. **Write a sentence describing the possible values of x.**

x can be any number between -5 and 4.

c. **Graph the solution set on the number line below.**



Exercise 3

Consider the compound inequality $-5 < 2x + 1 < 4$.

a. **Rewrite the inequality as a compound statement of inequality.**

$2x + 1 > -5$ and $2x + 1 < 4$

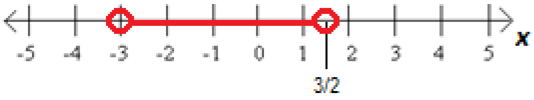
b. **Solve each inequality for x. Then, write the solution to the compound inequality.**

$x > -3$ and $x < \frac{3}{2}$ OR $-3 < x < \frac{3}{2}$

c. **Write a sentence describing the possible values of x.**

x can be any number between -3 and $\frac{3}{2}$.

d. **Graph the solution set on the number line below.**



Review Exercise 3 with students to demonstrate how to solve it without rewriting it.

- A friend of mine suggested I could solve the inequality as follows. Is she right?

$$-5 < 2x + 1 < 4$$
$$-5 - 1 < 2x + 1 - 1 < 4 - 1$$
$$-6 < 2x < 3$$
$$-3 < x < \frac{3}{2}$$

MP.3

Encourage students to articulate their thoughts and scrutinize each other's reasoning.

Point out to students that solving the two inequalities did not require any new skills. They are solved just as they learned in previous lessons.

Have students verify their solution by filling in a few test values.

Remind students that the solution can be written two ways: $x > -3$ and $x < \frac{3}{2}$

OR $-3 < x < \frac{3}{2}$.

Scaffolding:

- Remind students that when an inequality is multiplied or divided by a negative number, the direction of the inequality changes.

Exercises 4–5 (5 minutes)

Give students 4 minutes to work on Exercises 4 and 5. Then, review the results as a class. Again, point out to students that solving the two inequalities did not require any new skills. They are solved just as they learned in previous lessons. Have students verify their solutions by filling in a few test values.

Exercise 4

Given $x < -3$ or $x > -1$

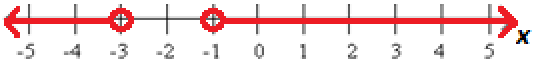
a. What must be true in order for the compound inequality to be a true statement?

One of the statements must be true, so either x has to be less than -3, or it has to be greater than -1. (In this case it is not possible that both are true.)

b. Write a sentence describing the possible values of x.

x can be any number that is less than -3 or any number that is greater than -1.

c. Graph the solution set on the number line below.



Exercise 5

Given $x + 4 < 6$ or $x - 1 > 3$

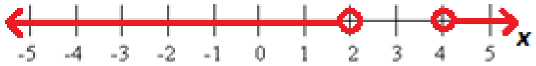
a. Solve each inequality for x. Then, write the solution to the compound inequality.

$x < 2$ or $x > 4$

b. **Write a sentence describing the possible values of x.**

x can be any number that is less than 2 or any number that is greater than 4.

c. **Graph the solution set on the number line below.**



Exercise 6 (14 minutes)

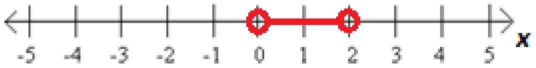
Have students work the exercises individually, with partners, or with small groups. Circulate around the room monitoring progress and offering guidance as needed. Make sure students are attending to the detail of correctly using open and closed endpoints.

Exercise 6

Solve each compound inequality for x and graph the solution on a number line.

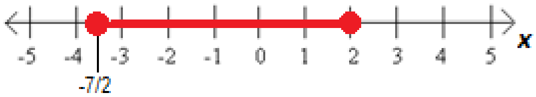
a. $x + 6 < 8$ and $x - 1 > -1$

$x < 2$ *and* $x > 0$ → $0 < x < 2$



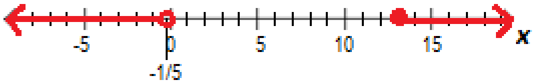
b. $-1 \leq 3 - 2x \leq 10$

$x \geq -\frac{7}{2}$ *and* $x \leq 2$ → $-\frac{7}{2} \leq x \leq 2$



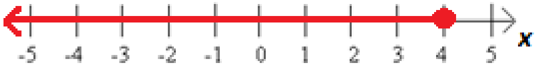
c. $5x + 1 < 0$ or $8 \leq x - 5$

$x < -\frac{1}{5}$ *or* $x \geq 13$



d. $10 > 3x - 2$ or $x = 4$

$x < 4$ *or* $x = 4$ → $x \leq 4$

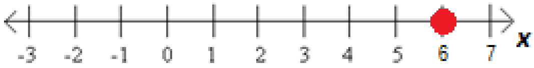


e. $x - 2 < 4$ or $x - 2 > 4$

$x < 6$ *or* $x > 6$ → $x \neq 6$

f. $x - 2 \leq 4$ and $x - 2 \geq 4$

$x = 6$



Debrief the exercise with the following questions:

- Look at the solution to part (f) closely. Remind students that both statements must be true. Therefore, the solution is only $x = 6$.
- How would the solution to part (f) change if the "and" was an "or"? Let this discussion lead in to Exercise 7.

Exercise 7 (9 minutes)

Have students work in groups to answer the questions. Students are exploring variations of previously seen problems. After completing the exercises, ask students to articulate how the problems differed from most of the other examples seen thus far.

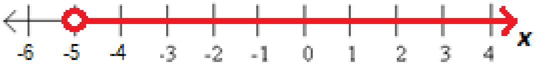
MP.1

Exercise 7

Solve each compound inequality for x and graph the solution on a number line. Pay careful attention to the inequality symbols and the "and" or "or" statements as you work.

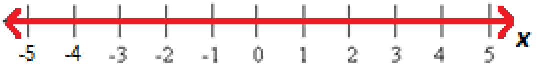
a. $1 + x > -4$ or $3x - 6 > -12$

$x > -5$



b. $1 + x > -4$ or $3x - 6 < -12$

x can be any real number.



c. $1 + x > 4$ and $3x - 6 < -12$

$x > 3$ and $x < -2$

No solution (empty set) since there are no numbers that satisfy both statements.

Have early finishers explore further with the following:

- Is it possible to write a problem separated by "or" that has no solution?
- Is it possible to have a problem separated by "and" that has a solution set consisting of all real numbers?

Closing (2 minutes)

For the first problem, students may have written the solution as $x > -5$ or $x > -2$. Look at the graph as a class and remind them that the solution is the set of all of the numbers included in either of the two solution sets (or the union of the two sets). Lead them to the idea that the solution is $x > -5$.

For the second problem, the two graphs overlap and span the entire number line. Lead them to the idea that the solution is all real numbers. Have students fill in a few test values to verify that any number will work.

For the third problem, the two graphs do not overlap. Remind them that the solution set is only the values that are in both of the individual solution sets. There is no number that will make both statements true. Lead them to the idea that there is no solution.

Read the questions at the end of the exploration and give students a few minutes to summarize their thoughts on the work in Exercise 7 independently. Call for a few volunteers to read their solutions.

Exit Ticket (4 minutes)

Name ____ Date____________________

Lesson 16: Solving and Graphing Inequalities Joined by "And" or "Or"

Exit Ticket

1. Solve each compound inequality for x and graph the solution on a number line.
   a. $9 + 2x < 17$ or $7 - 4x < -9$

   b. $6 \leq \frac{x}{2} \leq 11$

2.
   a. Give an example of a compound inequality separated by "or" that has a solution of all real numbers.

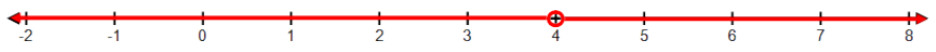
   b. Take the example from (a) and change the "or" to an "and." Explain why the solution set is no longer all real numbers. Use a graph on a number line as part of your explanation.

Exit Ticket Sample Solutions

1. Solve each compound inequality for x and graph the solution on a number line.

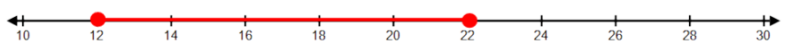
a. $9 + 2x < 17$ or $7 - 4x < -9$

$x < 4$ or $x > 4$ or $x \neq 4$



b. $6 \leq \frac{x}{2} \leq 11$

$12 \leq x \leq 22$



2.

a. **Give an example of a compound inequality separated by "or" that has a solution of all real numbers.**

Sample response: $x > 0$ or $x < 2$

b. **Take the example from (a) and change the "or" to an "and." Explain why the solution set is no longer all real numbers. Use a graph on a number line as part of your explanation.**

Sample response: $x > 0$ and $x < 2$

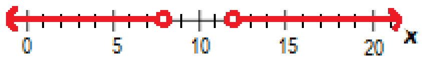
In the first example, only one of the inequalities needs to be true to make the compound statement true. Any number selected is either greater than 0 or less than 2 or both. In the second example, both inequalities must be true to make the compound statement true. This restricts the solution set to only numbers between 0 and 2.

Problem Set Sample Solutions

Solve each inequality for x and graph the solution on a number line.

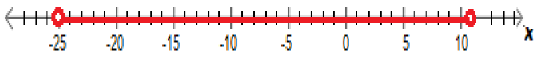
1. $x - 2 < 6$ or $\frac{x}{3} > 4$

$x < 8$ or $x > 12$



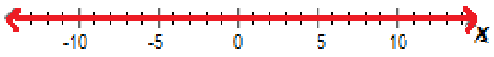
2. $-6 < \frac{x+1}{4} < 3$

$-25 < x < 11$



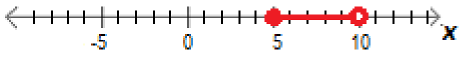
3. $5x \leq 21 + 2x$ or $3(x + 1) \geq 24$

$x \leq 7$ or $x \geq 7$ → all real numbers



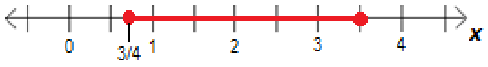
4. $5x + 2 \geq 27$ *and* $3x - 1 < 29$

$x \geq 5$ and $x < 10$ → $5 \leq x < 10$



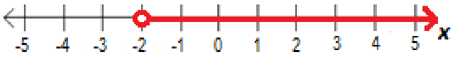
5. $0 \leq 4x - 3 \leq 11$

$\frac{3}{4} \leq x \leq \frac{7}{2}$



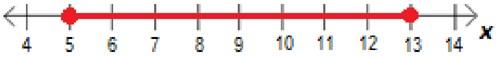
6. $2x > 8$ or $-2x < 4$

$x > 4$ or $x > -2$ → $x > -2$



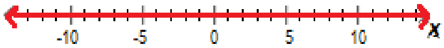
7. $8 \geq -2(x-9) \geq -8$

$5 \leq x \leq 13$



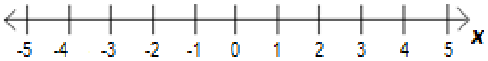
8. $4x + 8 > 2x - 10$ *or* $\frac{1}{3}x - 3 < 2$

$x > -9$ *or* $x < 15$ → *all real numbers*



9. $7 - 3x < 16$ and $x + 12 < -8$

$x > -3$ *and* $x < -20$ → *no solution*

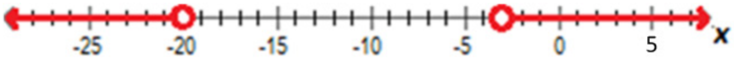


10. **If inequalities question 8 were joined by "and" instead of "or," what would the solution set become?**

$-9 < x < 15$

11. **If the inequalities in question 9 were joined by "or" instead of "and," what would the solution set become?**

$x > -3$ *or* $x < -20$



Lesson 17: Equations Involving Factored Expressions

Student Outcomes

- Students learn that equations of the form $(x-a)(x-b)=0$ have the same solution set as two equations joined by "or:" $x-a=0$ or $x-b=0$. Students solve factored or easily factorable equations.

Classwork

Exercise 1 (5 minutes)

Allow students a few minutes to complete only (a) through (d) of Exercise 1, either individually or in pairs.

Exercise 1

Solve each equation for x.

a. $x-10=0$

$\{10\}$

b. $\frac{x}{2}+20=0$

$\{-40\}$

c. **Demanding Dwight insists that you give him two solutions to the following equation:**

$$(x-10)\left(\frac{x}{2}+20\right)=0$$

Can you provide him with two solutions?

$\{10,-40\}$

MP.7 & MP.8

d. **Demanding Dwight now wants FIVE solutions to the following equation:**

$$(x-10)(2x+6)(x^2-36)(x^2+10)\left(\frac{x}{2}+20\right)=0$$

Can you provide him with five solutions?

$\{-40,-6,-3,6,10\}$

Do you think there might be a sixth solution?

There are exactly 5 *solutions.*

Discussion (5 minutes)

- If I told you that the product of two numbers is 20, could you tell me anything about the two numbers?
- Would the numbers have to be 4 and 5?
- Would both numbers have to be smaller than 20?
- Would they both have to be positive?

MP.7 & MP.8

- Is there much at all you could say about the two numbers.
    - *Not really. They have to have the same sign is about all we can say.*
- If I told you that the product of two numbers is zero, could you tell me anything about the two numbers?
    - *At least one of the numbers must be zero.*
- How could we phrase this mathematically?
    - *If $ab = 0$, then either $a = 0$ or $b = 0$ or $a = b = 0$.*
- This is known as the **zero-product property**.
- What if the product of three numbers is zero? What if the product of seven numbers is zero?
    - *If any product of numbers is zero, at least one of the terms in that product is zero.*

Exercise 1 (continued) (2 minutes)

Give students a few minutes to complete (e) and (f) and elicit student responses.

Consider the equation $(x-4)(x+3)=0$.

e. **Rewrite the equation as a compound statement.**

$x - 4 = 0$ *or* $x + 3 = 0$

f. **Find the two solutions to the equation.**

$\{-3, 4\}$

Scaffolding:

- Give early finishers this challenge: Write a factored equation that has the solution: $\{-5, -4\}$.

Examples 1–2 (5 minutes)

Work the two examples as a class.

Example 1

Solve $2x^2 - 10x = 0$, for x.

$\{0, 5\}$

Example 2

Solve $x(x-3) + 5(x-3) = 0$, for x.

$\{-5, 3\}$

Scaffolding:

- Remind students of the practice of applying the distribution property "backwards" that they saw in the Lesson 6 Problem Set. This practice is called factoring.

MP.6

Lead a discussion about the application of the distributive property, in the form of factoring polynomial expressions, when solving the equations in these two examples.

Students may want to divide both sides by x. Remind them that x is an unknown quantity that could be positive, negative, or zero. These cases need to be handled separately to get the correct answer. Here we will take a more familiar approach in the solution process, factoring.

Continue to emphasize the idea of rewriting the factored equation as a compound statement. Do not let students skip this step!

Exercises 2–7 (7 minutes)

Give students time to work on the problems individually. As students finish, have them work the problems on the board.

Answers are below.

Exercises 2–7

2. $(x+1)(x+2)=0$

$\{-2,-1\}$

3. $(3x-2)(x+12)=0$

$\left\{-12, \frac{2}{3}\right\}$

4. $(x-3)(x-3)=0$

$\{3\}$

5. $(x+4)(x-6)(x-10)=0$

$\{-4,6,10\}$

6. $x^2-6x=0$

$\{0,6\}$

7. $x(x-5)+4(x-5)=0$

$\{-4,5\}$

Example 3 (3 minutes)

Example 3

Consider the equation $(x-2)(2x-3)=(x-2)(x+5)$. Lulu chooses to multiply through by $\frac{1}{x-2}$ and gets the answer $x=8$. But Poindexter points out that $x=2$ is also an answer, which Lulu missed.

a. What's the problem with Lulu's approach?

You cannot multiply by $\frac{1}{x-2}$ because $x-2$ could equal 0, which means that you would be dividing by 0.

b. Use factoring to solve the original equation for x.

$$(x-2)(2x-3)-(x-2)(x+5)=0$$
$$(x-2)\big(2x-3-(x+5)\big)=0$$
$$(x-2)(x-8)=0$$
$$x-2=0 \text{ or } x-8=0$$
$$x=2 \text{ or } x=8$$

Work through the responses as a class.

- Emphasize the idea that multiplying by $\frac{1}{x-2}$ is a problem when $x-2$ equals 0.

Exercises 8–11 (10 minutes)

Give students time to work on Exercises 8–11 in pairs. Then, elicit student responses.

Remind students of the danger of multiplying both sides by a variable expression.

Exercises 8–11

8. Use factoring to solve the equation for x: $(x-2)(2x-3)=(x-2)(x+1)$.

$\{2,4\}$

Suggestion for Early Finishers:

- The problems seen in Exercise 9 are often called the difference of two squares. Give early finishers this challenge: $x^4 - 81 = (x^2)^2 - 9^2 = ?$

9. **Solve each of the following for x:**

   a. $x + 2 = 5$

   $\{3\}$

   b. $x^2 + 2x = 5x$

   $\{0, 3\}$

   c. $x(5x - 20) + 2(5x - 20) = 5(5x - 20)$

   $\{3, 4\}$

10.

   a. **Verify:** $(a - 5)(a + 5) = a^2 - 25$.

   $$a^2 + 5a - 5a - 25 = a^2 - 25$$
   $$a^2 - 25 = a^2 - 25$$

   b. **Verify:** $(x - 88)(x + 88) = x^2 - 88^2$.

   $$x^2 + 88x - 88x - 88^2 = x^2 - 88^2$$
   $$x^2 - 88^2 = x^2 - 88^2$$

   c. **Verify:** $A^2 - B^2 = (A - B)(A + B)$.

   $$A^2 - B^2 = A^2 + AB - AB - B^2$$
   $$A^2 - B^2 = A^2 - B^2$$

   d. **Solve for x:** $x^2 - 9 = 5(x - 3)$.

   $\{2, 3\}$

   e. **Solve for w:** $(w + 2)(w - 5) = w^2 - 4$.

   $\{-2\}$

11. **A string 60 inches long is to be laid out on a table-top to make a rectangle of perimeter 60 inches. Write the width of the rectangle as $15 + x$ inches. What is an expression for its length? What is an expression for its area? What value for x gives an area of largest possible value? Describe the shape of the rectangle for this special value of x.**

   Length: $15 - x$ *area:* $(15 - x)(15 + x)$

   The largest area is when $x = 0$. In this case, the rectangle is a square with length and width both equal to 15.

Discuss the results of Exercise 10.

Work through Exercise 11 as a class, explaining why $x = 0$ gives the largest area.

- Since $(15 + x)(15 - x) = 225 - x^2$ as x gets larger, $225 - x^2$ gets smaller until $x = 15$ at which point the area is zero. So the domain of x for this problem is $0 \leq x \leq 15$.
- How can we change the domain if we don't want to allow zero area?
  - *You can leave the* 15 *end of the interval open if you don't want to allow zero area.*

Closing (3 minutes)

Elicit student responses. Students should make notes of responses in the Lesson Summary rectangle.

- If the product of 4 numbers is zero, what do we know about the numbers? At least one of them must equal 0.
- What is the danger of dividing both sides of an equation by a variable factor? What should be done instead?

Lesson Summary

The zero-product property says that if $ab = 0$, then either $a = 0$ or $b = 0$ or $a = b = 0$.

Exit Ticket (5 minutes)

Name ____ Date____________________

Lesson 17: Equations Involving Factored Expressions

Exit Ticket

1. Find the solution set to the equation $3x^2 + 27x = 0$.

2. Determine if each statement is true or false. If the statement is false, explain why or show work proving that it is false.

   a. If $a = 5$, then $ac = 5c$.

   b. If $ac = 5c$, then $a = 5$.

Exit Ticket Solutions

1. **Find the solution set to the equation $3x^2 + 27x = 0$.**

   $3x(x+9) = 0$

   $3x = 0$ *or* $x + 9 = 0$

   $x = 0$ *or* $x = -9$

   solution set: $\{-9, 0\}$

2. **Determine if each statement is true or false. If the statement is false, explain why or show work proving that it is false.**

   a. **If $a = 5$, then $ac = 5c$.**

   True.

   b. **If $ac = 5c$, then $a = 5$.**

   False, a could equal 5 *or c could equal* 0.

   $ac = 5c$

   $ac - 5c = 0$

   $c(a-5) = 0$

   $c = 0$ *or* $a - 5 = 0$

   $c = 0$ *or* $a = 5$

Problem Set Solutions

1. **Find the solution set of each equation:**

   a. $(x-1)(x-2)(x-3) = 0$

   $\{1, 2, 3\}$

   b. $(x-16.5)(x-109) = 0$

   $\{16.5, 109\}$

   c. $x(x+7) + 5(x+7) = 0$

   $\{-7, -5\}$

   d. $x^2 + 8x + 15 = 0$

   $\{-5, -3\}$

   e. $(x-3)(x+3) = 8x$

   $\{-1, 9\}$

2. Solve $x^2 - 11x = 0$, for x.

   $\{0, 11\}$

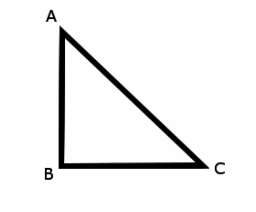
3. Solve $(p+3)(p-5) = 2(p+3)$, for p. What solution do you lose if you simply divide by $p+3$ to get $p-5=2$?

   $p = -3$ or $p = 7$. The lost solution is $p = -3$. We assumed $p+3$ was not zero when we divided by $p+3$; therefore, our solution was only complete for p values not equal to -3.

4. The square of a number plus 3 times the number is equal to 4. What is the number?

   Solve $x^2 + 3x = 4$, for x to obtain $x = -4$ or $x = 1$.

5. In the right triangle shown below, the length of side AB is x, the length of side BC is $x + 2$, and the length of the hypotenuse AC is $x + 4$. Use this information to find the length of each side. (Use the Pythagorean Theorem to get an equation, and solve for x.)



Use the Pythagorean Theorem to get the equation $x^2 + (x+2)^2 = (x+4)^2$. This is equivalent to $x^2 - 4x - 12 = 0$, and the solutions are -2 and 6. Choose 6 since x represents a length, and the lengths are

AB: 6
BC: 8
AC: 10

6. Using what you learned in this lesson, create an equation that has 53 and 22 as its only solutions.

   $(x-22)(x-53) = 0$

Lesson 18: Equations Involving a Variable Expression in the Denominator

Student Outcomes

- Students interpret equations like $\frac{1}{x} = 3$ as two equations "$\frac{1}{x} = 3$" and "$x \neq 0$" joined by "and." Students find the solution set for this new system of equations.

Classwork

Opening Exercise (5 minutes)

Allow students time to complete the warm up and then discuss the results.

MP.3

Opening Exercise

Nolan says that he checks the answer to a division problem by performing multiplication. For example, he says that $20 \div 4 = 5$ is correct because 5×4 is 20, and $\frac{3}{1/2} = 6$ is correct because $6 \times \frac{1}{2}$ is 3.

a. **Using Nolan's reasoning, explain why there is no real number that is the answer to the division problem $5 \div 0$.**

There is no number n such that $0 \times n = 5$.

b. **Quentin says that $\frac{0}{0} = 17$. What do you think?**

While it is true that $0 \times 17 = 0$, the problem is that by that principle $\frac{0}{0}$ could equal any number.

c. **Mavis says that the expression $\frac{5}{x+2}$ has a meaningful value for whatever value one chooses to assign to x. Do you agree?**

No, the expression does not have a meaningful value when $x = -2$.

d. **Bernoit says that the expression $\frac{3x-6}{x-2}$ always has the value 3 for whichever value one assigns to x. Do you agree?**

The expression does equal 3 for all values of x except $x = 2$.

Note that the problem with $\frac{0}{0}$ is that too many numbers pass Nolan's criterion! Have students change 17 to a different number. It still passes Nolan's multiplication check. Like $\frac{5}{0}$, it is a problematic notion. For this reason, we want to disallow the possibility of ever dividing by zero.

Point out that an expression like $\frac{5}{x+2}$ is really accompanied with the clause "under the assumption the denominator is not zero." So, $\frac{5}{x+2}$ should be read as a compound statement:

$$\frac{5}{x+2} \text{ and } x+2 \neq 0 \qquad \text{OR} \qquad \frac{5}{x+2} \text{ and } x \neq -2$$

Exercises 1–2 (5 minutes)

Give students a few minutes to complete the problems individually. Then, elicit answers from students.

Scaffolding:

- Remind students of the difference between dividing 0 by a number and dividing a number by 0.

Exercises 1–2

1. Rewrite $\frac{10}{x+5}$ as a compound statement.

$\frac{10}{x+5}$ and $x \neq -5$

2. Consider $\frac{x^2-25}{(x^2-9)(x+4)}$.

a. Is it permissible to let $x = 5$ in this expression?

Yes, $\frac{0}{144} = 0$.

b. Is it permissible to let $x = 3$ in this expression?

No, $-\frac{16}{0}$ is not permissible.

c. Give all the values of x that are *not* permissible in this expression.

$x \neq -3, 3, -4$

Examples 1–2 (10 minutes)

Work through the examples as a whole class.

Example 1

Consider the equation $\frac{1}{x} = \frac{3}{x-2}$.

a. Rewrite the equation into a system of equations.

$\frac{1}{x} = \frac{3}{x-2}$ and $x \neq 0$ and $x \neq 2$

b. Solve the equation for x, excluding the value(s) of x that lead to a denominator of zero.

$x = -1$ and $x \neq 0$ and $x \neq 2$ solution set: $\{-1\}$

Example 2

Consider the equation $\frac{x+3}{x-2} = \frac{5}{x-2}$.

a. **Rewrite the equation into a system of equations.**

$\frac{x+3}{x-2} = \frac{5}{x-2}$ and $x \neq 2$

b. **Solve the equation for x, excluding the value(s) of x that lead to a denominator of zero.**

$x = 2$ and $x \neq 2$ *solution set: $\emptyset$*

Emphasize the process of recognizing a rational equation as a system composed of the equation itself and the excluded value(s) of x. For Example 1, this is really the compound statement:

$$\frac{1}{x} = \frac{3}{x-2} \text{ and } x \neq 0 \text{ and } x - 2 \neq 0$$

- By the properties of equality, we can multiply through by non-zero quantities. Within this compound statement, x and $x - 2$ are nonzero, so we may write $x - 2 = 3x$ and $x \neq 0$ and $x - 2 \neq 0$, which is equivalent to $-2 = 2x$ and $x \neq 0$ and $x \neq 2$.
- All three declarations in this compound statement are true for $x = -1$. This is the solution set.

In Example 2, remind students of the previous lesson on solving equations involving factored expressions. Students will need to factor out the common factor and then apply the zero-product property.

- What happens in Example 2 when we have $x = 2$ and $x \neq 2$? Both declarations cannot be true. What can we say about the solution set of the equation?
  - *There is no solution.*

Exercises 3–11 (15 minutes)

Allow students time to complete the problems individually. Then, have students compare their work with another student. Make sure that students are setting up a system of equations as part of their solution.

Exercises 3–11

Rewrite each equation into a system of equations excluding the value(s) of x that lead to a denominator of zero; then, solve the equation for x.

3. $\frac{5}{x} = 1$

   $\frac{5}{x} = 1$ and $x \neq 0$

   $\{5\}$

4. $\frac{1}{x-5} = 3$

   $\frac{1}{x-5} = 3$ and $x \neq 5$

   $\left\{\frac{16}{3}\right\}$

5. $\frac{x}{x+1} = 4$

   $\frac{x}{x+1} = 4$ and $x \neq -1$

   $\left\{-\frac{4}{3}\right\}$

6. $\frac{2}{x} = \frac{3}{x-4}$

$\frac{2}{x} = \frac{3}{x-4}$ *and* $x \neq 0$ *and* $x \neq 4$

$\{-8\}$

7. $\frac{x}{x+6} = -\frac{6}{x+6}$

$\frac{x}{x+6} = -\frac{6}{x+6}$ *and* $x \neq -6$

No solution

8. $\frac{x-3}{x+2} = 0$

$\frac{x-3}{x+2} = 0$ *and* $x \neq -2$

$\{3\}$

9. $\frac{x+3}{x+3} = 5$

$\frac{x+3}{x+3} = 5$ *and* $x \neq -3$

No solution

10. $\frac{x+3}{x+3} = 1$

$\frac{x+3}{x+3} = 1$ *and* $x \neq -3$

All real numbers except -3

11. A baseball player's batting average is calculated by dividing the number of times a player got a hit by the total number of times the player was at bat. It is expressed as a decimal rounded to three places. After the first ten games of the season, Samuel had 12 hits off of 33 "at bats."

   a. What is his batting average after the first ten games?

   $$\frac{12}{33} \approx 0.364$$

   b. How many hits in a row would he need to get to raise his batting average to above 0.500?

   $$\frac{12+x}{33+x} > 0.500$$
   $$x > 9$$

   He would need 10 *hits in a row to be above* 0.500*.*

   c. How many "at bats" in a row without a hit would result in his batting average dropping below 0.300?

   $$\frac{12}{33+x} < 0.300$$
   $$x > 7$$

   If he went 8 *"at bats" in a row without a hit, he would be below* 0.300*.*

Ask:

- What was the difference between Exercises 9 and 10? How did that affect the solution set?
- Work through Exercises 11 as a class.

Closing (5 minutes)

Ask these questions after going over the exercises.

- When an equation has a variable in the denominator, what must be considered?
- When the solution to the equation is also an excluded value of x, then what is the solution set to the equation?

Exit Ticket (5 minutes)

Name ____ Date____________________

Lesson 18: Equations Involving a Variable Expression in the Denominator

Exit Ticket

1. Rewrite the equation $\frac{x-2}{x-9} = 2$ as a system of equations. Then, solve for x.

2. Write an equation that would have the restriction $x \neq -3$.

Exit Ticket Sample Solutions

1. Rewrite the equation $\frac{x-2}{x-9} = 2$ as a system of equations. Then, solve for x.

   $\frac{x-2}{x-9} = 2$ *and* $x \neq 9$

   $x - 2 = 2(x - 9)$

   $x - 2 = 2x - 18$

   $16 = x$

   $\{16\}$

2. Write an equation that would have the restriction $x \neq -3$.

   sample answer $\frac{4}{x+3} = 2$

Problem Set Sample Solutions

1. Consider the equation $\frac{10(x^2-49)}{3x(x^2-4)(x+1)} = 0$. Is $x = 7$ permissible? Which values of x are excluded? Rewrite as a system of equations.

   Yes, $x = 7$ *is permissible. The excluded values are* 0, ± 2, *and* -1. *The system is*

   $\frac{10(x^2-49)}{3x(x^2-4)(x+1)} = 0$ *and* $x \neq 0$ *and* $x \neq -2$ *and* $x \neq -1$ *and* $x \neq 2$.

2. Rewrite each equation as a system of equations excluding the value(s) of x that lead to a denominator of zero. Then solve the equation for x.

   a. $25x = \frac{1}{x}$

   System: $25x = \frac{1}{x}$ *and* $x \neq 0$; *solution set:* $\left\{\pm\frac{1}{5}\right\}$

   b. $\frac{1}{5x} = 10$

   System: $\frac{1}{5x} = 10$ *and* $x \neq 0$; *solution set:* $\left\{\frac{1}{50}\right\}$

   c. $\frac{x}{7-x} = 2x$

   System: $\frac{x}{7-x} = 2x$ *and* $x \neq 7$; *solution set:* $\left\{0, \frac{13}{2}\right\}$

   d. $\frac{2}{x} = \frac{5}{x+1}$

   System: $\frac{2}{x} = \frac{5}{x+1}$ *and* $x \neq -1$ *and* $x \neq 0$; *solution set:* $\left\{\frac{2}{3}\right\}$

e. $\frac{3+x}{3-x} = \frac{3+2x}{3-2x}$

System: $\frac{3+x}{3-x} = \frac{3+2x}{3-2x}$ *and* $x \neq \frac{3}{2}$ *and* $x \neq 3$*; solution set:* $\{0\}$

3. Ross wants to cut a 40-foot rope into two pieces so that the length of the first piece divided by the length of the second piece is 2.

   a. Let x represent the length of the first piece. Write an equation that represents the relationship between the pieces as stated above.

$$\frac{x}{40-x} = 2$$

   b. What values of x are not permissible in this equation? Describe within the context of the problem, what situation is occurring if x were to equal this value(s). Rewrite as a system of equations to exclude the value(s).

   40 *is not a permissible value because it would mean the rope is still intact. System:* $\frac{x}{40-x} = 2$ *and* $x \neq 40$

   c. Solve the equation to obtain the lengths of the two pieces of rope. (Round to the nearest tenth if necessary.)

   First piece is $\frac{80}{3} \approx 26.7$ *feet long; second piece is* $\frac{40}{3} \approx 13.3$ *feet long.*

4. Write an equation with the restrictions $x \neq 14$, $x \neq 2$, and $x \neq 0$.

   Answers will vary. Sample equation: $\frac{1}{x(x-2)(x-14)} = 0$

5. Write an equation that has no solution.

   Answers will vary. Sample equation: $\frac{1}{x} = 0$

Lesson 19: Rearranging Formulas

Student Outcomes

- Students learn to think of some of the letters in a formula as constants in order to define a relationship between two or more quantities, where one is in terms of another, for example holding V in $V = IR$ as constant, and finding R in terms of I.

Classwork

Scaffolding:

- Before starting the warm-up, ask students to read the introduction and discuss other formulas they have seen in previous grades.
- As students work the warm-up and first few exercises, adjust the pacing depending on how well students are doing with Exercises 1(c) and 3.

Provide an introduction to the lesson:

- Formulas that relate two or more variable symbols such as $A = lw$, $D = rt$, or $a^2 + b^2 = c^2$ arise in different applications of mathematics, science, and other areas of study. These formulas have meaning based on a situation.
- However, even without an applied setting, formulas can stand on their own as a relationship between variables.
- You can use the equation-solving techniques from earlier lessons to rearrange formulas and solve for a specific variable symbol.

Exercise 1 (5 minutes)

Have students work independently. Monitor their progress on part (c) and have them review and correct their solutions with a partner.

Exercise 1

Solve each equation for x. For part (c), remember a variable symbol, like a, b, and c, represents a number.

a. $2x - 6 = 10$

$x = 8$

b. $-3x - 3 = -12$

$x = 3$

c. $ax - b = c$

$$ax - b = c$$
$$ax = b + c$$
$$x = \frac{b + c}{a}$$

Exercises 2–3 (5 minutes)

Exercise 2

Compare your work in parts (a) through (c) above. Did you have to do anything differently to solve for x in part (c)?

The process to solve all three equations is the same.

Exercise 3

Solve the equation $ax - b = c$ for a. The variable symbols x, b, and c represent numbers.

Solving for a is the same process as solving for x. $a = \frac{b+c}{x}$

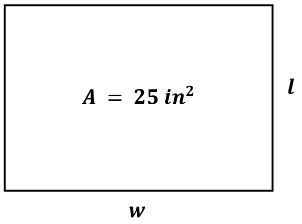
Debrief student responses to Exercises 2 and 3 as a whole class. Make sure to emphasize the points below.

- Variables are placeholders for numbers and as such have the same properties.
- When solving an equation with several variables, you use the same properties and reasoning as with single-variable equations.
- The equation in Exercise 3 holds as long as x does not equal 0 (division by 0 is undefined). Consider your result from Exercise 1(c). Does this equation hold for all values of the variables involved?
    - *No, it only holds if* $a \neq 0$.

Example 1 (5 minutes): Rearranging Familiar Formulas

Example 1: Rearranging Familiar Formulas

The area A of a rectangle is 25 in^2. The formula for area is $A = lw$.



- **If the width w is 10 inches, what is the length l?**

  $l = \frac{5}{2}$

- **If the width w is 15 inches, what is the length l?**

  $l = \frac{5}{3}$

- **Rearrange the area formula to solve for l.**

$$A = lw$$
$$\frac{A}{w} = \frac{lw}{w}$$
$$\frac{A}{w} = l \text{ or } l = \frac{A}{w}$$

- **Verify that the area formula, solved for l, will give the same results for l as having solved for l in the original area formula.**

$$A = lw$$
$$l = \frac{A}{w} = \frac{25}{10} = \frac{5}{2}$$
$$l = \frac{A}{w} = \frac{25}{15} = \frac{5}{3}$$

MP.3

Walk students through the solution to this problem. Have them write the reasons for each step in the equation solving process on their paper. Much of the work students will do in future classes will involve rearranging formulas to highlight a variable of interest. Begin to set the stage that solving for a variable *before* you plug in values is often easier than solving after you substitute the values, especially when the numbers are not user-friendly. If time permits, give them $A = 10.356$ and $w = 5\frac{3}{11}$ and ask them to solve for the length.

Exercise 4 (7 minutes)

Have students work in small groups or with a partner. Solving these exercises two ways will help students to further understand that rearranging a formula with variables involves the same reasoning as solving an equation for a single variable.

Exercise 4

Solve each problem two ways. First, substitute the given values and solve for the given variable. Then, solve for the given variable and substitute the given values.

a. **The perimeter formula for a rectangle is $p = 2(l + w)$, where p represents the perimeter, l represents the length, and w represents the width. Calculate l when $p = 70$ and $w = 15$.**

Sample responses:

Substitute and solve. $70 = 2(l + 15),\ l = 20$

Solve for the variable first: $l = \frac{p}{2} - w$

b. **The area formula for a triangle is $A = \frac{1}{2}bh$, where A represents the area, b represents the length of the base, and h represents the height. Calculate b when $A = 100$ and $h = 20$.**

$b = \frac{2A}{h}$, $b = 10$

Have one or two students present their solutions to the entire class.

Exercise 5 (7 minutes)

The next set of exercises increases slightly in difficulty. Instead of substituting, students solve for the requested variable. Have students continue to work in groups or with a partner. If the class seems to be getting stuck, solve part of one exercise as a whole class and then have them go back to working with their partner or group.

Have students present their results to the entire class. Look for valid solution methods that arrive at the same answer using a slightly different process to isolate the variable. For part (b–ii), students may need a reminder to use the square root to "undo" the square of a number. They learned about square roots and solving simple quadratic equations in Grade 8.

Exercise 5

Rearrange each formula to solve for the specified variable. Assume no variable is equal to 0.

a. **Given $A = P(1 + rt)$,**
   i. **Solve for P.**

$$P = \frac{A}{1 + rt}$$

   ii. **Solve for t.**

$$t = \left(\frac{A}{P} - 1\right) \div r$$

b. Given $K = \frac{1}{2}mv^2$,

i. Solve for m.

$$m = \frac{2K}{v^2}$$

ii. Solve for v.

$$v = \pm\sqrt{\frac{2K}{m}}$$

Example 2 (10 minutes): Comparing Equations with One Variable to Those With More Than One Variable

Demonstrate how to reverse the distributive property (factoring) to solve for x. Start by solving the related equation OR solve both equations at the same time, one line at a time. Encourage students to make notes justifying their reasoning on each step. Continue emphasizing that the process is the same because variables are just numbers whose values have yet to be assigned.

Scaffolding:

- In Example 2, some students may not need to solve the problems in the right column first. Others may need to start there before solving the literal equations in the left column.

Example 2

Equation Containing More Than One Variable	Related Equation
Solve $ax + b = d - cx$ for x. $ax + cx + b = d$ $ax + cx = d - b$ $x(a + c) = d - b$ $x = \frac{d - b}{a + c}$	Solve $3x + 4 = 6 - 5x$ for x. $3x + 5x + 4 = 6$ $3x + 5x = 6 - 4$ $x(3 + 5) = 2$ $8x = 2$ $x = \frac{2}{8} = \frac{1}{4}$
Solve for x. $\frac{ax}{b} + \frac{cx}{d} = e$ $bd\left(\frac{ax}{b} + \frac{cx}{d}\right) = bd(e)$ $dax + cbx = bde$ $x(da + cb) = bde$ $x = \frac{bde}{da + cb}$	Solve $\frac{2x}{5} + \frac{x}{7} = 3$ for x. $\frac{2x}{5} + \frac{x}{7} = 3$ $35\left(\frac{2x}{5} + \frac{x}{7}\right) = 35(3)$ $14x + 5x = 105$ $x(14 + 5) = 105$ $19x = 105$ $x = \frac{105}{19}$

Closing (2 minutes)

Review the Lesson Summary and close with these questions.

- How is rearranging formulas the same as solving equations that contain a single variable symbol?
- How is rearranging formulas different from solving equations that contain a single variable symbol?

As you wrap up, make sure students understand that while there is essentially no difference when rearranging formulas, it may seem more difficult and the final answers may appear more complicated because you cannot combine the variables into a single numerical expression like you can when you add, subtract, multiply, or divide numbers in the course of solving a typical equation.

Lesson Summary

The properties and reasoning used to solve equations apply regardless of how many variables appear in an equation or formula. Rearranging formulas to solve for a specific variable can be useful when solving applied problems.

Exit Ticket (4 minutes)

Name ____ Date____________________

Lesson 19: Rearranging Formulas

Exit Ticket

Given the formula $x = \frac{1+a}{1-a}$,

1. Solve for a when $x = 12$.

2. Rearrange the formula to solve for a.

Exit Ticket Sample Responses

Given the formula $= \frac{1+a}{1-a}$,

1. Solve for a when $x = 12$.

$$12 = \frac{1+a}{1-a}$$
$$12(1-a) = 1+a$$
$$12 - 12a = 1+a$$
$$11 = 13a$$
$$\frac{11}{13} = a$$

2. Rearrange the formula to solve for a.

$$x = \frac{1+a}{1-a}$$
$$x(1-a) = 1+a$$
$$x - xa = 1+a$$
$$x - 1 = a + xa$$
$$x - 1 = a(1+x)$$
$$\frac{x-1}{1+x} = a$$

Problem Set Sample Solutions

For Problems 1–8, solve for x.

1. $ax + 3b = 2f$

$x = \frac{2f - 3b}{a}$

2. $rx + h = sx - k$

$x = \frac{h+k}{s-r}$

3. $3px = 2q(r - 5x)$

$x = \frac{2qr}{3p + 10q}$

4. $\frac{x+b}{4} = c$

$x = 4c - b$

5. $\frac{x}{5} - 7 = 2q$

$x = 10q + 35$

6. $\frac{x}{6} - \frac{x}{7} = ab$

$x = 42ab$

7. $\frac{x}{m} - \frac{x}{n} = \frac{1}{p}$

$x = \frac{nm}{(n-m)p}$

8. $\frac{3ax+2b}{c} = 4d$

$x = \frac{4cd - 2b}{3a}$

9. Solve for m.

$t = \frac{ms}{m+n}$

$m = \frac{nt}{s-t}$

10. Solve for u.

$\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$

$u = \frac{vf}{v-f}$

11. Solve for s.

$A = s^2$

$s = \pm\sqrt{A}$

12. Solve for h.

$V = \pi r^2 h$

$h = \frac{V}{\pi r^2}$

13. Solve for m.

$T = 4\sqrt{m}$

$m = \frac{T^2}{16}$

14. Solve for d.

$F = G\frac{mn}{d^2}$

$d = \pm\sqrt{\frac{Gmn}{F}}$

15. Solve for y.

$ax + by = c$

$y = \frac{c - ax}{b}$

16. Solve for b_1.

$A = \frac{1}{2}h(b_1 + b_2)$

$b_1 = \frac{2A}{h} - b_2$

17. The science teacher wrote three equations on a board that relate velocity, v, distance traveled, d, and the time to travel the distance, t, on the board.

$v = \frac{d}{t}$ $t = \frac{d}{v}$ $d = vt$

Would you need to memorize all three equations or could you just memorize one? Explain your reasoning.

You could just memorize $d = vt$ since the other two equations are obtained from this one by solving for v and t.

Lesson 20: Solution Sets to Equations with Two Variables

Student Outcomes

- Students recognize and identify solutions to two-variable equations. They represent the solution set graphically. They create two variable equations to represent a situation. They understand that the graph of the line $ax + by = c$ is a visual representation of the solution set to the equation $ax + by = c$.

Classwork

Open the lesson with the following:

- When working with equations that contain more than one variable, there is often more than one solution. Solutions can be represented using ordered pairs when the equation contains two variables. All of the solutions to an equation are called the solution set.
- What do we know about the graph of equations of the form $ax + by = c$?
  - *We know from Grade 8, the graph of $ax + by = c$ is a line.*

Scaffolding:

- Circulate to make sure students are substituting the first coordinate for x and the second one for y.
- Create a class graph using a sheet of poster paper and give each student a sticky dot. Have them write the solution on the sticky dot and post theirs on the class graph. They can correct errors when their solution is not on the line.

Exercise 1 (5 minutes)

Exercise 1

a. **Circle all the ordered pairs (x, y) that are solutions to the equation $4x - y = 10$.**

[✓ $(3, 2)$] $(2, 3)$ [✓ $(-1, -14)$] $(0, 0)$ [✓ $(1, -6)$]

[✓ $(5, 10)$] [✓ $(0, -10)$] $(3, 4)$ $(6, 0)$ $(4, -1)$

b. **How did you decide whether or not an ordered pair was a solution to the equation?**

Most students will explain that they substituted and checked to see whether or not the equation was true.

Point out the two-variable equation in Exercise 1 and the possible solutions represented as ordered pairs. Working independently, students use their prior knowledge to verify which ordered pairs are solutions to an equation. Ask students to compare their solutions with a partner. Briefly share answers and give students a chance to revise their work or add to their written response to part (b).

Exercise 2 (15 minutes)

Students should work in groups on Exercise 2(a) only. After about 4 minutes, have each group share their solutions and their solution strategies with the entire class. Highlight the different approaches to finding solutions. Most groups will likely start by picking a value for either x or y and then deciding what the other variable should equal to make the number sentence true. Value groups that began to organize their solutions in a meaningful way, such as by increasing x-values. Groups may even organize their solutions in a graph.

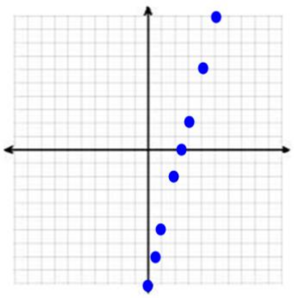
Exercise 2

a. **Discover as many additional solutions to the equation $4x - y = 10$ as possible. Consider the best way to organize all the solutions you have found. Be prepared to share the strategies you used to find your solutions.**

Sample answers: $(-5,-30)$; $(-2,-18)$; $(2,-2)$; $(4,6)$

b. **Now, find five more solutions where one or more variables are negative numbers or non-integer values. Be prepared to share the strategies you used to find your solutions.**

Sample answers: $(-4,-26)$; $(-3,-22)$; $\left(\frac{1}{2},-8\right)$; $\left(\frac{3}{2},-4\right)$; $\left(\frac{5}{2},0\right)$



c. **How many ordered pairs (x,y) will be in the solution set of the equation $4x - y = 10$?**

Infinitely many.

d. **Create a visual representation of the solution set by plotting each solution as a point (x,y) in the coordinate plane.**

e. **Why does it make sense to represent the solution to the equation $4x - y = 10$ as a line in the coordinate plane?**

Drawing the line is the only way to include all possible solutions. It would be impossible to plot every point that is a solution to the equation since there are infinitely many solutions.

Next, have the groups complete parts (b) through (d). Debrief the entire class by having each group share their work, or if time permits, create a class graph. Parts (c) and (e) are the most important. Students need to realize that listing all the solutions is impossible. A visual representation of a curve (in this case a line) is a way to include ALL possible solutions including those with fractional or irrational values. You could prove this to students by letting $x = \sqrt{2}$ and then solving for $y = 4\sqrt{2} - 10$. Then use a graphing calculator or graphing software to graph the line and find the value when $x = \sqrt{2}$.

Exercises 3–5 (15 minutes)

MP.2 & MP.6

These exercises ask students to create a linear equation in two variables that represents a situation. The equations in Exercises 3 and 4 are the same, but the domain of each variable is different due to the context. Require students to attend to precision in depicting answers that reflect the domain given the context. Are they using "let" statements to name each variable? Are they getting correct solutions to the equations? Notice whether or not they are using some of the strategies developed in the exercise to find their solutions. In Exercise 3, the graph should be a solid line that extends to the boundaries of the coordinate plane, but in Exercise 4, the graph should be discrete and only contain whole number values in the ordered pairs.

Scaffolding:

- Teach a method for getting solutions by selecting values for x and substituting them into the equation to find y. Have students organize the ordered pairs in a table.

Exercise 3–5

3. **The sum of two numbers is 25. What are the numbers?**

   a. **Create an equation using two variables to represent this situation. Be sure to explain the meaning of each variable.**

   Let $x =$ one number, and let $y =$ another number.

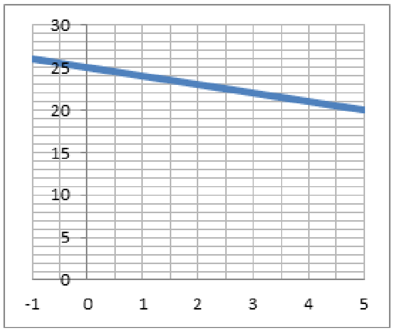
   Equation: $x + y = 25$

   b. **List at least six solutions to the equation you created in part (a).**

   Answers will vary but should be pairs of numbers whose sum is 25.

   c. **Create a graph that represents the solution set to the equation.**

   See graph to the right.



4. **Gia had 25 songs in a playlist composed of songs from her two favorite artists, Beyonce and Jennifer Lopez. How many songs did she have by each one in the playlist?**

   a. **Create an equation using two variables to represent this situation. Be sure to explain the meaning of each variable.**

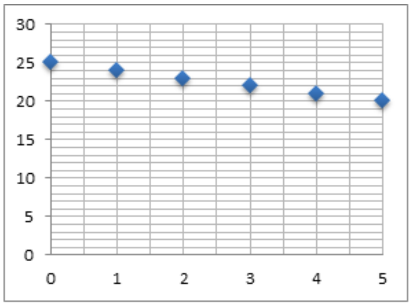
   Let $x =$ the number of Beyonce songs, and let $y =$ the number of Jennifer Lopez songs. Equation: $x + y = 25$

   b. **List at least three solutions to the equation you created in part (a).**

   Answers will vary but should be limited to pairs of whole numbers whose sum is 25.

   c. **Create a graph that represents the solution set to the equation.**

   See graph to the right.



5. **Compare your solutions to Exercises 3 and 4. How are they alike? How are they different?**

   The solution set to Exercise 3 is an infinite number of ordered pairs, and the graph is a solid line. The solution set to Exercise 4 is a finite set of ordered pairs that also happens to be in the solution set to Exercise 3. The graph is a discrete set of points that lie on the line $y = -x + 25$.

When you debrief these exercises with the whole class, be sure to reinforce that the domain depends on the situation. Review expectations for making a complete graph (scaling, labeling, etc.). Emphasize the difference between a discrete and continuous graph.

Closing (5 minutes)

Review the lesson summary by reading closely and posing the questions below.

- Is the graph of the line $y = 2x - 3$ the same as the solution set to the equation $y = 2x - 3$? Explain your reasoning.
    - *Yes. According to the lesson summary, the set of (x, y) points that forms the solution set will be points on the graph. Since the degree of x and y are both one, the resulting graph is a line.*
- Suppose I am using the equation to represent the following context: The number of trucks manufactured is always 3 fewer than twice the number of cars manufactured. Does it still make sense that every point on the line is a solution to my equation considering the context of my problem?
    - *No, only the points (x, y) that are in the domain of whole numbers (since they don't manufacture 0.3 of a car or truck) are solutions to the equation within this context.*
- Why is it useful to represent the solutions to a two-variable equation using a graph?
    - *If the domain of x and y are the real numbers, then it would be impossible to list all of the solutions as ordered pairs. A continuous graph implies that every x-value in the domain of the graph has a corresponding y-value that makes the equation true.*
    - *Even if the domain of x and y are integers or whole numbers or something other than the real numbers, it is helpful to extend a line along all the solution points and use that line to find other solutions within the domain (much more so than trying to draw out a bunch of dots along the ruler).*
    - *It is also cool and convenient that the graph of each class of equation has a distinctive shape on the x–y plane. The graphs of equations are usually visually striking and telling in a very efficient way. As we learn in future lessons, the 2-variable version of inequalities, solutions, systems, etc., each carries an interesting geometrical meaning in its 2D graph.*

Lesson Summary

An ordered pair is a solution to a two variable equation when each number substituted into its corresponding variable makes the equation a true-number sentence. All of the solutions to a two-variable equation are called the solution set.

Each ordered pair of numbers in the solution set of the equation corresponds to a point on the coordinate plane. The set of all such points in the coordinate plane is called the *graph of the equation*.

Exit Ticket (5 minutes)

Name ____ Date____________________

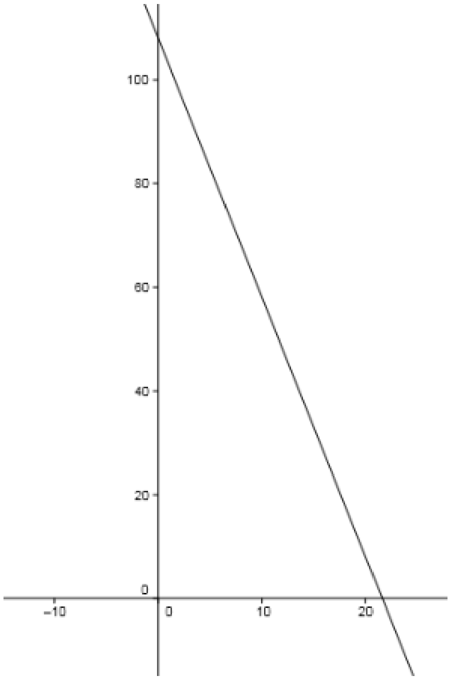
Lesson 20: Solution Sets to Equations with Two Variables

Exit Ticket

The Math Club sells hot dogs at a school fundraiser. The club earns $\$108$ and has a combination of five-dollar and one-dollar bills in its cash box. Possible combinations of bills are listed in the table below. Complete the table.

Number of five-dollar bills	Number of one-dollar bills	Total $= \$108$
19	13	$5(19) + 1(13) = 108$
16	28	
11	53	
4	88	

1. Find one more combination of ones and fives that totals $\$108$.



2. The equation $5x + 1y = 108$ represents this situation. A graph of the line $y = -5x + 108$ is shown. Verify that each ordered pair in the table lies on the line.

3. What is the meaning of the variables (x and y) and the numbers (1, 5, and 108) in the equation $5x + 1y = 108$?

Exit Ticket Sample Solutions

The Math Club sells hot dogs at a school fundraiser. The club earns $\$108$ and has a combination of five-dollar and one-dollar bills in its cash box. Possible combinations of bills are listed in the table below. Complete the table.

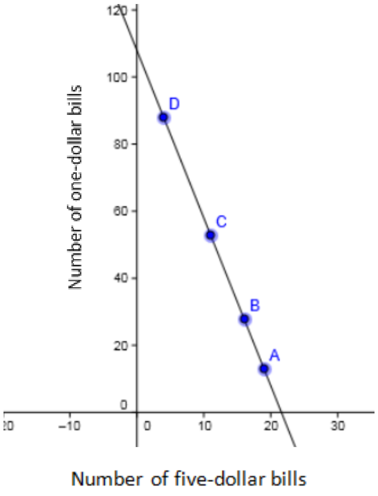
Number of five-dollar bills	Number of one-dollar bills	Total $= \$108$
19	13	$5(19) + 1(13) = 108$
16	28	$5(16) + 1(28) = 108$
11	53	$5(11) + 1(53) = 108$
4	88	$5(4) + 1(88) = 108$

1. **Find one more combination of ones and fives that totals $\$108$.**

   $(20, 8)$, which stands for 20 fives and 8 ones.

2. **The equation $5x + 1y = 108$ represents this situation. A graph of the line $y = -5x + 108$ is shown. Verify that each ordered pair in the table lies on the line.**

   Graph is shown with points plotted for the solutions. Verify by substituting the ordered pairs in the table into the equation for the line.



3. **What is the meaning of the variables (x and y) and the numbers (1, 5, and 108) in the equation $5x + 1y = 108$?**

   x is the number of 5-dollar bills; y is the number of 1-dollar bills; 1 is the value of a 1-dollar bill; 5 is the value of a 5-dollar bill, and 108 is the total dollar amount. If there are x fives, then $5x$ is the value of all the five-dollar bills.

Problem Set Sample Solutions

1. **Match each equation with its graph. Explain your reasoning.**

   Justifications will vary. Sample response: I identified points on each graph and substituted them into the equations. Some graphs had the same points like $(0, -6)$, so I needed to check the solutions with at least one other point.

   a. $y = 5x - 6$

   Graph 4

   b. $x + 2y = -12$

   Graph 5

   c. $2x + y = 4$

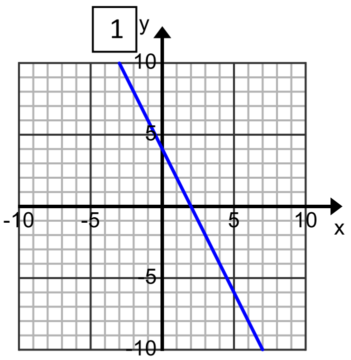
   Graph 1

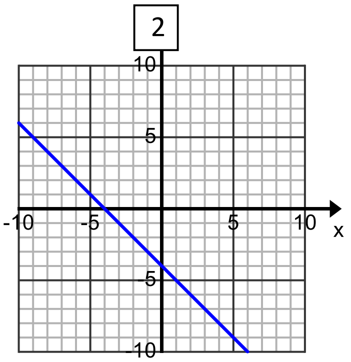
   d. $y = 3x - 6$

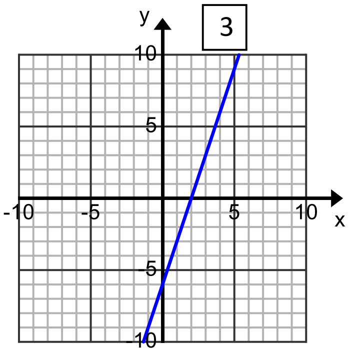
   Graph 3

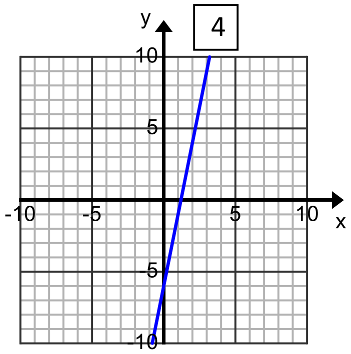
   e. $x = -y - 4$

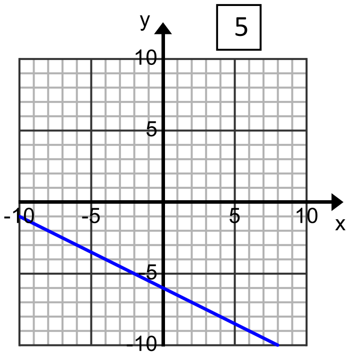
   Graph 2







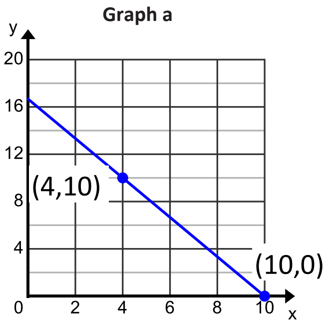


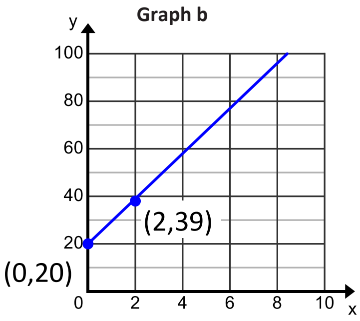


2. **Graph the solution set in the coordinate plane. Label at least two ordered pairs that are solutions on your graph.**

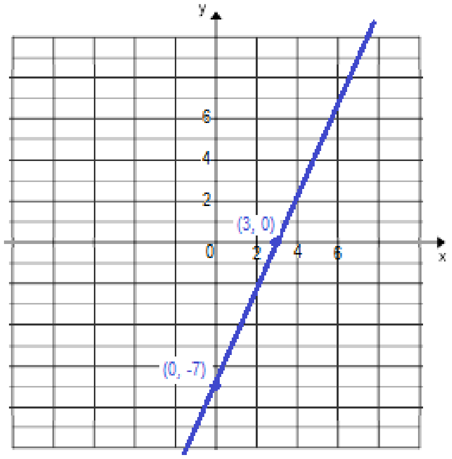
   a. $10x + 6y = 100$ b. $y = 9.5x + 20$ c. $7x - 3y = 21$ d. $y = 4(x + 10)$

   Solutions are shown below.

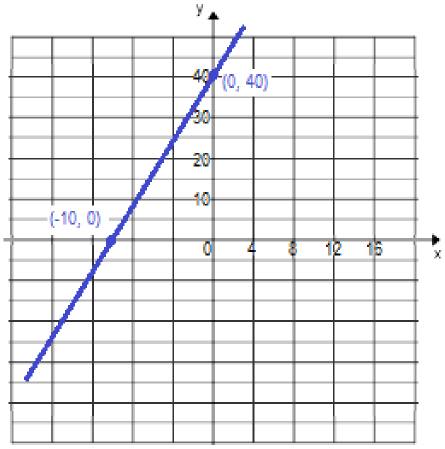




Graph c



Graph d



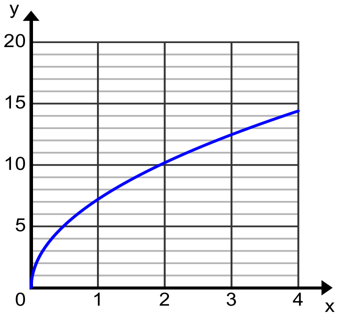
3. **Mari and Lori are starting a business to make gourmet toffee. They gather the following information from another business about prices for different amounts of toffee. Which equation and which graph are most likely to model the price, p, for x pounds of toffee? Justify your reasoning.**

Pounds x	Price p for x pounds
0.25	$\$3.60$
0.81	$\$6.48$
1	$\$7.20$
1.44	$\$8.64$

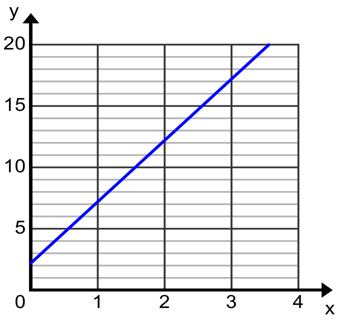
Equation A: $p = 5x + 2.2$

Equation B: $p = 7.2\sqrt{x}$

Graph 1



Graph 2



Graph 1 is a non-linear function. It fits the table data better and seems to be the correct equation. I estimated a few ordered pairs on the graph, and they were close matches to the values in the table. The points on the second graph do not match the table values at all for x-values larger than 1.

Lesson 21: Solution Sets to Inequalities with Two Variables

Student Outcomes

- Students recognize and identify solutions to two-variable inequalities. They represent the solution set graphically. They create two-variable inequalities to represent a situation.
- Students understand that a half-plane bounded by the line $ax + by = c$ is a visual representation of the solution set to a linear inequality, such as $ax + by < c$. They interpret the inequality symbol correctly to determine which portion of the coordinate plane is shaded to represent the solution.

Lesson Notes

Students explore an inequality related to the equation from the previous lesson's Exercises 1–2. Using the same equation will help students to distinguish the differences between solution sets and graphs of two-variable equations versus two-variable inequalities.

Materials

- Graph paper

Classwork

Consider opening the lesson with the following:

- When working with inequalities in one variable, you learned to graph the solution set on a number line. When working with inequalities with two variables, the solutions are also represented visually but in two-dimensions in the coordinate plane.

Exercise 1 (5 minutes)

Discuss the two-variable equation in Exercise 1 and the possible solutions represented as ordered pairs.

Have students work independently, using their prior knowledge to verify which ordered pairs are solutions to an equation (make a true number sentence).

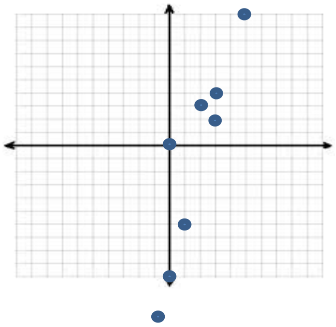
Exercise 1

a. **Circle each ordered pair (x, y) that is a solution to the equation $4x - y \leq 10$.**

i. [✓ $(3, 2)$] [✓ $(2, 3)$] [✓ $(-1, -14)$] [✓ $(0, 0)$] [✓ $(1, -6)$]

ii. [✓ $(5, 10)$] [✓ $(0, -10)$] [✓ $(3, 4)$] $(6, 0)$ $(4, -1)$

b. **Plot each solution as a point (x, y) in the coordinate plane.**



c. **How would you describe the location of the solutions in the coordinate plane?**

(Students may struggle to describe the points. Here is one possible description.) The points do not all fall on any one line, but if you drew a line through any two of the points, the others are not too far away from that line.

Ask students to compare their solutions with a partner. Briefly share answers and give students a chance to revise their work or add to their written response to part (a). Do not linger on part (c); the activity that follows will help to clarify their thinking.

Scaffolding

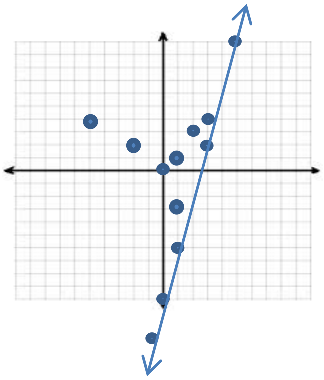
Pay attention to students who are still struggling to interpret the inequality symbols correctly. Perhaps creating a chart or adding terms to a word wall could serve as a reminder to the students.

Exercise 2 (10 minutes)

MP.1 Students should work in groups on part (a) only. After about 4 minutes, have each group share their solutions and their solution strategies with the entire class. Highlight the different approaches to finding solutions. Most groups will likely start by picking a value for either x or y and then deciding what the other variable should equal to make the number sentence true.

Exercise 2

a. **Discover as many additional solutions to the inequality $4x - y \leq 10$ as possible. Organize your solutions by plotting each solution as a point (x, y) in the coordinate plane. Be prepared to share the strategies used to find your solutions.**



(There are an infinite number of correct answers, as well as an infinite number of incorrect answers. Some sample correct answers are shown.)

$(1, 1)$, $(1, -3)$, $(-2, 2)$, $(-5, 4)$

b. **Graph the line $y = 4x - 10$. What do you notice about the solutions to the inequality $4x - y \leq 10$ and the graph of the line $y = 4x - 10$?**

All of the points are either on the line or to the left of (or above) the line.

c. **Solve the inequality for y.**

$y \geq 4x - 10$

d. **Complete the following sentence.**

If an ordered pair is a solution to $4x - y \leq 10$, then it will be located *on the line or above (or on the left side of)* **the line $y = 4x - 10$.**

Explain how you arrived at your conclusion.

I observed that all the points were on one side of the line, and then I tested some points on the other side of the line and found that all the points I tested from that side of the line were not solutions to the inequality.

Next have the groups complete parts (b)–(d). As they work, circulate around the room answering questions and providing support. Make sure that students reversed the inequality symbol when solving for y in part (c). Discuss the following:

- I noticed some of you wrote that all the points are on the left side of the line and others wrote that all the points are above the line. Are both of those descriptions correct?
- Now, look at your answer to part (c). When you solved the inequality for y, what does that statement seem to tell you?
  - *It tells you all the y-values have to be greater than or equal to something related to x.*
- Then which description would you say best correlates to the inequality we wrote in part (c)? Points to the left of the line or points above the line? Why?
  - *Points above the line because when we solved for y, we are describing where the y-values are in relation to the line, and y-values are plotted on the vertical axis; therefore, the words above and below are the accurate descriptors.*
- How can we depict the entire solution set of ALL the points above the line? When we worked with equations in one variable and graphed our solution set on the number line, how did we show what the solution set was?
  - *We colored it darker or shaded it. So we can just shade in the entire area above the line.*
- What about the line itself, is it part of the solution set?
  - *Yes.*
- What if it wasn't? What if the inequality was $y > 4x - 10$? How could we show that it is all the points except that line?
  - *We traditionally make the line a dashed line instead of a solid line to indicate that the points on the line are not part of the solution set.*

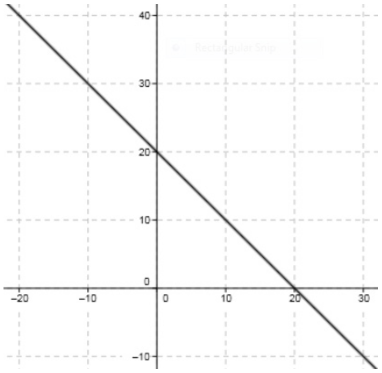
Before moving on, make sure students understand that any ordered pair in the solution set will be a point (x, y) that is located on (or above) the line because that is the portion of the coordinate plane where y is greater than or equal to the difference of $4x$ and 10.

Example 1 (10 minutes)

Example 1

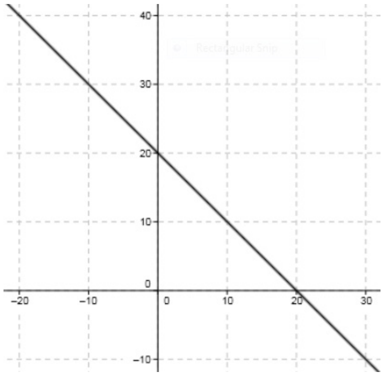
The solution to $x + y = 20$ is shown on the graph below.

a. **Graph the solution to $x + y \leq 20$.**



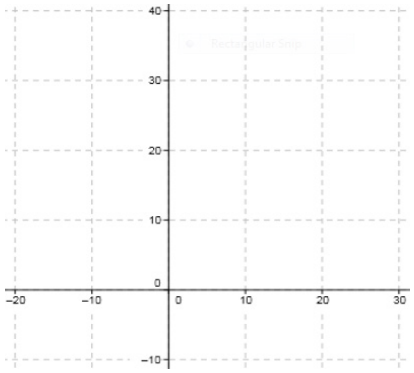
All points below the line should be shaded.

b. **Graph the solution to $x + y \geq 20$.**



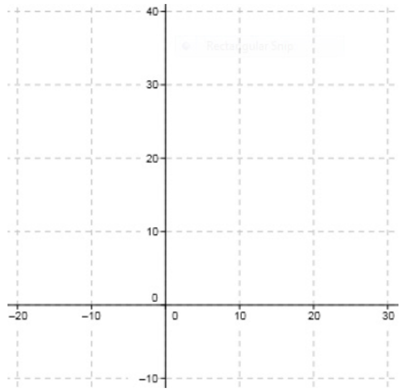
All points above the line should be shaded.

c. **Graph the solution to $x + y < 20$.**



The line should be dashed, and all points below the line should be shaded.

d. **Graph the solution to $x + y > 20$.**



The line should be dashed, and all points above the line should be shaded.

Exercises 3–5 (15 minutes)

Students will need graph paper for this portion of the lesson. Have students work individually to complete as much of Exercise 3 as they can in 8 minutes, reserving the final 7 minutes for comparing with a neighbor and debating any conflicting answers. Alternatively, differentiate by assigning only a subset of the problems most appropriate for each student or group of students. In any case, make the assignments in pairs so that students have someone with whom to compare answers. Students may struggle as they work on parts (f)–(j), graphing solutions to equations like $y = 5$; allow the students to struggle and discuss with each other. (Exercise 4 will revisit this idea with the entire class.) Students will rely on their experiences in Grade 8 as well as their explorations in Lessons 1–5 of this module to distinguish between the linear and non-linear inequalities and answer the question that concludes Exercise 3.

Allow students to debate and discuss. Guide them to the correct conclusion, and then review the definition of a half-plane that follows Exercise 3, clarifying for students that a **strict inequality** does not include the *or equal to* option. It must be either strictly *less than* or *greater than*.

Exercises 3–5

3. Using a separate sheet of graph paper, plot the solution sets to the following equations and inequalities:

- a. $x - y = 10$
- b. $x - y < 10$
- c. $y > x - 10$
- d. $y \geq x$
- e. $x \geq y$
- f. $y = 5$
- g. $y < 5$
- h. $x \geq 5$
- i. $y \neq 1$
- j. $x = 0$
- k. $x > 0$
- l. $y < 0$
- m. $x^2 - y = 0$
- n. $x^2 + y^2 > 0$
- o. $xy \leq 0$

Which of the inequalities in this exercise are *linear* inequalities?

Parts (a)–(l) are linear. Parts (m)–(o) are not.

a–c: Parts (b) and (c) are identical. In part (a), the solution is the graph of the line.

d–e: Both solution sets include the line $y = x$. Part (d) is the half-plane above the line, and part (e) is the half-plane below the line. When debriefing, ask students to share how they approached part (e).

f–i: These exercises focus on vertical and horizontal boundary lines. Emphasis should be placed on the fact that inequalities like part (h) are shaded to the left or to the right of the vertical line.

j–l: These exercises will help students to understand that $x = 0$ is the y-axis and $y = 0$ is the x-axis.

m–o: These exercises can serve as extension questions. For part (m), a curve separates the plane into two regions. In part (n), the solution is the entire coordinate plane except $(0, 0)$. In part (o), the solution is all points in quadrants 2 and 4, including both axes and the origin.

A half-plane is the graph of a solution set in the Cartesian coordinate plane of an inequality in two real number variables that is linear and strict.

4. Describe in words the half-plane that is the solution to each inequality.

a. $y \geq 0$

The half-plane lying above the x-axis and including the x-axis.

b. $x < -5$

The half plane to the left of the vertical line $x = -5$, not including the line $x = -5$.

c. $y \geq 2x - 5$

The line $y = 2x - 5$ and the half-plane lying above it.

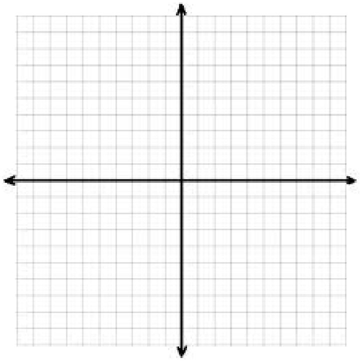
d. $y < 2x - 5$

The half-plane lying below the line $y = 2x - 5$.

5. **Graph the solution set to $x < -5$, reading it as an inequality in *one* variable, and describe the solution set in words. Then graph the solution set to $x < -5$ again, this time reading it as an inequality in *two* variables, and describe the solution set in words.**

Read in one variable: All real numbers less than -5. The graph will have an open circle at the endpoint -5 and extend as a ray to the left of -5 on the number line.

Read in two variables: All ordered pairs (x, y) such that x is less than -5. The graph will be a dashed vertical line through $x = -5$, and all points to the left of the line will be shaded.



Closing (2 minutes)

- Why is it useful to represent the solution to an inequality with two variables graphically?
- How does graphing the solution set of a one-variable inequality compare to graphing the solution set to a two-variable inequality?

Lesson Summary

An ordered pair is a solution to a two-variable inequality if, when each number is substituted into its corresponding variable, it makes the inequality a true number sentence.

Each ordered pair of numbers in the solution set of the inequality corresponds to a point on the coordinate plane. The set of all such points in the coordinate plane is called the graph of the inequality.

The graph of a linear inequality in the coordinate plane is called a half-plane.

Exit Ticket (3 minutes)

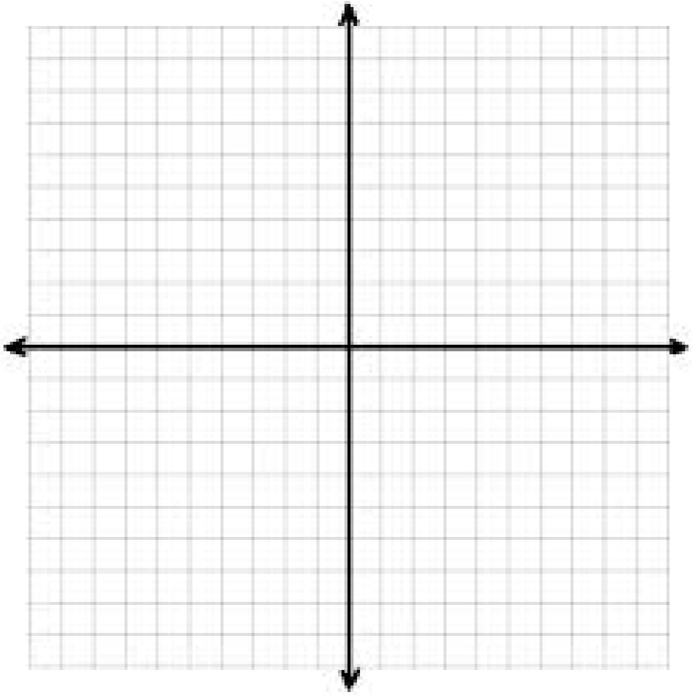
Name ____ Date____________________

Lesson 21: Solution Sets to Inequalities with Two Variables

Exit Ticket

What pairs of numbers satisfy the statement: The sum of two numbers is less than 10?

Create an inequality with two variables to represent this situation and graph the solution set.



Exit Ticket Sample Solution

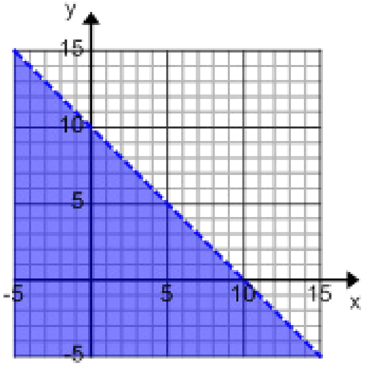
What pairs of numbers satisfy the statement: The sum of two numbers is less than 10?

Create an inequality with two variables to represent this situation and graph the solution set.

Let $x =$ one number, and let $y =$ a second number.

Inequality: $x + y < 10$

Graph the line $y = -x + 10$ using a dashed line and shade below the line.



Problem Set Sample Solutions

1. **Match each inequality with its graph. Explain your reasoning.**

   a. $2x - y > 6$

   Graph 2

   b. $y \leq 2x - 6$

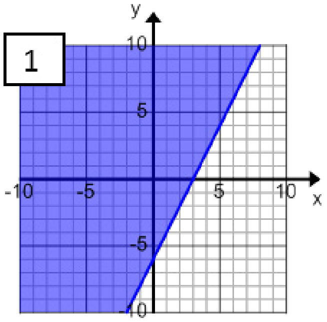
   Graph 4

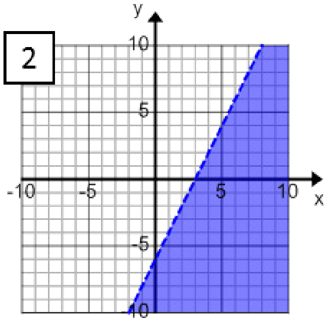
   c. $2x < y + 6$

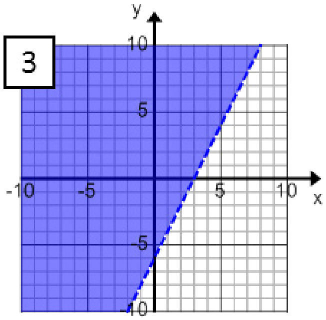
   Graph 3

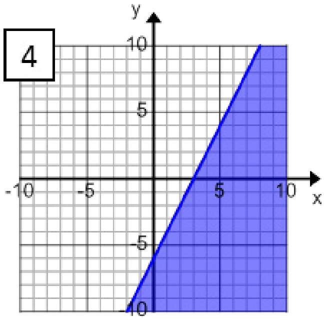
   d. $2x - 6 \leq y$

   Graph 1









Student explanations will vary. Sample response:

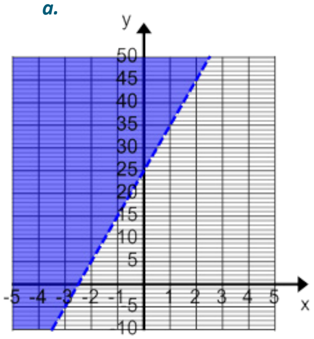
I re-arranged each equation and found that they were all the same except for the inequality symbol. The strict inequalities are the dashed lines, and the others are solid lines. When solved for y, you can decide the shading. Greater than is shaded above the line, and less than is shaded below the line.

2. **Graph the solution set in the coordinate plane. Support your answer by selecting two ordered pairs in the solution set and verifying that they make the inequality true.**

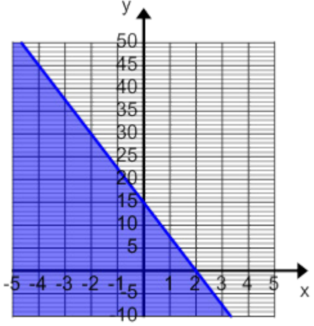
a.	$-10x + y > 25$	**b.**	$-6 \le y$	**c.**	$y \le -7.5x + 15$
d.	$2x - 8y \le 24$	**e.**	$3x < y$	**f.**	$2x > 0$

Solutions are graphed below for parts (a), (c), and (e).

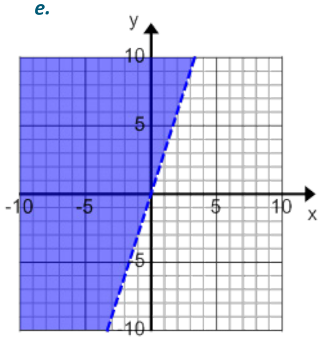
a.



c.



e.



3. **Marti sells tacos and burritos from a food truck at the farmers market. She sells burritos for $\$3.50$ each and tacos for $\$2.00$ each. She hopes to earn at least $\$120$ at the farmers market this Saturday.**

a. **Identify three combinations of tacos and burritos that will earn Marti more than $\$120$.**

Answers will vary. Answers to part (a) should be solutions to the inequality $3.5x + 2y > 120$.

b. **Identify three combinations of tacos and burritos that will earn Marti exactly $\$120$.**

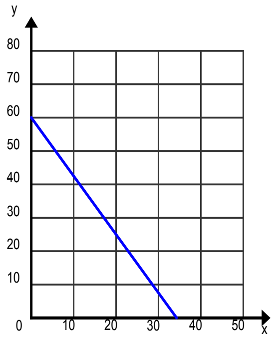
Answers will vary. Answers to part (b) should be solutions to the equation $3.5x + 2y = 120$.

c. **Identify three combinations of tacos and burritos that will *not* earn Marti at least $\$120$.**

Answers will vary. Answers to part (c) should not be solutions to the inequality or equation.

d. **Graph your answers to parts (a)–(c) in the coordinate plane and then shade a half-plane that contains all possible solutions to this problem.**

The graph shown for part (d) is shown to the right. Answers to part (a) should lie in the shaded half-plane. Answers to part (b) should lie on the line, and answers to part (c) should lie in the un-shaded half-plane.



e. **Create a linear inequality that represents the solution to this problem. Let x equal the number of burritos that Marti sells, and let y equal the number of tacos that Marti sells.**

$$3.5x + 2y \ge 120$$

f. **Are the points $(10, 49.5)$ a solution to inequality you created in part (e)? Explain your reasoning.**

The point would not be valid because it would not make sense in this situation to sell a fractional amount of tacos or burritos.

Lesson 22: Solution Sets to Simultaneous Equations

Student Outcomes

- Students identify solutions to simultaneous equations or inequalities; they solve systems of linear equations and inequalities either algebraically or graphically.

Classwork

Opening Exercise (8 minutes)

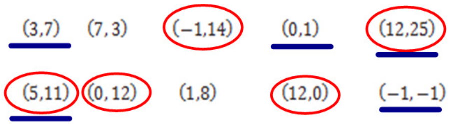
Allow students time to work on (a)–(d) individually. Then have students compare responses with a partner or share responses as a class.

Opening Exercise

Consider the following compound sentence: $x + y > 10$ and $y = 2x + 1$.

a. **Circle all the ordered pairs (x, y) that are solutions to the inequality $x + y > 10$.**

b. **Underline all the ordered pairs (x, y) that are solutions to the equation $y = 2x + 1$.**



c. **List the ordered pair(s) (x, y) from above that are solutions to the compound sentence $x + y > 10$ and $y = 2x + 1$.**

$(5, 11)$ *and* $(12, 25)$

d. **List three additional ordered pairs that are solutions to the compound sentence $x + y > 10$ and $y = 2x + 1$.**

$(4, 9)$, $(6, 13)$, *and* $(7, 15)$

Ask:

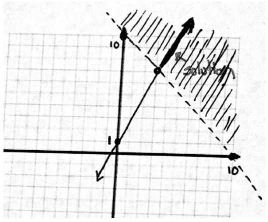
- How many possible answers are there to part (d)?
- Can anyone come up with a non-integer solution?

Discuss that just as they saw with compound equations in one variable, solving pairs of equations in two variables linked by AND is given by common solution points.

- How does the solution set change if the inequality is changed to $x + y \geq 10$?
  - *The point* $(3, 7)$ *would be added to the solution set.*

Have students complete (e) and (f) in pairs and discuss responses.

e. Sketch the solution set to the inequality $x + y > 10$ and the solution set to $y = 2x + 1$ on the same set of coordinate axes. Highlight the points that lie in BOTH solution sets.



f. Describe the solution set to $x + y > 10$ and $y = 2x + 1$.

All points that lie on the line $y = 2x + 1$ and above the line $y = -x + 10$.

- Which gives a more clear idea of the solution set: the graph or the verbal description?
  - *Answers could vary. The verbal description is pretty clear, but later in the lesson we will see systems with solution sets that would be difficult to describe adequately without a graph.*

Example 1 (7 minutes)

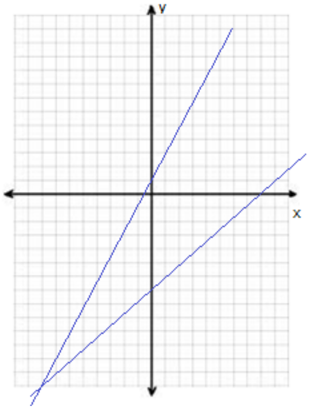
In Grade 8, students solved systems of linear equations both graphically and algebraically (using both substitution and elimination techniques), so this should primarily be a review. Work through the example as a class, introducing the notation shown as indicating a *system of equations*, wherein the two or more equations given are understood to be compound statements connected with an "and". Also convey that the word *simultaneous* from the title of the lesson is another way of saying that all equations must be true, simultaneously.

Example 1

Solve the following system of equations.

$$\begin{cases} y = 2x + 1 \\ x - y = 7 \end{cases}$$

Graphically:



Algebraically:

$$x - (2x + 1) = 7$$
$$x = -8$$

$$y = 2(-8) + 1$$
$$y = -15$$

Solution: $(-8, -15)$

Reinforce that even though the "and" is not stated explicitly, it is implied when given a system of equations. Problems written using this notation are asking one to find the solution(s) where $y = 2x + 1$ *and* $x - y = 17$. Work the problem using substitution. The elimination method is reviewed in the next lesson.

Exercise 1 (10 minutes)

Have students complete Exercise 1 individually.

Exercise 1

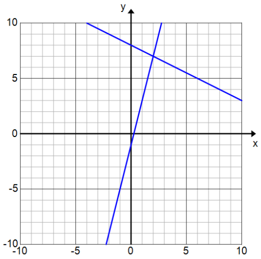
Solve each system first by graphing and then algebraically.

a. $\begin{cases} y = 4x - 1 \\ y = -\frac{1}{2}x + 8 \end{cases}$

$y = 4x - 1$

$y = -\frac{1}{2}x + 8$

$(2, 7)$

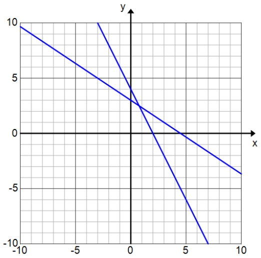


b. $\begin{cases} 2x + y = 4 \\ 2x + 3y = 9 \end{cases}$

$2x + y = 4$

$2x + 3y = 9$

$\left(\frac{3}{4}, \frac{5}{2}\right)$

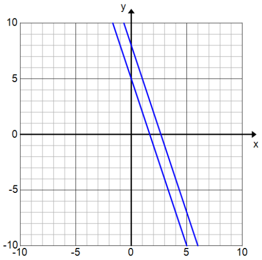


c. $\begin{cases} 3x + y = 5 \\ 3x + y = 8 \end{cases}$

$3x + y = 5$

$3x + y = 8$

No solution



Scaffolding:

In Algebra II, students will solve systems containing three unknowns. Challenge early finishers with this problem:

- If $x + y = 1$ and $y + z = 2$ and $x + z = 3$, find x, y, and z.
  - *Answer:* $(1,0,2)$.

As students finish, have them put both the graphical and algebraic approaches on the board for parts (a)–(c) or display student work using a document camera. Discuss as a class.

- Were you able to find the exact solution from the graph?
  - *Not for part (b).*
- Solving by graphing sometimes only yields an approximate solution.
- How can you tell when a system of equations will have no solution from the graph?
  - *The graphs do not intersect. For linear systems, this occurs when the lines have the same slope but have different y-intercepts, which means the lines will be parallel.*
- What if a system of linear equations had the same slope and the same y-intercept?
  - *There would be an infinite number of solutions (all points that lie on the line).*

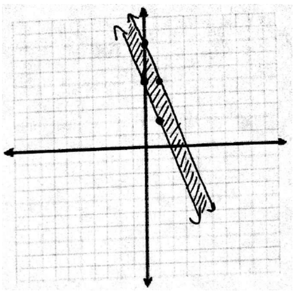
Example 2 (5 minutes)

Example 2

Now suppose the system of equations from Exercise 1(c) was instead a system of inequalities:

$$\begin{cases} 3x + y \geq 5 \\ 3x + y \leq 8 \end{cases}$$

Graph the solution set.



MP.7

- How did the solution set change from Exercise 1(c) to Example 2? What if we changed the problem to $3x + y \leq 5$ and $3x + y \geq 8$?
    - *There would be no solution.*

The solution to a system of inequalities is where their shaded regions intersect. Let this idea lead into Example 3.

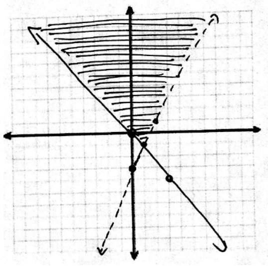
Example 3 (5 minutes)

Instruct students to graph and shade the solution set to each inequality in two different colored pencils. Give them a few minutes to complete this individually. Then discuss the solution to the system as a class.

Example 3

Graph the solution set to the system of inequalities.

$2x - y < 3$ and $4x + 3y \geq 0$



- Where does the solution to the system of inequalities lie?
    - *Where the shaded regions overlap.*
- What is true about all of the points in this region?
    - *These points are the only ones that satisfy both inequalities.*

Verify this by testing a couple of points from the shaded region and a couple of points that are not in the shaded region to confirm this idea to students.

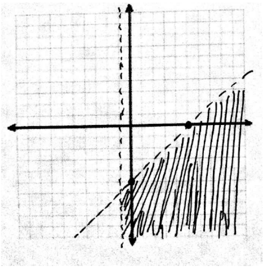
Exercise 2 (8 minutes)

Have students complete Exercise 2 individually and then compare their answers with a neighbor.

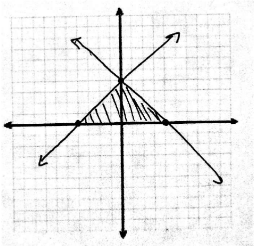
Exercise 2

Graph the solution set to each system of inequalities.

a. $\begin{cases} x - y > 5 \\ x > -1 \end{cases}$



b. $\begin{cases} y \leq x + 4 \\ y \leq 4 - x \\ y \geq 0 \end{cases}$



- Could you express the solution set of a system of inequalities without using a graph?
  - *Yes, using set notation, but a graph makes it easier to visualize and conceptualize which points are in the solution set.*
- How can you check your solution graph?
  - *Test a few points to confirm that the points in the shaded region satisfy all the inequalities.*

Closing (2 minutes)

- What are the different ways to solve a system of equations?
  - *Graphically, algebraically, or numerically (using a table).*
- Explain the limitations of solving a system of equations graphically.
  - *It is always subject to inaccuracies associated with reading graphs, so we are only able to approximate an intersection point.*
- Explain the limitations of expressing the solution to a system of inequalities without using a graph.
  - *It is difficult to describe the solution set without simply restating the problem in set notation, which is hard to visualize or conceptualize.*

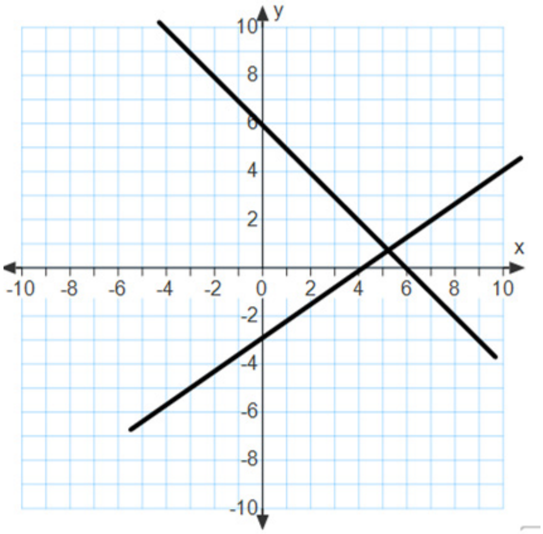
Exit Ticket (5 minutes)

Name ____ Date____________________

Lesson 22: Solution Sets to Simultaneous Equations

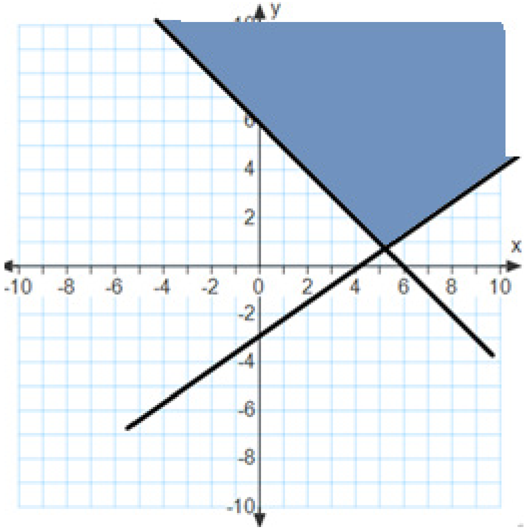
Exit Ticket

1. Estimate the solution to the system of equations whose graph is shown to the right.



2. Write the two equations for the system of equations and find the exact solution to the system algebraically.

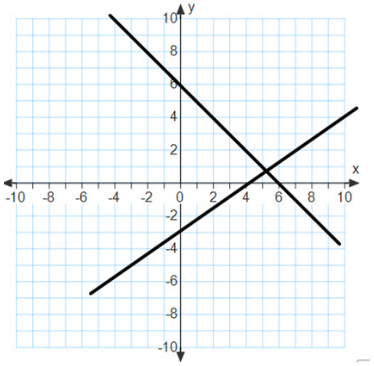
3. Write a system of inequalities that represents the shaded region on the graph shown to the right.



Exit Ticket Sample Solutions

1. **Estimate the solution to the system of equations whose graph is shown.**

   $(5.2, 0.9)$



2. **Write the two equations for the system of equations and find the exact solution to the system algebraically.**

$$\begin{cases} y = -x + 6 \\ y = \frac{3}{4}x - 3 \end{cases}$$

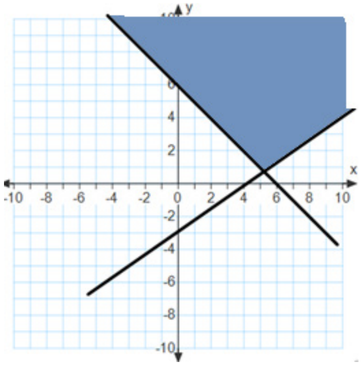
$$\frac{3}{4}x - 3 = -x + 6$$
$$\frac{7}{4}x = 9$$
$$x = \frac{36}{7}$$

$$y = -\frac{36}{7} + 6 = \frac{6}{7}$$

$$\left(\frac{36}{7}, \frac{6}{7}\right)$$

3. **Write a system of inequalities that represents the shaded region on the graph shown to the right.**

$$\begin{cases} y \geq -x + 6 \\ y \geq \frac{3}{4}x - 3 \end{cases}$$



Problem Set Sample Solutions

1. **Solve the following system of equations first by graphing and then algebraically.**

$$\begin{cases} 4x + y = -5 \\ x + 4y = 12 \end{cases}$$

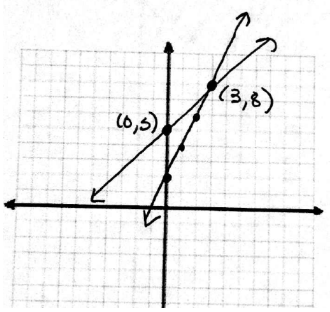
$$\left(\frac{-32}{15}, \frac{53}{15}\right)$$

2.

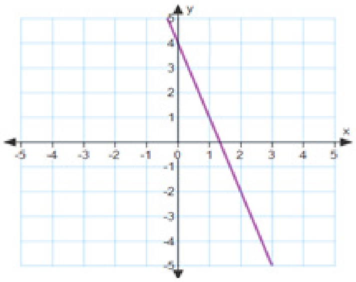
   a. **Without graphing, construct a system of two linear equations where $(0, 5)$ is a solution to the first equation but is not a solution to the second equation, and $(3, 8)$ is a solution to the system.**

   The first equation must be $y = x + 5$; the second equation could be any equation that is different from $y = x + 5$, and whose graph passes through $(3, 8)$; for example, $y = 2x + 2$ will work.

b. **Graph the system and label the graph to show that the system you created in part (a) satisfies the given conditions.**



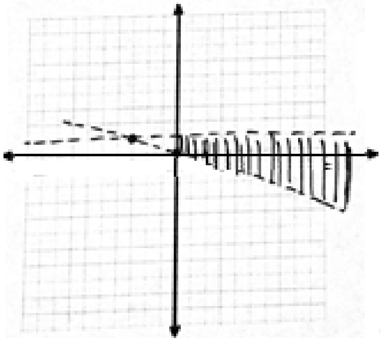
3. **Consider two linear equations. The graph of the first equation is shown. A table of values satisfying the second equation is given. What is the solution to the system of the two equations?**



x	-4	-2	0	2	4
y	-26	-18	-10	-2	6

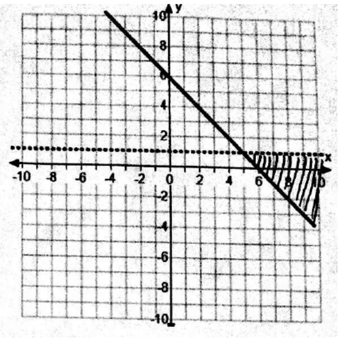
The form of the second equation can be determined exactly to be $y = 4x - 10$. Since the first equation is only given graphically, one can only estimate the solution graphically. The intersection of the two graphs appears to occur at $(2, -2)$. This may not be exactly right. Solving a system of equations graphically is always subject to inaccuracies associated with reading graphs.

4. **Graph the solution to the following system of inequalities:** $\begin{cases} x \geq 0 \\ y < 2 \\ x + 3y > 0 \end{cases}$



5. **Write a system of inequalities that represents the shaded region of the graph shown.**

$\begin{cases} y \geq -x + 6 \\ y < 1 \end{cases}$



6. **For each question below, provide an explanation or an example to support your claim.**

   a. **Is it possible to have a system of equations that has no solution?**

   Yes, for example, if the equations' graphs are parallel lines.

   b. **Is it possible to have a system of equations that has more than one solution?**

   Yes, for example, if the equations have the same graph, or in general, if the graphs intersect more than once.

   c. **Is it possible to have a system of inequalities that has no solution?**

   Yes, for example, if the solution sets of individual inequalities, represented by shaded regions on the coordinate plane, do not overlap.

Lesson 23: Solution Sets to Simultaneous Equations

Student Outcomes

- Students create systems of equations that have the same solution set as a given system.
- Students understand that adding a multiple of one equation to another creates a new system of two linear equations with the same solution set as the original system. This property provides a justification for a method to solve a system of two linear equations algebraically.

Lesson Notes

Students explore standard **A.REI.C.5** in great detail. They have already developed proficiency with solving a system of two linear equations. This lesson delves into why the *elimination method* works and further enhances student understanding of equivalence.

Classwork

Opening Exercise (3 minutes)

This should go very quickly. Expect students to substitute 3 for x and 4 for y into both equations. If students struggle with this piece, you may need to reinforce what it means when an ordered pair is a solution to simultaneous equations and continue to reinforce that notion throughout the lesson.

Opening Exercise

Here is a system of two linear equations. Verify that the solution to this system is $(3, 4)$.

Equation A1: $y = x + 1$

Equation A2: $y = -2x + 10$

Substitute 3 for x and 4 for y into both equations.

$4 = 3 + 1$ ***is a true equation.***

$4 = -2(3) + 10$ ***is a true equation.***

Exploratory Challenge (23 minutes)

Students should work in groups to complete parts (a)–(e) for about 5–7 minutes. Have one or more groups share their solutions with the class. These first questions get students thinking about ways to create a new system of equations with the same solution set as the Opening Exercise. Expect a variety of responses from groups as they create their new systems.

Scaffolding:

If groups are struggling to get started, remind them that $(3, 4)$ must be a solution to their new equations. Then ask them to consider how to make a line that includes that point. Encourage them to use the grid or work with graph paper.

Exploratory Challenge

a. **Write down another system of two linear equations whose solution is $(3, 4)$. This time make sure both linear equations have a positive slope.**

Equation B1: $y = x + 1$

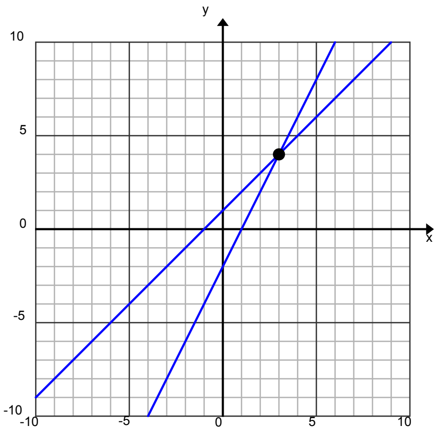
Equation B2: $y = 2(x - 3) + 4$

b. **Verify that the solution to this system of two linear equations is $(3, 4)$.**

$4 = 3 + 1$ *and* $4 = 2(3 - 3) + 4$

$4 = 4$ $4 = 4$

c. **Graph equation B1 and B2.**



d. **Are either B1 or B2 equivalent to the original A1 or A2? Explain your reasoning.**

Yes, I used the same B1. B2 is a different equation because it has a different slope. The only thing all the equations have in common is the point $(3, 4)$.

e. **Add A1 and A2 to create a new equation C1. Then, multiply A1 by 3 to create a new equation C2. Why is the solution to this system also $(3, 4)$? Explain your reasoning.**

Equation C1: $2y = -x + 11$

Equation C2: $3y = 3x + 3$

If you substitute 3 for x and 4 for y, both equations are true, so $(3, 4)$ is a solution. When A1 was multiplied by 3, it did not create a new equation. Both equations had $(3, 4)$ as a solution; therefore, when we add the equations, $(3, 4)$ will still be a solution because of the addition property of equality.

Hold a class discussion before moving on to parts (f)–(i). When you debrief and discuss, be sure to highlight the different approaches. Begin to distinguish between solutions where students created a new system by simply multiplying one equation or both by a constant factor and those who create two new equations that both contain the point $(3, 4)$.

- What different approaches did groups use to solve this problem?
  - *I used guess and check.*
  - *I started with the point $(3, 4)$ and realized I could pick any slope I wanted. I moved left 3 and down 4 until I got to the $(0, 0)$; therefore, my equation would be $y = \frac{4}{3}x$.*
- When you multiplied one equation by a constant, did it actually create a different linear equation? When you added two equations together, did it actually create a different equation?
  - *Multiplying by a constant doesn't create a different equation because the slope and y-intercept are the same. Adding two equations together does create a new equation because the slope is different.*

Move on to parts (f)–(i). These questions specifically direct students to consider creating a new system by multiplying one equation by a constant and adding it to another. Students are considering whether or not this is a valid way to generate a system with the same solution. Have each group record their answer to part (i) on the board to show that this method works regardless of the number by which you multiply.

The following system of equations was obtained from the original system by adding a multiple of equation A2 to equation A1.

Equation D1: $y = x + 1$

Equation D2: $3y = -3x + 21$

f. **What multiple of A2 was added to A1 to create D2?**

A2 was multiplied by 2 *and then added to A1.*

g. **What is the solution to the system of two linear equations formed by D1 and D2?**

The solution is still $(3, 4)$*. I checked by substituting* $(3, 4)$ *into both equations.*

h. **Is D2 equivalent to the original A1 or A2? Explain your reasoning.**

No, the slope of D2 is -1*. Neither of the original equations had that slope.*

i. **Start with equation A1. Multiply it by a number of your choice and add the result to equation A2. This creates a new equation E2. Record E2 below to check if the solution is** $(3, 4)$**.**

Equation E1: $y = x + 1$

Equation E2: $5y = 2x + 14$

I multiplied A1 by 4 *to get* $4y = 4x + 4$*. Adding it to A2 gives* $5y = 2x + 14$*. We already know* $(3, 4)$ *is a solution to* $y = x + 1$*. Substituting into E2 gives* $5(4) = 2(3) + 14$*, which is a true equation. Therefore,* $(3, 4)$ *is a solution to this new system.*

Wrap up the discussion by emphasizing the following:

- Will this method of creating a new system work every time? Why does it work?
  - *This method will always work because multiplying by a constant is a property of equality that keeps the point of intersection (i.e., the solution set) the same.*
- Who said you can add two equations like that? Left-hand side to left-hand side; right-hand side to right-hand side? How do you know that the solutions are not changed by that move?

Example 1 (4 minutes): Why Does the Elimination Method Work?

Students will see how **A.REI.C.5** provides a justification for solving a system by elimination. Choose a multiple that will eliminate a variable when the two equations are added together. Be sure to emphasize that this process generates a new system of two linear equations where one of the two equations contains only a single variable and is thus easy to solve.

Example 1: Why Does the Elimination Method Work?

Solve this system of linear equations algebraically.

ORIGINAL SYSTEM	NEW SYSTEM	SOLUTION
$2x + y = 6$		
$x - 3y = -11$		

ORIGINAL SYSTEM	*NEW SYSTEM*		*SOLUTION*
$2x + y = 6$	$6x + 3y = 18$		$x = 1$
$x - 3y = -11$	$x - 3y = -11$	$7x = 7$	$2(1) + y = 6$ *so* $y = 4$

Multiply the first equation by 3 *and add it to the second. Solve the new system.* $(1, 4)$

- Why did I multiply by the number 3?
  - *Multiplying by* 3 *allows one to generate* $3y$ *to eliminate the* $-3y$ *in the other equation when both equations are added together; it leads to an equation in* x *only. Selecting this number strategically created a new system where one equation had only a single variable.*
- Could I have selected a different number and created a system that was easy to solve?
  - *Yes, you could have multiplied the second equation by* -2 *to create a system that eliminated* x*.*

Exercises 1–2 (8 minutes)

Both of these exercises mimic the example. Students should be able to work quickly through them since they learned to solve systems by elimination in Grade 8.

Exercises 1–2

1. Explain a way to create a new system of equations with the same solution as the original that eliminates variable y from one equation. Then determine the solution.

ORIGINAL SYSTEM	NEW SYSTEM	SOLUTION
$2x + 3y = 7$		
$x - y = 1$		

Multiply the second equation by 3*, and add it to the first one.*

ORIGINAL SYSTEM	*NEW SYSTEM*	*SOLUTION*
$2x + 3y = 7$	$2x + 3y = 7$	$x = 2$
$x - y = 1$	$\underline{+(3x - 3y = 3)}$	$2(2) + 3y = 7$*, so* $y = 1$
	$5x = 10$	$(2, 1)$

2. **Explain a way to create a new system of equations with the same solution as the original that eliminates variable x from one equation. Then determine the solution.**

ORIGINAL SYSTEM	NEW SYSTEM	SOLUTION
$2x + 3y = 7$		
$x - y = 1$		

Multiply the second equation by -2, and add it to the first one.

ORIGINAL SYSTEM	*NEW SYSTEM*	*SOLUTION*
$2x + 3y = 7$	$2x + 3y = 7$	$y = 1$
$x - y = 1$	$\underline{+(-2x + 2y = -2)}$	$2x + 3(1) = 7$, *so* $x = 2$
	$5y = 5$	$(2, 1)$

Closing (2 minutes)

Close with a reminder that this lesson was about proving that a technique to solve a system of equations is valid.

- There are many ways to generate systems of equations that have the same solution set, but the technique explored in Exercises 1 and 2 is especially helpful if you are trying to solve a system algebraically.

Exit Ticket (5 minutes)

Name ____ Date____________________

Lesson 23: Solution Sets to Simultaneous Equations

Exit Ticket

The sum of two numbers is 10 and the difference is 6. What are the numbers?

1. Create a system of two linear equations to represent this problem.

2. What is the solution to the system?

3. Create a new system of two linear equations using the methods described in part (i) of the Exploratory Challenge. Verify that the new system has the same solution.

Exit Ticket Sample Solutions

The sum of two numbers is 10 and the difference is 6. What are the numbers?

1. Create a system of two linear equations to represent this problem.

$x+y=10$ and $x-y=6$

2. What is the solution to the system?

$x=8$ and $y=2$

3. Create a new system of two linear equations using the methods described in part (i) of the Exploratory Challenge. Verify that the new system has the same solution.

$$4(x\ +\ y\ =\ 10) \rightarrow \qquad \begin{array}{r} 4x\ +\ 4y\ =\ 40 \\ +\quad (x-y\ =\ 6) \\ \hline 5x\ +\ 3y\ =\ 46 \end{array}$$

Solution to $x+y=10$ and $5x+3y=46$ is still $(8,2)$

$8+2=10$ $\qquad$ $5(8)+3(2)=46$

Problem Set Sample Solutions

Try to answer the following without solving for x and y first:

1. If $3x+2y=6$ and $x+y=4$, then

a. $2x+y=?$ $\qquad$ **b.** $4x+3y=?$

a. $2x+y=2$ $\qquad$ *b.* $4x+3y=10$

Answers in parts (a) and (b) are obtained by adding and subtracting the two original equations WITHOUT actually solving for x and y first.

The solution $(-2,6)$ satisfies all four equations.

2. You always get the same solution no matter which two of the four equations you choose from Problem 1 to form a system of two linear equations. Explain why this is true.

The reason is that the 3rd equation is the difference of the 1st and the 2nd; the 4th equation is the sum of the 1st and the 2nd. When we add (or subtract) two equations to create a new equation, no new (or independent) information is created. The 3rd and 4th equations are thus not independent of the 1st and the 2nd. They still contain the solution common to their parent equations, the 1st and 2nd.

3. Solve the system of equations $\begin{cases} y=\frac{1}{4}x \\ y=-x+5 \end{cases}$ by graphing. Then, create a new system of equations that has the same solution. Show either algebraically or graphically that the systems have the same solution.

Solution is $(4,1)$.

One example of a second system: $\begin{cases} y=\frac{2}{3}x-\frac{5}{3} \\ y=4x-15 \end{cases}$

4. Without solving the systems, explain why the following systems must have the same solution.

System (i): $4x - 5y = 13$ **System (ii):** $8x - 10y = 26$

$3x + 6y = 11$ $x - 11y = 2$

The first equation in system (ii) is created by multiplying the first equation in system (i) by 2. *The second equation in system (ii) is creating by subtracting the two equations from system (i). Neither of these actions will change the solution to the system. Multiplying and adding equations are properties of equality that keep the point(s) of intersection (which is the solution set) the same.*

Solve each system of equations by writing a new system that eliminates one of the variables.

5. $2x + y = 25$
$4x + 3y = 9$

$$\begin{cases} -4x - 2y = -50 \\ 4x + 3y = 9 \end{cases}$$

$y = -41$

$(33, -41)$

6. $3x + 2y = 4$
$4x + 7y = 1$

$$\begin{cases} 12x + 8y = 16 \\ -12x - 21y = -3 \end{cases}$$

$y = -1$

$(2, -1)$

Lesson 24: Applications of Systems of Equations and Inequalities

Student Outcomes

- Students use systems of equations or inequalities to solve contextual problems and interpret solutions within a particular context.

Lesson Notes

This lesson introduces students to the idea of using systems to solve various application problems in order to prepare them for more extensive modeling tasks that they encounter in Topic D.

Classwork

Opening Exercise (8 minutes)

MP.3 Have students brainstorm this problem in groups. Allow groups to share different approaches to solving the problem (*i.e.*, guess and check, making a table, or algebraically). Encourage students to critique the various approaches.

What were the advantages or disadvantages to the various approaches? Lead students through the algebraic approach. Then, discuss the following:

Opening Exercise

In Lewis Carroll's *Through the Looking Glass,* Tweedledum says, "The sum of your weight and twice mine is 361 pounds." Tweedledee replies, "The sum of your weight and twice mine is 362 pounds." Find both of their weights.

Let x = the number of pounds Tweedledee weighs, and let y = the number of pounds Tweedledum weighs.

$x + 2y = 361$

$y + 2x = 362$

Tweedledum weighs 120 pounds, and Tweedledee weighs 121 pounds.

Discussion (5 minutes)

- Could we solve the problem above using only Tweedledum's sentence?
  - *No. There are two unknowns, and as we saw in earlier lessons, the equation $x + 2y = 361$ has an infinite number of solutions.*
- In a situation where there are two unknowns, how many equations do we need to write in order to solve the system?
  - *Two equations.*

- If I told you I was holding 20 coins that were some mix of dimes and quarters, could you tell me anything about how many of each I have?
  - *You could only list possible combinations (1 dime and 19 quarters, 2 dimes and 18 quarters, etc.).*
- What other piece of information would be useful in determining how many of each type of coin I was holding?
  - *The total amount of money, how many more or fewer quarters than dimes, etc.*

Example 1 (5 minutes)

Let the discussion lead into Example 1. Work through the example as a class. Make sure students specify the variables being used in the equations. Discuss the various ways of solving (i.e., graphically, making a table, algebraically). In the previous lesson, we did not solve systems by making a table. Demonstrate how this might be a useful technique in the following situation.

Example 1

Lulu tells her little brother, Jack, that she is holding 20 coins all of which are dimes and quarters. They have a value of $\$4.10$. She says she will give him the coins if he can tell her how many of each she is holding. Solve this problem for Jack.

Let d = the number of dimes, and let q = the number of quarters.

$d + q = 20$

$0.10d + 0.25q = 4.10$

Lulu is holding 6 dimes and 14 quarters.

Scaffolding:

Allow students to decide how they want to solve the problem. However, encourage them to look at other approaches. Discuss the limitations of the various methods.

Exploratory Challenge (20 minutes)

Have students work in groups on part (a). Then, discuss responses as a class.

MP.6 Emphasize that regardless of how each group chooses to solve the problem, every group should specify the variables and clearly label the graph.

MP.1 Then have students work in groups on part (b). Guide them through the problem as needed. This is a fairly complicated problem, so students may need assistance. Encourage the students to persevere and to break up the problem into manageable pieces.

Exploratory Challenge

a. **At a state fair, there is a game where you throw a ball at a pyramid of cans. If you knock over all of the cans, you win a prize. The cost is 3 throws for $\$1$, but if have you an armband, you get 6 throws for $\$1$. The armband costs $\$10$.**

   i. **Write two cost equations for the game in terms of the number of throws purchased, one without an armband and one with.**

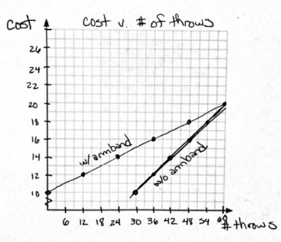
   Let x = number of throws, and let C = cost.

   Without armband: $C = \frac{1}{3}x$

   With armband: $C = \frac{1}{6}x + 10$

ii. **Graph the two cost equations on the same graph. Be sure to label the axes and show an appropriate scale.**

See graph at right.



iii. **Does it make sense to buy the armband?**

Only if you want 60 *or more throws*

Point out the constraints of x. Without the armband, x must be a multiple of 3; with the armband, x must be a multiple of 6.

Remind students about discrete and continuous graphs. The graphs of each equation should actually be discrete rather than continuous. Discuss why other points on the graph would not make sense for this scenario.

b. **A clothing manufacturer has $1,000$ yd. of cotton to make shirts and pajamas. A shirt requires 1 yd. of fabric, and a pair of pajamas requires 2 yd. of fabric. It takes 2 hr. to make a shirt and 3 hr. to make the pajamas, and there are $1,600$ hr. available to make the clothing.**

i. **What are the variables?**

Number of shirts made and number of pajamas made.

ii. **What are the constraints?**

How much time the manufacturer has and how much material is available.

iii. **Write inequalities for the constraints.**

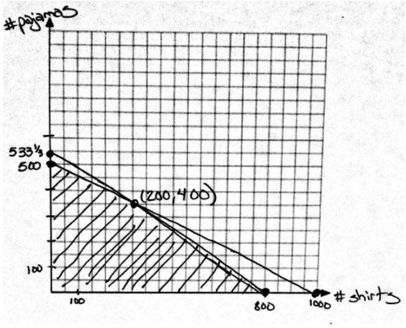
Let $x =$ *number of shirts, and let* $y =$ *number of pajamas.*

$x \geq 0$ *and* $y \geq 0$

$x + 2y \leq 1000$

$2x + 3y \leq 1600$

iv. **Graph the inequalities and shade the solution set.**



v. **What does the shaded region represent?**

The various combinations of shirts and pajamas that it would be possible for the manufacturer to make.

Scaffolding:

- Students may need help graphing the system. Point out that it is easier to find the x- and y-intercepts in this problem than to rearrange the inequality.
- This branch of mathematics is called linear programming. Have early finishers research it.

vi. **Suppose the manufacturer makes a profit of $\$10$ on shirts and $\$18$ on pajamas. How would it decide how many of each to make?**

The manufacturer wants to make as many as possible, so the maximum should be at one of the endpoints of the shaded region.

vii. **How many of each should the manufacturer make, assuming he will sell all the shirts and pajamas he makes?**

Profit $= 10x + 18y$

Possible points	*Profit*
$(0, 500)$	$\$9,000$
$(200, 400)$	$\$9,200$
$(800, 0)$	$\$8,000$

He should make 200 *shirts and* 400 *pairs of pajamas for a maximum profit.*

- Why does this scenario call for inequalities rather than equations?
  - *He cannot exceed the amount of time or material available but does not necessarily have to use all of it.*
- The shaded region in a problem of this type is sometimes called the *feasible region*. Why does this name make sense?
  - *This is the region that represents the number of shirts and pajamas that he can feasibly make given the constraints.*

Students should intuitively believe that the maximum profit should be at one of the endpoints of the shaded region (which is true because he is maximizing the given resources). However, you can have students test other points to prove that intersection point is, in fact, the maximum.

Closing (3 minutes)

Recap the steps followed in solving these problems. Do not have students copy the steps, just discuss the strategy, both specifically for this problem and then making generic descriptions (e.g.*,* identified the variables, created equations or inequalities based on the constraints of the problem, decided on the best method for solving, interpreted the solution).

Exit Ticket (4 minutes)

Name ____ Date____________________

Lesson 24: Applications of Systems of Equations and Inequalities

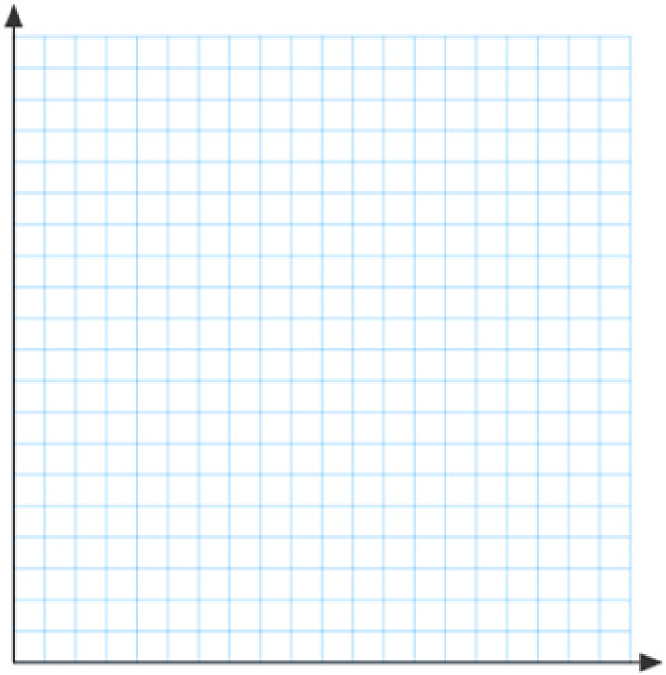
Exit Ticket

Andy's Cab Service charges a $\$6$ fee plus $\$0.50$ per mile. His twin brother Randy starts a rival business where he charges $\$0.80$ per mile but does not charge a fee.

1. Write a cost equation for each cab service in terms of the number of miles.

2. Graph both cost equations.

3. For what trip distances should a customer use Andy's Cab Service? For what trip distances should a customer use Randy's Cab Service? Justify your answer algebraically, and show the location of the solution on the graph.



Exit Ticket Sample Solutions

Andy's Cab Service charges a $\$6$ fee plus $\$0.50$ per mile. His twin brother Randy starts a rival business where he charges $\$0.80$ per mile but does not charge a fee.

1. **Write a cost equation for each cab service in terms of the number of miles.**

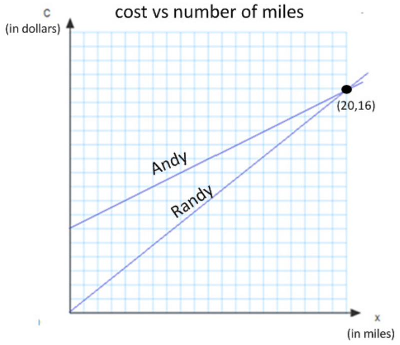
   Let $x =$ number of miles, and let $C =$ cost.

   Andy's: $C = 0.5x + 6$

   Randy's: $C = 0.8x$

2. **Graph both cost equations.**

   See graph.



3. **For what trip distances should a customer use Andy's Cab Service? For what trip distances should a customer use Randy's Cab Service? Justify your answer algebraically, and show the location of the solution on the graph.**

   $$0.5x + 6 = 0.8x$$
   $$x = 20$$

   If the trip is less than 20 miles, use Randy's. If the trip is more than 20 miles use Andy's. If the trip is exactly 20 miles, either choice will result in the same cost.

Problem Set Sample Solutions

1. **Find two numbers such that the sum of the first and three times the second is 5 and the sum of second and two times the first is 8.**

   The two numbers are $\frac{19}{5}$ *and* $\frac{2}{5}$.

2. **A chemist has two solutions: a 50% methane solution and an 80% methane solution. He wants 100 ml of a 70% methane solution. How many ml of each solution does he need to mix?**

   The chemist should use $33\frac{1}{3}$ *mL of the* 50% *solution and* $66\frac{2}{3}$ *mL of the* 80% *solution.*

3. **Pam has two part time jobs. At one job, she works as a cashier and makes $\$8$ per hour. At the second job, she works as a tutor and makes $\$12$ per hour. One week she worked 30 hours and made $\$268$. How many hours did she spend at each job?**

   She worked at the cashier job for 23 hours and tutored for 7 hours.

4. **A store sells Brazilian coffee for $\$10$ per lb. and Columbian coffee for $\$14$ per lb. If the store decides to make a 150-lb. blend of the two and sell it for $\$11$ per lb., how much of each type of coffee should be used?**

   They should use $112\frac{1}{2}$ *lb. of Brazilian coffee and* $37\frac{1}{2}$ *lb. of Columbian coffee.*

5. A potter is making cups and plates. It takes her 6 min. to make a cup and 3 min. to make a plate. Each cup uses $\frac{3}{4}$ lb. of clay, and each plate uses 1 lb. of clay. She has 20 hr. available to make the cups and plates and has 250 lb. of clay.

   a. What are the variables?

      c = # of cups made

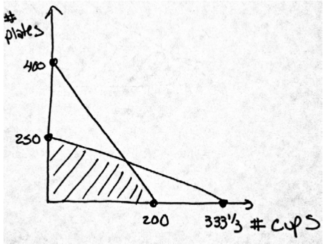
      p = # of plates made

   b. Write inequalities for the constraints.

      $c \geq 0$ *and* $p \geq 0$ *and* $\frac{1}{10}c + \frac{1}{20}p \leq 20$ *and* $\frac{3}{4}c + p \leq 250$

   c. Graph and shade the solution set.

      See graph at right.



   d. If she makes a profit of $\$2$ on each cup and $\$1.50$ on each plate, how many of each should she make in order to maximize her profit?

      120 cups and 160 plates

   e. What is her maximum profit?

      $\$480$

Topic D:

Creating Equations to Solve Problems

N-Q.A.1, A-SSE.A.1, A-CED.A.1, A-CED.A.2, A-REI.B.3

Focus Standard:	N-Q.A.1	Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays.
	A-SSE.A.1	Interpret expressions that represent a quantity in terms of its context.★
	A-SSE.A.1a	Interpret parts of an expression, such as terms, factors, and coefficients.
	A-SSE.A.1b	Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret $P(1+r)^n$ as the product of P and a factor not depending on P.*
	A-CED.A.1	Create equations and inequalities in one variable and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.*
	A-CED.A.2	Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.
	A-REI.B.3	Solve linear equations and inequalities in one variable, including equations with coefficients represented by letters.
Instructional Days:	4	
Lesson 25:	Solving Problems in Two Ways—Rules and Algebra (M)[1]	
Lessons 26–27:	Recursive Challenge Problem—The Double and Add 5 Game (M, M)	
Lesson 28:	Federal Income Tax (M)	

In this topic, students are introduced to the modeling cycle (see page 61 of the Common Core Learning Standards) through problems that can be solved using equations and inequalities in one variable, systems of equations, and graphing. From the CCLS (page 61):

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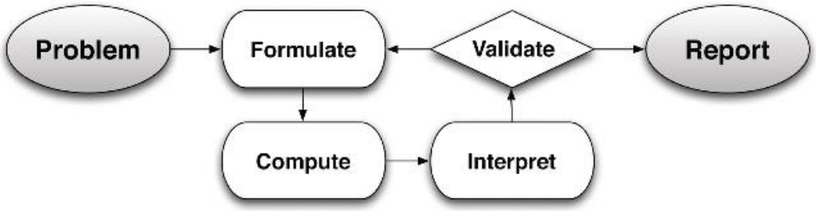
[1] Lesson Structure Key: **P**-Problem Set Lesson, **M**-Modeling Cycle Lesson, **E**-Exploration Lesson, **S**-Socratic Lesson

> Modeling links classroom mathematics and statistics to everyday life, work, and decision-making.
>
> 
>
>
> The basic modeling cycle is summarized in the diagram. It involves (1) identifying variables in the situation and selecting those that represent essential features; (2) formulating a model by creating and selecting geometric, graphical, tabular, algebraic, or statistical representations that describe relationships between the variables; (3) analyzing and performing operations on these relationships to draw conclusions; (4) interpreting the results of the mathematics in terms of the original situation; (5) validating the conclusions by comparing them with the situation and then either improving the model; (6) or if it is acceptable, reporting on the conclusions and the reasoning behind them. Choices, assumptions, and approximations are present throughout this cycle.

The first lesson introduces parts of the modeling cycle using problems and situations that students have encountered before: creating linear equations, tape diagrams, rates, systems of linear equations, graphs of systems, etc.

The next lesson, *The Double and Add 5 Game*, employs the modeling cycle in a mathematical context. In this 2-day lesson, students formulate a model and build an equation to represent the model (in this case, converting a sequence defined recursively to an explicit formula). After they play the game in a specific case, “double and add 5,” they have to interpret the results of the mathematics in terms of the original model and validate whether their model is acceptable. Then they use the model to analyze and report on a problem that is too difficult to do “by hand” without the model.

Finally, Lesson 28 serves as a signature lesson on modeling as students take on the very real-life example of understanding federal marginal income tax rates (i.e., the progressive income tax brackets). Students are provided the current standard deduction tables per dependent or marital status and the marginal income tax table per marital filing status. For a specific household situation (e.g., married filing jointly with two dependents), students determine equations for the total Federal Income Tax for different income intervals, graph the piecewise-defined equations, and answer specific questions about the total effective rate for different income levels. All elements of the modeling cycle occur as students analyze the information to find, for example, roughly how much their favorite famous performer paid in federal taxes last year

Lesson 25: Solving Problems in Two Ways—Rates and Algebra

Student Outcomes

- Students investigate a problem that can be solved by reasoning quantitatively and by creating equations in one variable.
- Students compare the numerical approach to the algebraic approach.

Classwork

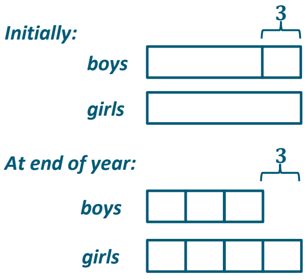
Exercise 1 (10 minutes)

Exercise 1

MP.2

a. **Solve the following problem first using a tape diagram and then using an equation: In a school choir, $\frac{1}{2}$ of the members were girls. At the end of the year, 3 boys left the choir, and the ratio of boys to girls became 3:4. How many boys remained in the choir?**

Using a tape diagram:



1 unit $= 3$ boys

3 units $= 9$ boys

There are 9 boys in the choir.

Using an equation:

# of boys initially: b

# of girls initially: b

# of boys at the end of the year: $b-3$

Ratio of boys to girls at the end of the year: $b-3:b=3:4$

Therefore,

$$\frac{b-3}{b}=\frac{3}{4}$$

$$b\cdot\left(\frac{b-3}{b}\right)=\left(\frac{3}{4}\right)\cdot b$$

$$b-3=\frac{3}{4}b$$

$$b-3-\frac{3}{4}b=\frac{3}{4}b-\frac{3}{4}b$$

$$\frac{1}{4}b-3=0$$

$$\frac{1}{4}b-3+3=0+3$$

$$4\left(\frac{1}{4}b\right)=(3)4$$

$$b=12$$

The number of boys remaining in the choir is $12-3$, or 9.

b. **Which problem solution, the one using a tape diagram or the one using an equation, was easier to set up and solve? Why?**

Answers will vary. Most should say the tape diagram since it requires significantly less work than solving the problem algebraically. Point out that this may not always be the case.

- Modify the question so that the tape diagram solution would definitely not be the easier way to solve the problem.
- Sample question: In a school choir, $\frac{6}{11}$ of the members were girls. At the end of the year, 3 boys had left the choir, and the ratio of the number of boys to the number of girls was $3:4$. How many boys remained in the choir?

Mathematical Modeling Exercise/Exercise 2 (27 minutes)

MP.1

The following problem is nontrivial for students. Please study the different solution types carefully before working the problem with your students. Always let students work (individually or in groups of two) on each solution type for 5–10 minutes before summarizing with the entire class. While they are working, walk around and help, looking for students or groups who could present a correct solution (or could present a solution with your guidance). If no one solves the problem, that's okay. They should now better understand the challenges the problem presents as they work through it with you.

Exercise 2

Read the following problem:

All the printing presses at a print shop were scheduled to make copies of a novel and a cookbook. They were to print the same number of copies of each book, but the novel had twice as many pages as the cookbook. All of the printing presses worked for the first day on the larger book, turning out novels. Then, on day two, the presses were split into two equally sized groups. The first group continued printing copies of the novel and finished printing all the copies by the evening of the second day. The second group worked on the cookbook but did not finish by evening. One printing press, working for two additional full days, finished printing the remaining copies of the cookbooks. If all printing presses printed pages (for both the novel and cookbook) at the same constant rate, how many printing presses are there at the print shop?

Analyze the problem with your students, rereading the problem together as you answer questions like:

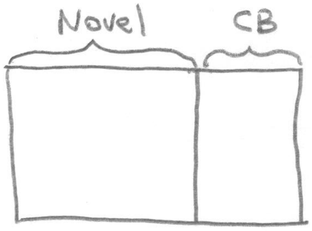
- What is this story about?
    - *Printing novels and cookbooks*
- Will more novels be printed than cookbooks or less?
    - *Neither. The same number of each will be printed.*
- How many pages does the novel have compared to the cookbook?
    - *Twice as many as the cookbook*
- How much longer would one printing press take to print a novel versus a cookbook?
    - *Twice as long*
- Is it important to know how many pages each book has or how many of each book will need to be printed? Why?
    - *No. Answers will vary.*
- How many of the printing presses are used to print the novel on the first day?
    - *All of them.*
- How many of the printing presses are used to print the novel on the second day?
    - *One half of them.*

- How many of the printing presses are used to print the cookbook on the second day?
  - *One half of them.*
- How many printing presses are used to print the remainder of the cookbooks on the third and fourth days?
  - *One.*
- How many printing presses could be used to print the remainder of the cookbooks in one day instead of two?
  - *Two printing presses could have finished the cookbooks on the third day.*
- What are we trying to find in this problem?
  - *The number of printing presses at the print shop.*

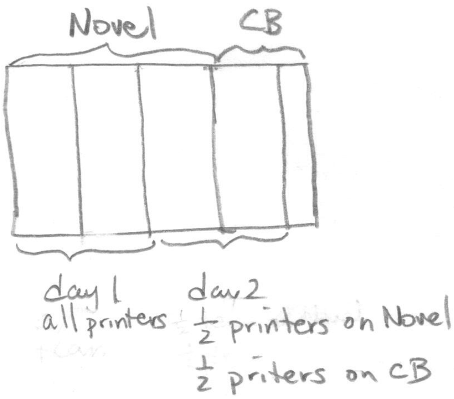
a. Solve the problem working with rates to setup a tape diagram or an area model.

One of the keys to a (somewhat) simple solution using an area model is recognizing that the novel has twice as many pages as the cookbook, so the area that represents the novel job should be twice the size as the area used to represent the cookbook job.

Draw a rectangular region to represent the work needed to complete the novel job and another rectangular region, half the size, to represent the work needed to complete the cookbook job.

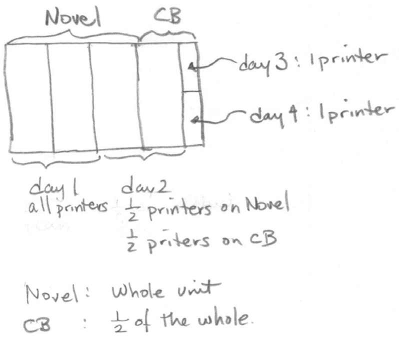


On the first day, all the printing presses were used on printing copies of the novel, and on the second day, half completed the novel while half worked on the cookbook.

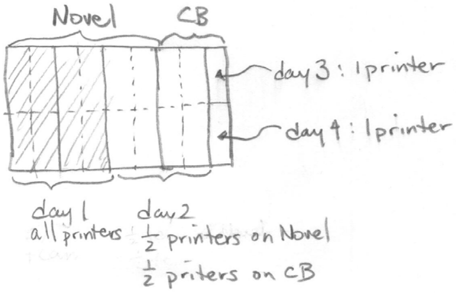


Students can test conjectures on why the novel job is split into thirds. For example, if all printing presses completed $\frac{1}{2}$ of the novel job on day 1, then $\frac{1}{2}$ of the printing presses would only complete $\frac{1}{4}$ of the job on day 2. Therefore, only $\frac{3}{4}$ of the novel job would be complete by the end of day two. But the story says the whole novel job is complete by the end of day two (i.e., something is wrong with that conjecture).

Once students understand that the novel job must be split into thirds, label the novel job, the "whole unit," and each third unit as the fractional unit, "$\frac{1}{3}$ of the whole." The cookbook job is then "$\frac{1}{2}$ of the whole." The same $\frac{1}{3}$ unit must have also been completed on the cookbook job on day 2, leaving $\frac{1}{2} - \frac{1}{3} = \frac{1}{6}$ of the whole unit left for days 3 and 4.



Each of the small rectangles in the picture above represents how much of the whole job (i.e., novel job) one printing press can do in one day. Drawing in the small rectangle units into the rest of the diagram, one sees that eight printing presses worked on the novel job on day 1, which was all of the printing presses. Hence, there are eight printing presses at the print shop! Students now know there are eight printing presses at the workshop. This should help them confirm their solution paths in Exercise 2(b)!



b. Solve the problem by setting up an equation.

MP.2

Start by asking students to write down all variables for the quantities they see in the problem, giving an appropriate letter for each. While it does not matter if they work with the number of copies of cookbooks or the number of pages per cookbook, guide them to use the number of copies of cookbooks as you walk around your class. (Tell students not to worry about writing down too many letters—there is no need to write all of them.)

Be careful not to write on the board, printing presses $= x$. Such a statement abuses how the $=$ symbol should be used. Such statements do not make sense—printing presses aren't numbers! It is a good idea to require students to develop the habit of always properly labeling their variables using short descriptions (e.g., number of printing presses: x). This habit can help them significantly in setting up equations and solving algebra problems.

Number of printing presses: x. (This is what we wish to find!)

Number of copies of cookbooks printed by one printing press in one day: r

Number of copies of novels printed by one printing press in one day: $\frac{1}{2}r$ (Only half the number of novels can be produced each day.)

Number of copies in the cookbook job: c

Number of copies in the novel job: c

Next, ask students to write down important expressions using the variables above. There are many valid possible expressions they can write down that can be used to solve the problem. Here is one possible list:

Novels:

Number of copies of novels printed in one day by the entire print shop: $\frac{1}{2}rx$.

Number of copies of novels printed in one day by half of the entire print shop: $\frac{1}{4}rx$.

Total number of copies of novels printed: $\frac{1}{2}rx + \frac{1}{4}rx$, ***or*** $\frac{3}{4}rx$.

Cookbooks:

Number of copies of cookbooks printed in one day by half of the entire print shop: $\frac{1}{2}rx$.

Number of copies of cookbooks printed in two days by one printing press: $2r$.

Total number of copies of cookbooks printed: $\frac{1}{2}rx + 2r$.

Since the number of cookbooks and novels is the same, we can equate the expressions of the totals above to get:

$$\frac{3}{4}rx = \frac{1}{2}rx + 2r.$$

Since r is known not to be zero, we can divide both sides of the equation by r, and solve the resulting equation for x:

$$\frac{3}{4}x = \frac{1}{2}x + 2$$

$$\frac{3}{4}x - \frac{1}{2}x = \frac{1}{2}x + 2 - \frac{1}{2}x$$

$$\frac{1}{4}x = 2$$

$$4\left(\frac{1}{4}x\right) = 4(2)$$

$$x = 8$$

Check by substituting 8 ***back into the original equation. The left hand side is*** $\frac{3}{4}r(8) = 6r$, ***which is equal to the right hand side,*** $\frac{1}{2}r(8) + 2r = 4r + 2r = 6r$.

There are 8 ***printing presses in the print shop!***

MP.1 Because there are a number of ways to setup a correct equation, students will inevitably create several solution paths. If time permits, explore the different solution paths with students. Help them see that their solution is just as valid as the teacher's.

Closing (3 minutes)

Pose the following questions to your students:

- How can you tell that this problem was not about a real situation? That is, how can you tell it was completely made up?
  - *The clue that this is a made up problem is the following question: Why would the number of printing presses not be known in this situation?*
- In a real-life situation, not only would the number of printing presses be known, but the constant speed in which they print pages would be known, too. Plus, knowing the page quantity for each book would certainly be a factor in scheduling the job, as well as knowing exactly how many books each client wanted in total.
- The problem was made up to ask the following Exit Ticket question.

Exit Ticket (5 minutes)

Name ____ Date____________________

Lesson 25: Solving Problems in Two Ways—Rates and Algebra

Exit Ticket

Suppose we know that the print shop had 8 printing presses and each printing press runs at a constant speed of $5{,}000$ pages per hour for 6 hours a day.

1. Compute the total number of pages printed for the cookbook job, and the total number of pages printed for the novel job following the schedule and situation described in Exercise 2.

2. Describe a scenario where it would make sense for the job scheduler to schedule both jobs as described in Exercise 2.

BONUS

3. If the novel was 500 pages and the cookbook 250 pages, how many copies of each were printed?

Exit Ticket Sample Solutions

Suppose we know that the print shop had 8 printing presses, and each printing press runs at a constant speed of $5,000$ pages per hour for 6 hours a day.

1. Compute the total number of pages printed for the cookbook job and the total number of pages printed for the novel job following the schedule and situation described in Exercise 2(a).

   Number of pages printed by a printing press in 1 *day:* $30,000$
   Number of pages printed by 8 *printing presses in* 1 *day:* $240,000$
   Number of pages printed by 8 *printing presses in* 2 *days:* $480,000$
   Number of pages 1 *printing press prints in* 2 *days:* $60,000$
   Number of pages printed for both jobs: $60,000 + 480,000 = 540,000$.

   Ratio of number of pages of the novel to the number of pages of the cookbook: $2:1$.
   Therefore, $360,000$ *pages for the novel and* $180,000$ *for the cookbook.*

2. Describe a scenario where it would make sense for the job scheduler to schedule both jobs as described in Exercise 2(a).

   The client for the novel job wants the print job completed by midweek, whereas the cookbook does not need to be completed until the end of the week.

3. BONUS: if the novel was 250 pages and the cookbook 125 pages, how many copies of each were printed?

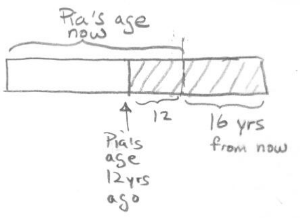
   $1,440$ *copies of each were printed.*

Problem Set Sample Solutions

1. Solve the following problems first using a tape diagram and then by setting up an equation. For each, give your opinion on which solution method was easier. Can you see the connection(s) between the two methods? What does each "unit" in the tape diagram stand for?

   a. 16 years from now, Pia's age will be twice her age 12 years ago. Find her present age.

      Tape:



1 unit = 12+16 = 28
28 + 12 = 40
Pia is 40 years old.

Equation:

Pia's age now: p years old.

$p+16 = 2(p-12)$

$p+16 = 2p-24$

$p+16+24 = 2p-24+24$

$p+40 = 2p$

$p+40-p = 2p-p$

$40 = p$

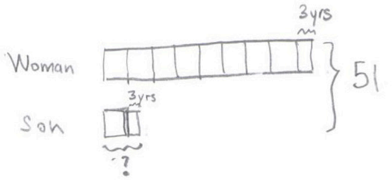
Check: $40+16 = 2(40-12)$

$56 = 56$ ✓

Pia is 40 years old.

b. **The total age of a woman and her son is 51 years. Three years ago, the woman was eight times as old as her son. How old is her son now?**

Tape:



9 units + 6 = 51

9 units = 45

1 unit = 5

5+3=8

Her son is 8yrs old.

Equation:

Son's age: s yrs old

Woman's age: $51-s$

Eqn: $51-s-3 = 8(s-3)$

Solve: $48-s = 8s-24$

$48-s+s = 8s-24+s$

$48 = 9s-24$

$48+24 = 9s-24+24$

$72 = 9s$

$8 = s$

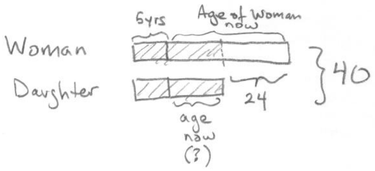
Check: If son is 8yrs old now, then the woman is 43yrs old.

Three yrs ago: woman - 40, son - 5 $\big\}$ ×8!

The son is 8years old.

c. **Five years from now, the sum of the ages of a woman and her daughter will be 40 years. The difference in their present age is 24 years. How old is her daughter now?**

Tape:



Shaded part is a unit.
2units + 24 = 40
2units = 16
1 unit = 8
8 − 5 = 3

The daughter is 3 years old.

Equation:

Daughter's age now: d yrs old
Woman's age now: d+24

Eqn: $(d+5)+(d+24+5)=40$

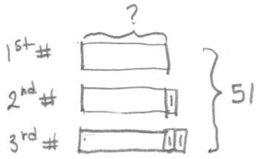
Solve: $2d+34=40$
$2d+34-34=40-34$
$2d=6$
$\frac{1}{2}(2d)=\frac{1}{2}(6)$
$d=3$

Check: Woman's age now: 27 yrs
In 5 yrs, woman is 32, daughter is 8 } sum is 40 ✓

Daughter is 3 years old.

d. **Find three consecutive integers such that their sum is 51.**

Tape:



3units + 3 = 51
3units = 48
1unit = 16

The consecutive numbers are 16, 17, 18.

Equation:

1st number : x
2nd number : $x+1$
3rd number : $x+2$

Eqn: $x + (x+1) + (x+2) = 51$

Soln: $3x + 3 = 51$
$3x + 3 - 3 = 51 - 3$
$3x = 48$
$\frac{1}{3}(3x) = \frac{1}{3}(48)$
$x = 16$

Check: $16 + 17 + 18 = 51$ ✓

The three consecutive numbers are 16, 17, 18.

2. **Solve the following problems by setting up an equation or inequality.**

   a. **If two numbers represented by $(2m + 1)$ and $(2m + 5)$ have a sum of 74, find m.**

Eqn: $(2m+1) + (2m+5) = 74$

Solve: $4m + 6 = 74$
$4m + 6 - 6 = 74 - 6$
$4m = 68$
$\frac{1}{4}(4m) = \frac{1}{4}(68)$
$m = 17$

Check: $2 \cdot 17 + 1 + 2 \cdot 17 + 5 = 68 + 6 = 74$ ✓

The number m is 17.

   b. **Find two consecutive even numbers such that the sum of the smaller number and twice the greater number is 100.**

1st number : x
2nd number : $x+2$

Eqn: $x + 2(x+2) = 100$

Solve: $x + 2x + 4 = 100$
$3x + 4 = 100$
$3x = 96$
$\frac{1}{3}(3x) = \frac{1}{3}(96)$
$x = 32$

Check: $32 + 2(34) = 32 + 68 = 100$ ✓

The numbers are 32 and 34.

c. **If 9 is subtracted from a number, and the result is multiplied by 19, the product is 171. Find the number.**

Number : n

Eqn: $19(n-9)=171$

Solve $\frac{1}{19}(19(n-9))=\frac{1}{19}(171)$

$n-9=9$

$n=18$

Check: $19(18-9)=19\cdot 9=171$ ✓

The number is 18.

d. **The product of two consecutive whole numbers is less than the sum of the square of the smaller number and 13.**

1st number: m
2nd number: $m+1$

Eqn: $m(m+1)<m^2+13$

Solve: $m^2+m<m^2+13$

$m^2+m-m^2<m^2+13-m^2$

$m<13$

Since m is a whole number: $0\leq m<13$

Check: $0\cdot 1<13$ ✓
$1\cdot 2<1^2+13$ ✓
$2\cdot 3<2^2+13$ ✓
$\vdots$
$12\cdot 13<12^2+13$ ✓
$13\cdot 14<13^2+13$ ⊗

The first whole number is in the set $\{0,1,2,\ldots,12\}$.

3. **The length, 18 meters, is the answer to the following question.**
"The length of a rectangle is three meters longer than its width. The area of the rectangle is 270 square meters. What is the length of the rectangle?"

Rework this problem: Write an equation using L as the length (in meters) of the rectangle that would lead to the solution of the problem. Check that the answer above is correct by substituting 18 for L in your equation.

Length (in meters): L

Width (in meters): $L-3$

Equation: $L(L-3)=270$

Check: $18(18-3)=18\cdot 15=270$

4. **Jim tells you he paid a total of $\$23,078.90$ for a car, and you would like to know the price of the car before sales tax so that you can compare the price of that model of car at various dealers. Find price of the car before sales tax if Jim bought the car in each of the following states:**

   a. **Arizona, where the sales tax is 6.6%.**

      Solving $x(1+0.066) = 23078.90$ results in $x = 21,650$. The car costs $\$21,650$.

   b. **New York, where the sales tax is 8.25%.**

      Solving $x(1+0.0825) = 23078.90$ results in $x = 21,320$. The car costs $\$21,320$.

   c. **A state where the sales tax is $s\%$.**

      Solving $x(1+\frac{s}{100}) = 23078.90$ results in $x = \frac{2,307,890}{100+s}$. For a sales tax of s%, the car costs $\frac{2,307,890}{100+s}$ dollars.

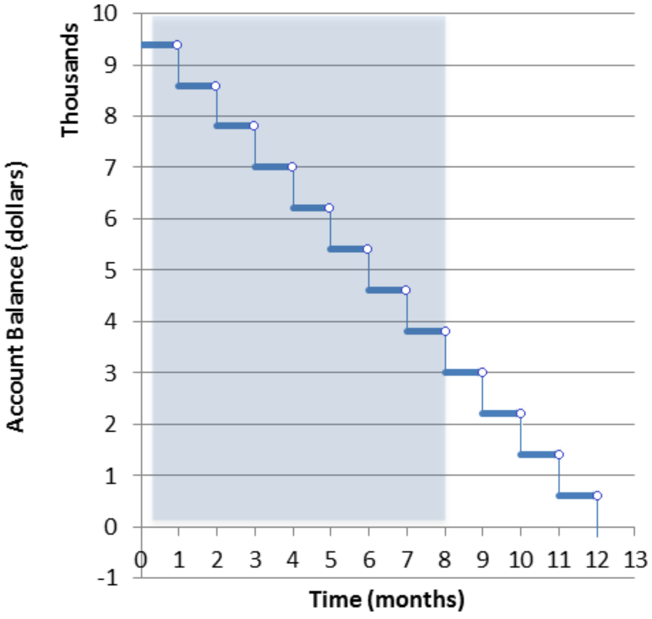
5. **A checking account is set up with an initial balance of $\$9,400$, and $\$800$ is removed from the account at the end of each month for rent (no other user transactions occur on the account).**

   a. **Write an inequality whose solutions are the months, m, in which the account balance is greater than $\$3,000$. Write the solution set to your equation by identifying all of the solutions.**

      For m a non-negative real number, m satisfies the inequality, $9400 - 800m > 3000$. For real numbers m, the solution set is $0 \leq m < 8$.

   b. **Make a graph of the balance in the account after m months and indicate on the plot the solutions to your inequality in part (a).**

      Students can create a step-function (like below), or simply plot the points (with no lines drawn). Drawing a straight line through the points, however, does not accurately reflect the information in the problem.



6. **Axel and his brother like to play tennis. About three months ago they decided to keep track of how many games they have each won. As of today, Axel has won 18 out of the 30 games against his brother.**

   a. **How many games would Axel have to win in a row in order to have a 75% winning record?**

      Solving $18 + n = 0.75(30 + n)$ results in $n = 18$. He would have to win 18 games.

   b. **How many games would Axel have to win in a row in order to have a 90% winning record?**

      Solving $18 + n = 0.90(30 + n)$ results in $n = 90$. He would have to win 90 games.

   c. **Is Axel ever able to reach a 100% winning record? Explain why or why not.**

      No. A 100% winning record would mean solving the equation $18 + n = 1(30 + n)$, which has no solutions.

   d. **Suppose that after reaching a winning record of 90% in part (b), Axel had a losing streak. How many games in a row would Axel have to lose in order to drop down to a winning record of 60% again?**

      Solving, $108 = 0.60(120 + n)$, results in $n = 60$. He would have to lose 60 games.

7. **Omar has $\$84$ and Calina has $\$12$. How much money must Omar give to Calina so that Calina will have three times as much as Omar?**

   a. **Solve the problem above by setting up an equation.**

      Solution 1:

      Amount Omar gives: x dollars.
      Amount Omar has after giving: $84 - x$.
      Amount Calina has after: $12 + x$.

      Equation: $12 + x = 3(84 - x)$.

      Solve:

$$\begin{aligned} 12 + x &= 252 - 3x \\ 12 + x + 3x &= 252 - 3x + 3x \\ 12 + 4x &= 252 \\ 12 + 4x - 12 &= 252 - 12 \\ 4x &= 240 \\ \frac{1}{4}(4x) &= \frac{1}{4}(240) \\ x &= 60 \end{aligned}$$

      Check: Omar now has $\$24$ and Calina has $\$72$, which is three times as much as Omar.

      Solution 2:

      A few students might notice that if Calina has three times as much as Omar afterwards, and if we let y be the amount that Omar has after, then $y + 3y = 96$, or $y = 24$. Now it is easy to find out how much Omar gave.

   b. **In your opinion, is this problem easier to solve using an equation or using a tape diagram? Why?**

      Most likely, students will say it is easier to solve with an equation because it is easier to set up and solve. (They may show an attempt at drawing a tape diagram, for example.) However, students who used the second solution may respond that the tape diagram is easier.

Lesson 26: Recursive Challenge Problem—The Double and Add 5 Game

Student Outcomes

- Students learn the meaning and notation of recursive sequences in a modeling setting.
- Following the modeling cycle, students investigate the *double and add 5* game in a simple case in order to understand the statement of the main problem.

Lesson Notes

The *double and add 5* game is *loosely* related to the Collatz conjecture—an *unsolved* conjecture in mathematics named after Lothar Collatz, who first proposed the problem in 1937. The conjecture includes a recurrence relation, *triple and add 1,* as part of the problem statement. A worthwhile activity for you and your class is to read about the conjecture online.

Students begin by playing the *Double and Add 5* game in a simple situation. Given a number, double it and add 5. The result of round 2 is the double of the result of Round 1 plus 5, and so on. The goal of the game is to find the smallest starting whole number, a_0, that produces a number 100 or greater in three rounds or fewer (Answer: $a_0 = 9$). Students are then exposed to the more difficult challenge of finding the smallest starting whole number that produces a number $1{,}000$ or greater in three rounds or fewer. To solve this problem, the notation of recursive sequences and recursive relations are explained, and students formalize the problem in terms of an equation, solve, interpret their answer, and validate (answer: $a_0 = 121$).

Classwork

MP.4

This challenging two-day modeling lesson (see page 61 of CCLS) about recursive sequences runs through the *problem, formulate, compute, interpret, validate, report* modeling cycle. This modeling activity involves playing a game and describing the mathematical process in the game using a recurrence relation in order to solve a harder version of the game. Please read through both lessons before planning out your class time.

Example 1 (7 minutes)

This activity describes the process so students can be given the problem statement. Introduce it by stating that you want to create an interesting sequence by *doubling and adding 5.*

Work through the table below with your students on the board to explain the meaning of the following:

- starting number,
- double and add 5,
- result of round one,
- result of round two, and so on.

Here is what the table looks like at the beginning:

	Number	Double and add 5	
starting number→	1	$1 \cdot 2 + 5 = 7$	← result of round 1
	7	$7 \cdot 2 + 5 = 19$	← result of round 2

and here is the completed table:

Example 1

Fill in the *doubling and adding* 5 below:

	Number	**Double and add 5**	
starting number→	**1**	$1 \cdot 2 + 5 = 7$	**← result of round 1**
	7	$7 \cdot 2 + 5 = 19$	**← result of round 2**
	19	$19 \cdot 2 + 5 = 43$	**← result of round 3**
	43	$43 \cdot 2 + 5 = 91$	**← result of round 4**
	91	$91 \cdot 2 + 5 = 187$	**← result of round 5**

Exercise 1 (5 minutes)

Have students complete the tables in Exercise 1. Walk around the classroom to ensure they are completing the tables correctly and understand the process.

Exercise 1

Complete the tables below for the given starting number.

Number	**Double and add 5**
2	$2 \cdot 2 + 5 = 9$
9	$9 \cdot 2 + 5 = 23$
23	$23 \cdot 2 + 5 = 51$

Number	**Double and add 5**
3	$3 \cdot 2 + 5 = 11$
11	$11 \cdot 2 + 5 = 27$
27	$27 \cdot 2 + 5 = 59$

Mathematical Modeling Exercise/Exercise 2 (15 minutes)

MP.1

(Problem statement of the modeling cycle.) State the following (starter) problem to students, and let them wrestle with it until they find a solution (or at least a strategy for finding the solution):

> **Exercise 2**
>
> **Given a starting number, double it and add 5 to get the result of Round 1. Double the result of Round 1 and add 5, and so on. The goal of the game is to find the smallest starting whole number that produces a result of 100 or greater in three rounds or fewer.**

Walk around the class, observe student work, and give advice, such as:

- Does starting with 10 produce a result of 100 or greater in Round 3?
  - *Yes.*
- Why will all numbers greater than 10 work? Can you find a smaller starting number that also works?
- Do you see any patterns in the tables you have already created?
  - *As the starting number increases by* 1*, the result of Round 3 increases by* 8*.*
- Yes, 9 works. Is it the smallest?
  - *Yes. When I start with* 8*, I need four rounds to get past* 100*.*

After 8 minutes, show (or have a student show) that 9 is the correct answer by showing the tables:

Number	***Double and add 5***
8	$8 \cdot 2 + 5 = 21$
21	$21 \cdot 2 + 5 = 47$
47	$47 \cdot 2 + 5 = 99$*, no!*

Number	***Double and add 5***
9	$9 \cdot 2 + 5 = 23$
23	$23 \cdot 2 + 5 = 51$
51	$51 \cdot 2 + 5 = 107$*, yes!*

MP.1

Invite students to share other methods for finding the answer. For example, some may have *worked the problem backwards*: If 100 is reached in three rounds, then $\frac{100-5}{2} = 47.5$ must have been reached in two rounds, and $\frac{47.5-5}{2} = 21.25$ must been reached after the first round, which means the starting number is greater than $\frac{21.25-5}{2} = 8.125$, or the whole number 9.

Tell students that the next goal is to solve the same problem, but find the smallest number that results in $1{,}000$ in three rounds or fewer:

> Given a starting number, double it and add 5 to get the result of Round 1. Double the result of Round 1 and add 5, and so on. The goal of the game is to find the smallest starting whole number that produces a result of $1{,}000$ or greater in three rounds or fewer.

This problem is not as easy as the starter problem to solve by guess-and-check. To solve this problem, guide students to *formulate* an equation. But first, you will need to explain how mathematicians create and describe recursive sequences.

Let a_1 be the number of the result of Round 1. We can also label the result of Round 2 as a_2, and so on. Ask, "How could we label the starting number?" Guide them to label the starting number as a_0. Then write an equation in terms of a_0 and a_1 in the table (that is still on the board) like this:

Number	Double and add 5	Equation
$a_0 = 5$	$5 \cdot 2 + 5 = 15$	$a_0 \cdot 2 + 5 = a_1$
$a_1 = 15$	$15 \cdot 2 + 5 = 35$	
$a_2 = 35$	$35 \cdot 2 + 5 = 75$	

Ask students to help you complete and extend the table as follows:

Number	Double and add 5	Equation
$a_0 = 5$	$5 \cdot 2 + 5 = 15$	$a_0 \cdot 2 + 5 = a_1$
$a_1 = 15$	$15 \cdot 2 + 5 = 35$	$a_1 \cdot 2 + 5 = a_2$
$a_2 = 35$	$35 \cdot 2 + 5 = 75$	$a_2 \cdot 2 + 5 = a_3$
$a_3 = 75$		
a_i		$a_i \cdot 2 + 5 = a_{i+1}$
a_{i+1}		

Highlight on the board that the ordered list of *terms* 5, 15, 35, 75,… can be described by an *initial value,* $a_0 = 5$, and a *recurrence relation*, $a_{i+1} = 2a_i + 5$, for $i \geq 0$. Written as follows:

$$\begin{cases} a_0 = 5 \\ a_{i+1} = 2a_i + 5, i \geq 0 \end{cases}$$

Tell them that this is an example of a *recursively-defined sequence,* or simply, a *recursive sequence.*

- Have students mentally use the recurrence relation to find the next term after 75. Is it the *double and add 5 rule?*

Ask:

- What other terms have we studied so far that are defined recursively?
    - *Algebraic expressions, polynomial expressions, monomials*

Teacher note: Terms that are defined recursively often use the term itself in the statement of the definition, but the definition of the term is not considered *circular*. Circularity does not arise in recursively defined terms because they always start with a well-defined set of *base* examples, and then the definition describes how to generate new examples of the term from those base examples, which, by reiterating further, can then be used to generate all other examples of the term. The base examples prevent the definition from being circular. For recursive sequences, the base example(s) is just the initial value(s). For algebraic expressions, the well-defined base examples are numerical symbols and variable symbols.

Exercise 3 (10 minutes)

Ask students:

Exercise 3

Using a generic initial value,a_0, and the recurrence relation, $a_{i+1} = 2a_i + 5$, for $i \geq 0$, find a formula for a_1, a_2, a_3, a_4 in terms of a_0.

Let students work individually or in pairs. Visit each group and ask questions that lead students to the following:

$a_1 = 2a_0 + 5$,
$a_2 = 2a_1 + 5 = 2(2a_0 + 5) + 5 = 4a_0 + 15$,
$a_3 = 2a_2 + 5 = 2(2 \cdot 2a_0 + 15) + 5 = 8a_0 + 35$,
$a_4 = 2a_3 + 5 = 2(2^3 \cdot a_0 + 35) + 5 = 16a_0 + 75$.

Closing (5 minutes)

Discuss the following definitions in the student materials:

Vocabulary

Sequence: A *sequence* can be thought of as an ordered list of elements. The elements of the list are called the *terms of the sequence.*

For example, (P, O, O, L) is a sequence that is different than (L, O, O, P). Usually the terms are *indexed* (and therefore ordered) by a subscript starting at either 0 or 1: $a_1, a_2, a_3, a_4, \ldots$. The "…" symbol indicates that the pattern described is regular, that is, the next term is a_5, and the next is a_6, and so on. In the first example, $a_1 = P$ is the first term, $a_2 = O$ is the second term, and so on. Both finite and infinite sequences exist everywhere in mathematics. For example, the infinite decimal expansion of $\frac{1}{3} = 0.333333333\ldots$ can be represented as the sequence, $(0.3, 0.33, 0.333, 0.3333, \ldots)$.

Recursive Sequence: An example of a *recursive sequence* is a sequence that is defined by (1) specifying the values of one or more initial terms and (2) having the property that the remaining terms satisfy a recurrence relation that describes the value of a term based upon an algebraic expression in numbers, previous terms, or the index of the term.

The sequence generated by initial term, $a_1 = 3$, and recurrence relation, $a_n = 3a_{n-1}$, is the sequence $(3, 9, 27, 81, 243, \ldots)$. Another example, given by the initial terms, $a_0 = 1$, $a_1 = 1$, and recurrence relation, $a_n = a_{n-1} + a_{n-2}$, generates the famed *Fibonacci sequence* $(1, 1, 2, 3, 5, \ldots)$.

Exit Ticket (3 minutes)

Name ____ Date____________________

Lesson 26: Recursive Challenge Problem—The Double and Add 5 Game

Exit Ticket

The following sequence was generated by an initial value a_0 and recurrence relation $a_{i+1} = 2a_i + 5$, for $i \geq 0$.

1. Fill in the blanks in the sequence:

   $(\_\_\_\_\_\_\_\_,\ 29,\ \_\_\_\_\_\_\_\_,\ \_\_\_\_\_\_\_\_,\ \_\_\_\_\_\_\_\_,\ 539,\ 1083).$

2. In the sequence above, what is a_0? What is a_5?

Exit Ticket Sample Solutions

The following sequence was generated by an initial value a_0 and recurrence relation $a_{i+1} = 2a_i + 5$, for $i \geq 0$.

1. **Fill in the blanks in the sequence:**

   $($ __12__, $29,$ __63__, __131__, __267__, $539, 1083)$

2. **In the sequence above, what is a_0? What is a_5?**

   $a_0 = 12$, $a_5 = 539$

Problem Set Sample Solutions

1. **Write down the first 5 terms of the recursive sequences defined by the initial values and recurrence relations below:**

   a. $a_0 = 0$ and $a_{i+1} = a_i + 1$, for $i \geq 0$,

   $(0, 1, 2, 3, 4)$

   b. $a_1 = 1$ and $a_{i+1} = a_i + 1$, for $i \geq 1$,

   $(1, 2, 3, 4, 5)$

   c. $a_1 = 2$ and $a_{i+1} = a_i + 2$, for $i \geq 1$,

   $(2, 4, 6, 8, 10)$

   d. $a_1 = 3$ and $a_{i+1} = a_i + 3$, for $i \geq 1$,

   $(3, 6, 9, 12, 15)$

   e. $a_1 = 2$ and $a_{i+1} = 2a_i$, for $i \geq 1$,

   $(2, 4, 8, 16, 32)$

   f. $a_1 = 3$ and $a_{i+1} = 3a_i$, for $i \geq 1$,

   $(3, 9, 27, 81, 243)$

   g. $a_1 = 4$ and $a_{i+1} = 4a_i$, for $i \geq 1$,

   $(4, 16, 64, 256, 1024)$

   h. $a_1 = 1$ and $a_{i+1} = (-1)a_i$, for $i \geq 1$,

   $(1, -1, 1, -1, 1)$

   i. $a_1 = 64$ and $a_{i+1} = \left(-\frac{1}{2}\right)a_i$, for $i \geq 1$,

   $(64, -32, 16, -8, 4)$

2. Look at the sequences you created in Problems 1(b) through 1(d). How would you define a recursive sequence that generates multiples of 31?

   $a_1 = 31$ and $a_{i+1} = a_i + 31$, for $i \geq 1$

3. Look at the sequences you created in problems 1(e) through 1(g). How would you define a recursive sequence that generates powers of 15?

   $a_1 = 15$ and $a_{i+1} = 15a_i$, for $i \geq 1$

4. The following recursive sequence was generated starting with an initial value of a_0, and the recurrence relation $a_{i+1} = 3a_i + 1$, for $i \geq 0$. Fill in the blanks of the sequence

   (__10__, __31__, 94, __283__, 850, __2551__)

5. For the recursive sequence generated by initial value, a_0, and recurrence relation, $a_{i+1} = a_i + 2$, for $i \geq 0$, find a formula for a_1, a_2, a_3, a_4 in terms of a_0. Describe in words what this sequence is generating.

   $a_1 = a_0 + 2,$
   $a_2 = a_0 + 4,$
   $a_3 = a_0 + 6,$
   $a_4 = a_0 + 8$

   It finds the next consecutive even or odd numbers after a_0, depending on whether a_0 is even or odd.

6. For the recursive sequence generated by initial value, a_0, and recurrence relation, $a_{i+1} = 3a_i + 1$, for $i \geq 0$, find a formula for a_1, a_2, a_3, a_4 in terms of a_0.

   $a_1 = 3 \cdot a_0 + 1,$
   $a_2 = 9a_0 + 4,$
   $a_3 = 27a_0 + 13,$
   $a_4 = 81a_0 + 40$

Lesson 27: Recursive Challenge Problem—The Double and Add 5 Game

Student Outcomes

- Students learn the meaning and notation of recursive sequences in a modeling setting.
- Students use recursive sequences to model and answer problems.
- Students create equations and inequalities to solve a modeling problem.
- Students represent constraints by equations and inequalities and interpret solutions as viable or non-viable options in a modeling context.

Lesson Notes

The *double and add 5* game is *loosely* related to the Collatz conjecture—an *unsolved* conjecture in mathematics named after Lothar Collatz, who first proposed the problem in 1937. The conjecture includes a recurrence relation, *triple and add 1,* as part of the problem statement. A worthwhile activity for you and your class is to read about the conjecture online.

Students begin by playing the *Double and Add* 5 game in a simple situation. Given a number, double it and add 5. The result of round two is the double of the result of round one, plus 5, and so on. The goal of the game is to find the smallest starting whole number, a_0, that produces a number 100 or greater in three rounds or fewer (answer: $a_0 = 9$). Students are then exposed to the more difficult challenge of finding the smallest starting whole number that produces a number $1{,}000$ or greater in three rounds or fewer (answer: $a_0 = 121$). To solve this problem, the notation of recursive sequences and recursive relations are explained, and students formalize the problem in terms of an equation, solve, interpret their answer, and validate.

Classwork

This challenging two-day modeling lesson (see page 61 of CCLS) about recursive sequences runs through the *problem, formulate, compute, interpret, validate, report* modeling cycle. This modeling activity involves playing a game and describing the mathematical process in the game using a recurrence relation in order to solve a more difficult version of the game. This part two lesson picks up where the last lesson left off—in this lesson students formulate, compute, interpret, validate, and report on their answers to the *Double and Add 5* game problem stated in the previous lesson.

Recall the statement of the problem from the last lesson for your students:

- Given a starting number, double it and add 5 to get the result of round one. Double the result of round one and add 5, and so on. The goal of the game is to find the smallest starting whole number that produces a result of $1{,}000$ or greater in three rounds or fewer.

Example 1 (10 minutes)

The repeat of this example from the previous lesson speaks to the value and importance of students doing this work. This time require students to work individually to complete the task. Visit students as needed and ask questions that lead students to the correct formulas.

Example 1

Review Exercise 3 from the previous lesson: Using a generic initial value, a_0, and the recurrence relation, $a_{i+1} = 2a_i + 5$, for $i \geq 0$, find a formula for a_1, a_2, a_3, a_4 in terms of a_0.

$a_1 = 2a_0 + 5$,
$a_2 = 2a_1 + 5 = 2(2a_0 + 5) + 5 = 2^2 \cdot a_0 + 15$,
$a_3 = 2a_2 + 5 = 2(2 \cdot 2a_0 + 15) + 5 = 2^3 \cdot a_0 + 35$,
$a_4 = 2a_3 + 5 = 2(2^3 \cdot a_0 + 35) + 5 = 2^4 \cdot a_0 + 75$.

Mathematical Modeling Exercise/Exercise 1 (15 minutes)

MP.2 & MP.4

(Formulation step of the modeling cycle) Ask students: Using one of the four formulas from Example 1, write an inequality that, if solved for a_0, will lead to finding the smallest starting whole number for the *Double and Add* 5 game that produces a result of 1,000 or greater in three rounds or fewer.

Exercise 1

Using one of the four formulas from Example 1, write an inequality that, if solved for a_0, will lead to finding the smallest starting whole number for the *double and add* 5 game that produces a result of 1,000 or greater in 3 rounds or fewer.

This exercise is loaded with phrases that students will need to interpret correctly in order to formulate an equation (*do not* expect this to be easy for them). Start with simple questions and build up:

- What does a_2 mean in terms of rounds?
  - *The result of round two*
- Write what the statement, "produce a result of 1,000 or greater in two rounds," means using a term of the sequence.
  - *The result of round two, a_2, must be greater than or equal to 1,000. Ask students to write the equation, $a_2 \geq 1000$, for that statement.*
- After replacing a_2 in the inequality, $a_2 \geq 1000$, with the expression in terms of a_0, what do the numbers a_0 that satisfy the inequality, $4a_0 + 15 \geq 1000$, mean?
  - *The numbers a_0 that satisfy the inequality are the starting numbers for the Double and Add 5 game that produce a result of 1,000 or greater in two rounds or fewer. The "or fewer" in the previous sentence is important and can be understood by thinking about the question, "Do we need two rounds to reach 1,000, starting with number 999? 800? 500?"*

Let students solve for a_0 in $4a_0 + 15 \geq 1000$, and let them find the smallest whole number a_0 for exactly two rounds *(Answer: 247)*. Then continue with your questioning:

- What inequality in terms of a_0 would you write down to find the smallest starting number for the *Double and Add* 5 game that produces a result of 1,000 or greater in three rounds or fewer?
  - $8a_0 + 35 \geq 1000$

Exercise 2 (10 minutes)

(Compute, interpret, validate steps of the modeling cycle) Tell students:

MP.2 & MP.3

Exercise 2

Solve the inequality derived in Exercise 1. Interpret your answer, and validate that it is the solution to the problem. That is, show that the whole number you found results in $1,000$ or greater in three rounds, but the previous whole number takes four rounds to reach $1,000$.

$$\begin{aligned} 8a_0 + 35 &\geq 1000 \\ 8a_0 + 35 - 35 &\geq 1000 - 35 \\ 8a_0 &\geq 965 \\ \frac{1}{8}(8a_0) &\geq \frac{1}{8}(965) \\ a_0 &\geq \frac{965}{8} \end{aligned}$$

Students should write or say something similar to the following response: I interpret $a_0 \geq \frac{965}{8}$ or $a_0 \geq 120.625$ as the set of all starting numbers that reach $1,000$ or greater in three rounds or fewer. Therefore, the smallest starting whole number is 121. To validate, I checked that starting with 121 results in $1,003$ after three rounds, whereas 120 results in 995 after three rounds.

Exercise 3 (5 minutes)

(This exercise cycles through the modeling cycle again.) Ask students:

Exercise 3

Find the smallest starting whole number for the *Double and Add* 5 game that produces a result of $1,000,000$ or greater in four rounds or fewer.

$$\begin{aligned} 16 \cdot a_0 + 75 &\geq 1,000,000 \\ 16a_0 + 75 - 75 &\geq 1,000,000 - 75 \\ 16a_0 &\geq 999,925 \\ \frac{1}{16}(16a_0) &\geq \frac{1}{16}(999,925) \\ a_0 &\geq \frac{999,925}{16} \end{aligned}$$

Students should write or say something similar to the following response: I interpreted $a_0 \geq \frac{999,925}{16}$ or $a_0 \geq 62,495.3125$ as the set of all starting numbers that reach $1,000,000$ or greater in four rounds or fewer. Therefore, the smallest starting whole number is $62,496$. To validate, I checked that starting with $62,496$ results in $1,000,011$ after four rounds, whereas $62,495$ results in $999,995$ after four rounds.

Lesson Summary

The formula, $a_n = 2^n(a_0 + 5) - 5$, describes the n^{th} term of the *double and add 5* game in terms of the starting number a_0 and n. Use this formula to find the smallest starting whole number for the *double and add 5* game that produces a result of $10,000,000$ or greater in 15 rounds or fewer.

Exit Ticket (5 minutes)

Use the Exit Ticket to have students report their findings (the report step of the modeling cycle).

Name ____ Date_____________________

Lesson 27: Recursive Challenge Problem—The Double and Add 5 Game

Exit Ticket

Write a *brief* report about the answers you found to the *Double and Add 5* game problems. Include justifications for why your starting numbers are correct.

Exit Ticket Sample Responses

MP.3

Write a *brief* report about the answers you found to the *Double and Add 5* game problems. Include justifications for why your starting numbers are correct.

Results for finding the smallest starting number in the Double and Add 5 *game:*

1. *Reaching 100 in three rounds or fewer: The starting number 9 results in 107 in round three. The starting number 8 results in 99 in round three, requiring another round to reach 100. Numbers 1–8 take more than three rounds to reach 100.*

2. *Reaching $1,000$ in three rounds or fewer: The starting number 121 results in $1,003$ in round three. The starting number 120 results in 995 in round three, requiring another round to reach $1,000$. All other whole numbers less than 120 take more than three rounds to reach $1,000$.*

3. *Reaching $1,000,000$ in four rounds or fewer: The starting number $62,496$ results in $1,000,011$ in round four. The starting number $62,495$ results in $999,995$ in round four, requiring another round to reach $1,000,000$. All other whole numbers less than $62,495$ take more than four rounds to reach $1,000,000$.*

Problem Set Sample Solutions

1. Your older sibling came home from college for the weekend and showed you the following sequences (from her homework) that she claimed were generated from initial values and recurrence relations. For each sequence, find an initial value and recurrence relation that describes the sequence. (Your sister showed you an answer to the first problem.)

a. $(0, 2, 4, 6, 8, 10, 12, 14, 16, \ldots)$

$a_1 = 0$ *and* $a_{i+1} = a_i + 2$ *for* $i \geq 1$

b. $(1, 3, 5, 7, 9, 11, 13, 15, 17, \ldots)$

$a_1 = 1$ *and* $a_{i+1} = a_i + 2$ *for* $i \geq 1$

c. $(14, 16, 18, 20, 22, 24, 26, \ldots)$

$a_1 = 14$ *and* $a_{i+1} = a_i + 2$ *for* $i \geq 1$

d. $(14, 21, 28, 35, 42, 49, \ldots)$

$a_1 = 14$ *and* $a_{i+1} = a_i + 7$ *for* $i \geq 1$

e. $(14, 7, 0, -7, -14, -21, -28, -35, \ldots)$

$a_1 = 14$ *and* $a_{i+1} = a_i - 7$ *for* $i \geq 1$

f. $(2, 4, 8, 16, 32, 64, 128, \ldots)$

$a_1 = 2$ *and* $a_{i+1} = 2a_i$ *for* $i \geq 1$

g. $(3, 6, 12, 24, 48, 96, \ldots)$

$a_1 = 3$ and $a_{i+1} = 2a_i$ for $i \geq 1$

h. $(1, 3, 9, 27, 81, 243, \ldots)$

$a_1 = 1$ and $a_{i+1} = 3a_i$ for $i \geq 1$

i. $(9, 27, 81, 243, \ldots)$

$a_1 = 9$ and $a_{i+1} = 3a_i$ for $i \geq 1$

2. **Answer the following questions about the recursive sequence generated by initial value, $a_1 = 4$, and recurrence relation, $a_{i+1} = 4a_i$ for $i \geq 1$.**

a. **Find a formula for a_1, a_2, a_3, a_4, a_5 in terms of powers of 4.**

$a_1 = 4^1$

$a_2 = 4^2$

$a_3 = 4^3$

$a_4 = 4^4$

$a_5 = 4^5$

b. **Your friend, Carl, says that he can describe the n^{th} term of the sequence using the formula, $a_n = 4^n$. Is Carl correct? Write one or two sentences using the recurrence relation to explain why or why not.**

Yes. The recurrence relation, $a_{i+1} = 4a_i$ for $i \geq 0$, means that the next term in the sequence is always 4 times larger than the current term, i.e., one more power of 4. Therefore, the n^{th} term will be n powers of 4, or 4^n.

3. **The expression, $2^n(a_0 + 5) - 5$, describes the n^{th} term of the *double and add 5* game in terms of the starting number a_0 and n. Verify that it does describe the n^{th} term by filling out the tables below for parts (b) through (e). (The first table is done for you.)**

a. **Table for $a_0 = 1$**

n	$2^n(a_0 + 5) - 5$
1	$2^1 \cdot 6 - 5 = 7$
2	$2^2 \cdot 6 - 5 = 19$
3	$2^3 \cdot 6 - 5 = 43$
4	$2^4 \cdot 6 - 5 = 91$

b. **Table for $a_0 = 8$**

n	$2^n(a_0 + 5) - 5$
1	$2^1 \cdot 13 - 5 = 21$
2	$2^2 \cdot 13 - 5 = 47$
3	$2^3 \cdot 13 - 5 = 99$
4	$2^4 \cdot 13 - 5 = 203$

c. **Table for $a_0 = 9$**

n	$2^n(a_0 + 5) - 5$
2	$2^2 \cdot 14 - 5 = 51$
3	$2^3 \cdot 14 - 5 = 107$

d. Table for $a_0 = 120$

n	$2^n(a_0+5)-5$
3	$2^3 \cdot 125 - 5 = 995$
4	$2^4 \cdot 125 - 5 = 1995$

e. Table for $a_0 = 121$

n	$2^n(a_0+5)-5$
2	$2^2 \cdot 126 - 5 = 499$
3	$2^3 \cdot 126 - 5 = 1003$

4. Bilbo Baggins stated to Samwise Gamgee, "Today, Sam, I will give you $\$1$. Every day thereafter for the next 14 days, I will take the previous day's amount, double it and add $\$5$, and give that new amount to you for that day."

   a. How much did Bilbo give Sam on day 15? (Hint: You don't have to compute each term.)

      $a_{15} = 2^{15}(1+5) - 5 = 196,603$. *Bilbo gave Sam* $\$196,603$ *on day* 15.

   b. Did Bilbo give Sam more than $\$350,000$ altogether?

      Yes. He gave $\$98,299$ *on day* 14*,* $\$49,147$*, on day* 13*,* $\$24,571$ *on day* 12*, and so on.*

5. The formula, $a_n = 2^{n-1}(a_0+5) - 5$, describes the n^{th} term of the *Double and Add 5* game in terms of the starting number a_0 and n. Use this formula to find the smallest starting whole number for the *Double and Add 5* game that produces a result of $10,000,000$ or greater in 15 rounds or fewer.

   Solving $2^{14}(a_0+5) - 5 \geq 10,000,000$ *for* a_0 *results in* $a_0 \geq 300.1759\ldots$

   Hence, 301 *is the smallest starting whole number that will reach* $10,000,000$ *in* 15 *rounds or fewer.*

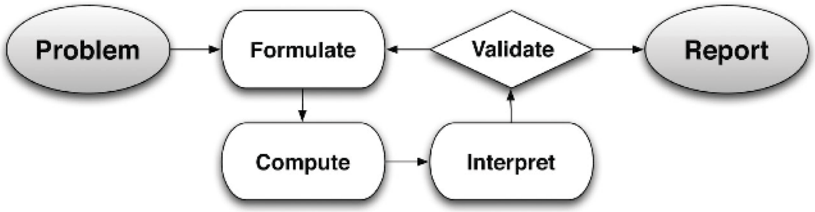
Lesson 28: Federal Income Tax

Student Outcomes

- Students create equations and inequalities in one variable and use them to solve problems.
- Students create equations in two or more variables to represent relationships between quantities and graph equations on coordinate axes with labels and scales.
- Students represent constraints by inequalities and interpret solutions as viable or non-viable options in a modeling context.

Lesson Notes

This real-life descriptive modeling lesson (see page 61 of the CCLS or page 71 of the CCSS) is about using inequalities and graphs to understand the progressive federal tax system. Like the last lesson, this lesson again runs through the *problem, formulate, compute, interpret, validate, report* modeling cycle, but unlike the difficult modeling lesson on the *Double and Add 5* game, more autonomy can be given to students in this lesson. You might want to include more discussion of the words and process used in the modeling cycle:



Materials

Please ensure that each student has a copy of the tax tables (on the next page) in their student materials or as a handout. Students will need a calculator and (your call) a spreadsheet program.

Classwork

Mathematical Modeling Exercise

Formulating the Problem (15 minutes)

Tell students: The federal income tax is not calculated by summing up all that an individual earns and then taking a fixed percentage of that income. Instead, the federal tax system is progressive. That means the more an individual makes, the greater the percentage of it is taxed. In this lesson, we will analyze our tax system, graph the federal income tax versus income, and use the graph to compute effective tax rates for families with different incomes.

Important Tax Tables for this Lesson

Exemption Deductions for Tax Year 2013

Exemption Class	Exemption Deduction
Single	$\$3,900$
Married	$\$7,800$
Married with 1 child	$\$11,700$
Married with 2 children	$\$15,600$
Married with 3 children	$\$19,500$

Standard Deductions Based Upon Filing Status for Tax Year 2013

Filing Status	Standard Deduction
Single	$\$6,100$
Married filing jointly	$\$12,200$

Federal Income Tax for Married Filing Jointly for Tax Year 2013

If taxable income is over--	But not over--	The tax is:	Plus the Marginal Rate	Of the amount over--
$\$0$	$\$17,850$	10%		$\$0$
$\$17,850$	$\$72,500$	$\$1,785.00$	15%	$\$17,850$
$\$72,500$	$\$146,400$	$\$9,982.50$	25%	$\$72,500$
$\$146,400$	$\$223,050$	$\$28,457.50$	28%	$\$146,400$
$\$223,050$	$\$398,350$	$\$49,919.50$	33%	$\$223,050$
$\$398,350$	$\$450,000$	$\$107,768.50$	35%	$\$398,350$
$\$450,000+$		$\$125,846.00$	39.6%	$\$450,000$

Taxable Income: The U.S. government considers the *income* of a family (or individual) to include the sum of any money earned from a husband's or wife's jobs, and money made from their personal businesses or investments. The taxes for a household (i.e., an individual or family) are not computed from the income; rather, they are computed from the household's taxable income. For many families, the household's *taxable income* is simply the household's income minus exemption deductions and minus standard deductions:

(taxable income) $=$ ***(income)*** $-$ ***(exemption deduction)*** $-$ ***(standard deduction)***

All of the problems we will model in this lesson will use this equation to find a family's taxable income. The only exception is if the family's taxable income is less than zero, in which case we will say that the family's taxable income is just $\$0$.

MP.2

Use this formula and the tables above to answer the following questions about taxable income:

Exercise 1

Find the taxable income of a single person with no kids, who has an income of $\$55,000$.

$55,000 - 3,900 - 6,100 = 45,000$. The family's taxable income is $\$45,000$.

Exercise 2

Find the taxable income of a married couple with two children, who have a combined income of $\$55,000$.

$55,000 - 15,600 - 12,200 = 27,200$. The family's taxable income is $\$27,200$.

Exercise 3

Find the taxable income of a married couple with one child, who have a combined income of $\$23,000$.

$23,000 - 11,700 - 12,200 = -900$. The family's taxable income is $\$0$.

Federal Income Tax and the Marginal Tax Rate: Below is an example of how to compute the federal income tax of a household using the Federal Income Tax table above.

Example 1

Compute the Federal Income Tax for the situation described in Exercise 1 (a single person with no kids making $\$55,000$).

From the answer in Exercise 1, the taxable income is $\$45,000$. Looking up $\$45,000$ in the tax table above, we see that $\$45,000$ corresponds to the second row because it is between $\$17,850$ and $\$72,500$:

If taxable income is over--	But not over--	The tax is:	Plus the Marginal Rate	Of the amount over--
$\$17,850$	$\$72,500$	$\$1,785.00$	15%	$\$17,850$

To calculate the tax, add $\$1,785$ plus 15% of the amount of $\$45,000$ that is over $\$17,850$. Since $45,000 - 17,850 = 27,150$, and 15% of $27,150$ is $\$4,072.50$, the total federal income tax on $\$45,000$ of taxable income is $\$5,857.50$.

Exercise 4

Compute the Federal Income Tax for a married couple with two children making $\$127,800$.

The taxable income is $\$127,800 - \$15,600 - \$12,200 = \$100,000$.

Looking up $\$100,000$ in the tax table, we see that $\$100,000$ corresponds to the third row because it is between $\$72,500$ and $\$146,000$:

If taxable income is over—	But not over—	The tax is:	Plus the Marginal Rate	Of the amount over—
$\$72,500$	$\$146,400$	$\$9,982.50$	25%	$\$72,500$

To calculate the tax, add $\$9982.50$ plus 25% of the amount over $\$72,500$. Since $100,000 - 72,500 = 27,500$, we take 25% of $27,500$ to get $\$6875$. Thus, the total federal income tax on $\$100,000$ of taxable income is $\$16,857.50$.

Taxpayers sometimes misunderstand *marginal tax* to mean: "If my taxable income is $\$100{,}000$, and my marginal tax rate is 25%, my federal income taxes are $\$25{,}000$." This statement is not true—they would not owe $\$25{,}000$ to the federal government. Instead, a marginal income tax charges a progressively higher tax rate for successively greater levels of income. Therefore, they would really owe:

- **10% on the first $\$17{,}850$, or $\$1{,}785$ in taxes for the interval from $\$0$ to $\$17{,}850$;**
- **15% on the next $\$54{,}650$, or $\$8{,}197.50$ in taxes for the interval from $\$17{,}850$ to $\$72{,}500$;**
- **25% on the last $\$27{,}500$, or $\$6{,}875.00$ in taxes for the interval from $\$72{,}500$ to $\$100{,}000$;**

for a total of $\$16{,}857.50$ of the $\$100{,}000$ of taxable income. Thus, their *effective federal income tax rate* is 16.8575%, not 25% as they claimed. Note that the tax table above incorporates the different intervals so that only one calculation needs to be made (the answer to this problem is the same as the answer in Exercise 5).

Statement, Formulation, and Analysis of Problem (15 minutes)

Students are now ready to formulate and create a graph of federal income taxes versus income.

Exercise 5

The creation of the table and the graph involves many of the ideas that students have been learning throughout this module. The hope here is that they can work through this problem on their own (or in groups of two) with *minimum help from you.* However, since these tax terms are new, you may need to walk around the room and help explain words like income, taxable income, exemption, standard deduction, and federal income tax (as well as marginal tax rate, filing status, and deduction).

MP.1 & MP.4

Exercise 5

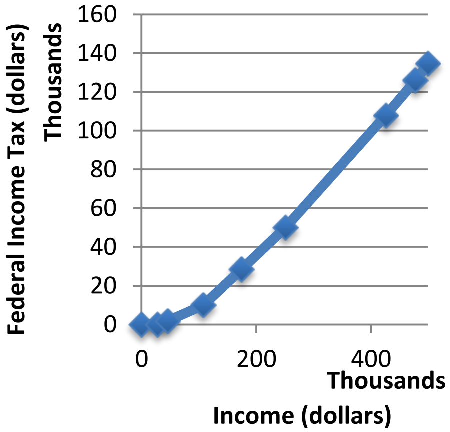
Create a table and a graph of federal income tax versus income for a married couple with two children between $\$0$ of income and $\$500{,}000$ of income.

The first step in creating the graph is to determine the equation for taxable income. A married couple with two children has a standard deduction of $\$12{,}200$ and an exemption deduction of $\$15{,}600$, for a total deduction of $\$27{,}800$. If we let the real number, TI, stand for the family's taxable income, and the real number, I, stand for the family's income, we get the following equation for taxable income:

$$TI = \begin{cases} I - 27800 & I \geq 27800 \\ 0 & 0 \leq I < 27800 \end{cases}$$

Help students to create the following table using the intervals in the federal income tax table:

Income ($)	Taxable Income ($)	Federal Income Tax ($)
0	0	0
27,800	0	0
45,650	17,850	1785.00
100,300	72,500	9,982.50
174,200	146,400	28,457.50
250,850	223,050	49,919.50
426,150	398,350	107,768.50
477,800	450,000	125,846.00
500,000	472,200	134,637.20



Use column 1 and column 3 in this table to create the graph on the right.

Exercise 6

Interpret and validate the graph you created in Exercise 5. Does your graph provide an approximate value for the federal income tax you calculated in Exercise 4?

Yes. The graph suggests that the federal income tax for a married couple with two children with an income of $\$127,800$ ***should be between*** $\$15,000$ ***and*** $\$20,000$***, which is close to the actual amount of*** $\$16,857.50$.

Exercise 7

Use the table you created in Exercise 5 to report on the effective federal income tax rate for a married couple with two children, who makes:

a. $\$27,800$
b. $\$45,650$
c. $\$500,000$

Note to teacher: Answer the first two incomes with your class, using them as examples to explain the meaning of effective federal income tax rate. Let them find the effective federal income tax rate for $500,000 as an exercise.

The effective federal income tax rate is found by writing the number (federal income tax)/(income) as a percentage. The effective federal income tax rate for a married couple with two children making:

a. $\$27,800$ ***is*** 0%,
b. $\$45,650$ ***is about*** 4%,
c. $\$500,000$ ***is about*** 27%.

Exit Ticket (5 minutes)

Adjust this problem based upon the remaining class time.

Name ____ Date____________________

Lesson 28: Federal Income Tax

Exit Ticket

A famous movie actress made $\$10$ million last year. She is married and has no children, and her husband does not earn any income. Assume that she computes her taxable income using the following formula:

$$(\text{taxable income}) = (\text{income}) - (\text{exemptions}) - (\text{standard deductions})$$

Find her taxable income, her federal income tax, and her effective federal income tax rate.

Exit Ticket Sample Solutions

A famous movie actress made $\$10$ million last year. She is married and has no children, and her husband does not earn any income. Assume that she computes her taxable income using the following formula:

(taxable income) = (income) − (exemptions) − (standard deductions)

Find her taxable income, her federal income tax, and her effective federal income tax rate.

Taxable Income: $\$10,000,000 - \$7,800 - \$12,200 \ = \ \$9,980,000$

Federal Income Tax:

If taxable income is over—	*But not over—*	*The tax is:*	*Plus the Marginal Rate*	*Of the amount over—*
$\$450,000+$		$\$125,846.00$	39.6%	$\$450,000$

39.6% of $9,980,000 - \ 450,000$, or 39.6% of $9,530,000$, is $\$3,773,880$ in tax over the first $\$450,000$. Add the tax of $\$125,846$ on the first $\$450,000$ of taxable income, to get a total federal income tax of $\$3,899,726$.

Effective Federal Income Tax Rate: $\frac{3{,}899{,}726}{10{,}000{,}000} \cdot 100 \approx 39\%$

Problem Set Sample Solutions

Use the formula and tax tables given in the lesson to perform all computations.

1. **Find the taxable income of a married couple with two children, who have a combined income of $\$75,000$.**

   $\$47,200$

2. **Find the taxable income of a single person with no children, who has an income of $\$37,000$.**

   $\$27,000$

3. **Find the taxable income of a married couple with three children, who have a combined income of $\$62,000$.**

   $\$30,300$

4. **Find the federal income tax of a married couple with two children, who have a combined income of $\$100,000$.**

   $\$9,937.50$

5. **Find the federal income tax of a married couple with three children, who have a combined income of $\$300,000$.**

   $\$64,852$

6. **Find the effective federal income tax rate of a married couple with no children, who have a combined income of $\$34,000$.**

   4.1%

7. **Find the effective federal income tax rate of a married couple with one child who have a combined income of $\$250,000$.**

   $\sim 20.4\%$

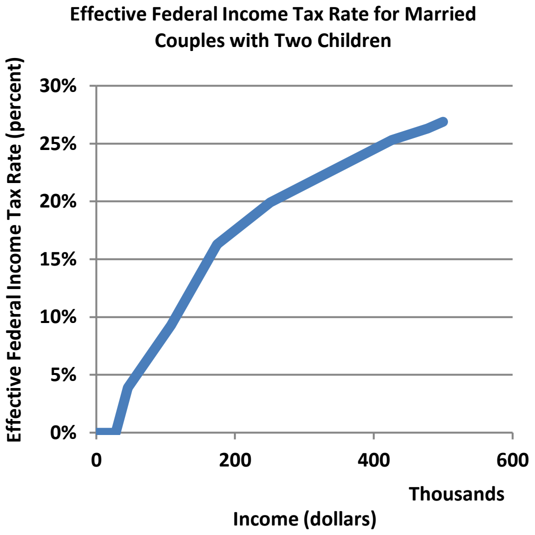
8. **The latest report on median household (family) income in the United States is $\$50,502$ per year. Compute the federal income tax and effective federal income tax rate for a married couple with three children, who have a combined income of $\$50,502$.**

   Federal income tax: $\$1,927.80$

   Effective federal income tax rate: $\sim 3.8\%$

9. **Extend the table you created in Exercise 6 by adding a column called, "Effective federal income tax rate." Compute the effective federal income tax rate to the nearest tenth for each row of the table, and create a graph that shows effective federal income tax rate versus income using the table.**

Income	*Taxable Income*	*Federal Income Tax*	*Effective Federal Income Tax Rate*
0	0	0	0%
$27,800$	0	0	0%
$45,650$	$17,850$	$1,785.00$	3.9%
$107,550$	$72,500$	$9,982.50$	9.3%
$174,200$	$146,400$	$28,457.50$	16.3%
$250,850$	$223,050$	$49,919.50$	19.9%
$426,150$	$398,350$	$107,768.50$	25.3%
$477,800$	$450,000$	$125,846.00$	26.3%
$500,000$	$472,200$	$134,637.20$	26.9%



Name ____ Date ____________________

1. Solve the following equations for x. Write your answer in set notation.

   a. $3x - 5 = 16$

   b. $3(x + 3) - 5 = 16$

   c. $3(2x - 3) - 5 = 16$

   d. $6(x + 3) - 10 = 32$

   e. Which two equations above have the same solution set? Write a sentence explaining how the properties of equality can be used to determine the pair without having to find the solution set for each.

2. Let c and d be real numbers.

   a. If $c = 42 + d$ is true, then which is greater: c or d, or are you not able to tell? Explain how you know your choice is correct.

   b. If $c = 42 - d$ is true, then which is greater: c or d, or are you not able to tell? Explain how you know your choice is correct.

3. If $a < 0$ and $c > b$, circle the expression that is greater:

$$a(b - c) \qquad \text{or} \qquad a(c - b)$$

Use the properties of inequalities to explain your choice.

4. Solve for x in each of the equations or inequalities below, and name the property and/or properties used:

   a. $\frac{3}{4}x = 9$

   b. $10 + 3x = 5x$

   c. $a + x = b$

   d. $cx = d$

   e. $\frac{1}{2}x - g < m$

   f. $q + 5x = 7x - r$

g. $\frac{3}{4}(x+2) = 6(x+12)$

h. $3(5-5x) > 5x$

5. The equation $3x + 4 = 5x - 4$ has the solution set $\{4\}$.

   a. Explain why the equation $(3x+4) + 4 = (5x-4) + 4$ also has the solution set $\{4\}$.

b. In part (a), the expression $(3x + 4) + 4$ is equivalent to the expression $3x + 8$. What is the definition of equivalent expressions? Why does changing an expression on one side of an equation to an equivalent expression leave the solution set unchanged?

c. When we square both sides of the original equation, we get the following new equation:

$$(3x + 4)^2 = (5x - 4)^2.$$

Show that 4 is still a solution to the new equation. Show that 0 is also a solution to the new equation but is not a solution to the original equation. Write a sentence that describes how the solution set to an equation may change when both sides of the equation are squared.

d. When we replace x by x^2 in the original equation, we get the following new equation:

$$3x^2 + 4 = 5x^2 - 4.$$

Use the fact that the solution set to the original equation is $\{4\}$ to find the solution set to this new equation.

6. The Zonda Information and Telephone Company (ZI&T) calculates a customer's total monthly cell phone charge using the formula,

$$C = (b + rm)(1 + t),$$

where C is the total cell phone charge, b is a basic monthly fee, r is the rate per minute, m is the number of minutes used that month, and t is the tax rate.

Solve for m, the number of minutes the customer used that month.

7. Students and adults purchased tickets for a recent basketball playoff game. All tickets were sold at the ticket booth—season passes, discounts, etc. were not allowed.

   Student tickets cost $\$5$ each, and adult tickets cost $\$10$ each. A total of $\$4{,}500$ was collected. 700 tickets were sold.

   a. Write a system of equations that can be used to find the number of student tickets, s, and the number of adult tickets, a, that were sold at the playoff game.

   b. Assuming that the number of students and adults attending would not change, how much more money could have been collected at the playoff game if the ticket booth charged students and adults the same price of $\$10$ per ticket?

   c. Assuming that the number of students and adults attending would not change, how much more money could have been collected at the playoff game if the student price was kept at $\$5$ per ticket and adults were charged $\$15$ per ticket instead of $\$10$?

8. Alexus is modeling the growth of bacteria for an experiment in science. She assumes that there are B bacteria in a Petri dish at 12:00 noon. In reality, each bacterium in the Petri dish subdivides into two new bacteria *approximately* every 20 minutes. However, for the purposes of the model, Alexus assumes that each bacterium subdivides into two new bacteria *exactly* every 20 minutes.

   a. Create a table that shows the total number of bacteria in the Petri dish at $\frac{1}{3}$ hour intervals for 2 hours starting with time 0 to represent 12:00 noon.

   b. Write an equation that describes the relationship between total number of bacteria T and time h in hours, assuming there are B bacteria in the Petri dish at $h = 0$.

   c. If Alexus starts with 100 bacteria in the Petri dish, draw a graph that displays the total number of bacteria with respect to time from 12:00 noon ($h = 0$) to 4:00 p.m. ($h = 4$). Label points on your graph at time $h = 0, 1, 2, 3, 4$.

d. For her experiment, Alexus plans to add an anti-bacterial chemical to the Petri dish at 4:00 p.m. that is supposed to kill 99.9% of the bacteria instantaneously. If she started with 100 bacteria at 12:00 noon, how many live bacteria might Alexus expect to find in the Petri dish right after she adds the anti-bacterial chemical?

9. Jack is 27 years older than Susan. In 5 years, he will be 4 times as old as she is.

   a. Find the present ages of Jack and Susan.

   b. What calculations would you do to check if your answer is correct?

10.

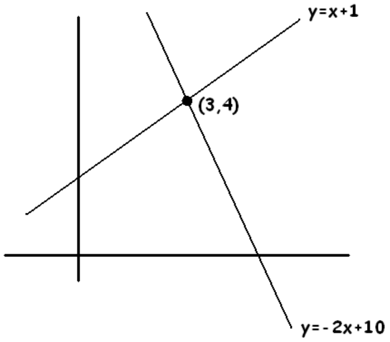
a. Find the product: $(x^2 - x + 1)(2x^2 + 3x + 2)$.

b. Use the results of part (a) to factor 21,112 as a product of a two-digit number and a three-digit number.

11. Consider the following system of equations with the solution $x = 3$, $y = 4$.

Equation A1: $y = x + 1$

Equation A2: $y = -2x + 10$



a. Write a unique system of two linear equations with the same solution set. This time make both linear equations have positive slope.

Equation B1: ________________

Equation B2: ________________

b. The following system of equations was obtained from the original system by adding a multiple of equation A2 to equation A1.

Equation C1: $y = x + 1$

Equation C2: $3y = -3x + 21$

What multiple of A2 was added to A1?

c. What is the solution to the system given in part (b)?

d. For any real number m, the line $y = m(x - 3) + 4$ passes through the point $(3,4)$.

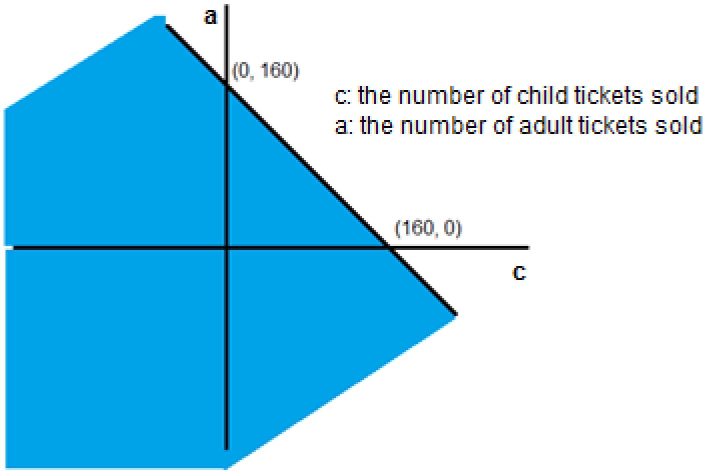
Is it certain, then, that the system of equations

Equation D1: $y = x + 1$

Equation D2: $y = m(x - 3) + 4$

has only the solution $x = 3$, $y = 4$? Explain.

12. The local theater in Jamie's home town has a maximum capacity of 160 people. Jamie shared with Venus the following graph and said that the shaded region represented all the possible combinations of adult and child tickets that could be sold for one show.



a. Venus objected and said there was more than one reason that Jamie's thinking was flawed. What reasons could Venus be thinking of?

b. Use equations, inequalities, graphs, and/or words to describe for Jamie the set of all possible combinations of adult and child tickets that could be sold for one show.

c. The theater charges $\$9$ for each adult ticket and $\$6$ for each child ticket. The theater sold 144 tickets for the first showing of the new release. The total money collected from ticket sales for that show was $\$1{,}164$. Write a system of equations that could be used to find the number of child tickets and the number of adult tickets sold, and solve the system algebraically. Summarize your findings using the context of the problem.

A Progression Toward Mastery					
Assessment Task Item		**STEP 1** **Missing or incorrect answer and little evidence of reasoning or application of mathematics to solve the problem.**	**STEP 2** **Missing or incorrect answer but evidence of some reasoning or application of mathematics to solve the problem.**	**STEP 3** **A correct answer with some evidence of reasoning or application of mathematics to solve the problem, <u>OR</u> an incorrect answer with substantial evidence of solid reasoning or application of mathematics to solve the problem.**	**STEP 4** **A correct answer supported by substantial evidence of solid reasoning or application of mathematics to solve the problem.**
1	**a–d** A-REI.A.1	Student gave a short incorrect answer or left the question blank.	Student showed at least one correct step, but the solution was incorrect.	Student solved the equation correctly (every step that was shown was correct) but did not express the answer as a solution set.	Student solved the equation correctly (every step that was shown was correct) and expressed the answer as a solution set.
	e A-SSE.A.1b A-REI.B.3	Student did not answer or answered incorrectly with something other than (b) and (d).	Student answered (b) and (d) but did not demonstrate solid reasoning in the explanation.	Student answered (b) and (d) but made minor misstatements in the explanation.	Student answered (b) and (d) and articulated solid reasoning in the explanation.
2	**a** A-CED.A.3	Student responded incorrectly or left the question blank.	Student responded correctly that (c) must be greater but did not use solid reasoning to explain the answer.	Student responded correctly that (c) must be greater but gave an incomplete or slightly incorrect explanation of why.	Student responded correctly that (c) must be greater and supported the statement with solid, well-expressed reasoning.
	b A-CED.A.3	Student responded incorrectly or left the question blank.	Student responded correctly that there is no way to tell but did not use solid reasoning to explain the answer.	Student responded correctly that there is no way to tell but gave an incomplete or slightly incorrect explanation of why.	Student responded correctly that there is no way to tell and supported the statement with solid, well-expressed reasoning.

3	**A-SSE.A.1b**	Student responded incorrectly or left the question blank.	Student responded correctly by circling the expression on the left but did not use solid reasoning to explain the answer.	Student responded correctly by circling the expression on the left but gave limited explanation or did not use the properties of inequality in the explanation.	Student responded correctly by circling the expression on the left and gave a complete explanation that used the properties of inequality.
4	**a–h** **A-REI.A.1** **A-REI.B.3**	Student answered incorrectly with no correct steps shown.	Student answered incorrectly but had one or more correct steps.	Student answered correctly but did not correctly identify the property or properties used.	Student answered correctly and correctly identified the property or properties used.
5	**a** **A-REI.A.1**	Student did not answer or demonstrated incorrect reasoning throughout.	Student demonstrated only limited reasoning.	Student demonstrated solid reasoning but fell short of a complete answer or made a minor misstatement in the answer.	Student answer was complete and demonstrated solid reasoning throughout.
	b **A-REI.A.1**	Student did not answer or did not demonstrate understanding of what the question was asking.	Student made more than one misstatement in the definition.	Student provided a mostly correct definition with a minor misstatement.	Student answered completely and used a correct definition without error or mis-statement.
	c **A-REI.A.1**	Student made mistakes in both verifications and demonstrated incorrect reasoning or left the question blank.	Student conducted both verifications but fell short of articulating reasoning to answer the question.	Student conducted both verifications and articulated valid reasoning to answer the question but made a minor error in the verification or a minor misstatement in the explanation.	Student conducted both verifications without error and articulated valid reasoning to answer the question.
	d **A-REI.A.1**	Student answered incorrectly or did not answer.	Student identified one or both solutions but was unable to convey how the solutions could be found using the fact that 4 is a solution to the original equation.	Student identified only one solution correctly but articulated the reasoning of using the solution to the original equation to find the solution to the new equation.	Student identified both solutions correctly and articulated the reasoning of using the solution to the original equation to find the solution to the new equation.

6	**A-CED.A.4**	Student did not answer or showed no evidence of reasoning.	Student made more than one error in the solution process but showed some evidence of reasoning.	Student answer showed valid steps but with one minor error.	Student answered correctly.
7	**a–c** **A-CED.A.3**	Student was unable to answer any portion correctly.	Student answered one part correctly or showed some evidence of reasoning in more than one part.	Student showed solid evidence of reasoning in every part but may have made minor errors.	Student answered every part correctly and demonstrated and expressed valid reasoning throughout.
8	**a** **A-CED.A.2**	Student provided no table or a table with multiple incorrect entries.	Student provided a data table that was incomplete or had more than one minor error.	Student provided a data table that was complete but may have had one error or slightly inaccurate headings.	Student provided a data table that was complete and correct with correct headings.
	b **A-CED.A.2**	Student provided no equation or an equation that did not represent exponential growth.	Student provided an incorrect equation but one that modeled exponential growth.	Student provided a correct answer in the form of $T = B(2)^{3h}$.	Student provided a correct answer in the form of $T = B8^{h}$ or in more than one form, such as $T = B(2)^{3h}$ and $T = B8^{h}$.
	c **A-CED.A.2**	Student provided no graph or a grossly inaccurate graph.	Student provided a graph with an inaccurate shape but provided some evidence of reasoning in labeling the axes and/or data points.	Student created a graph with correct general shape but may have left off or made an error on one or two axes or data points.	Student created a complete graph with correctly labeled axes and correctly labeled data points (or a data table) showing the values for $h = 0, 1, 2, 3, 4$.
	d **A-CED.A.2**	Student provided no answer or an incorrect answer with no evidence of reasoning in arriving at the answer.	Student provided limited evidence of reasoning and an incorrect answer.	Student answered that 409.6 bacteria would be alive.	Student answered that 410, or about 410, bacteria would be alive.
9	**a** **A-CED.A.1**	Student wrote incorrect equations or did not provide equations.	Student answers were incorrect, but at least one of the equations was correct. Student may have made a gross error in the solution, made more	Both equations were correct, but student made a minor mistake in finding the solution.	Both equations were correct and student solved them correctly to arrive at the answer that Jack is 31 and Susan is 4.

			than one minor error in the solution process, or may have had one of the two equations incorrect.		
	b **A-REI.B.3**	Student did not answer or gave a completely incorrect answer.	Student articulated only one of the calculations correctly.	Student articulated the two calculations but with a minor mis-statement in one of the descriptions.	Student articulated both calculations correctly.
10	**a–b** **A-APR.A.1**	Student work was blank or demonstrated no understanding of multiplication of polynomials, nor how to apply part (a) to arrive at an answer for part (b).	Student made more than one error in the multiplication but demonstrated some understanding of multiplication of polynomials. Student may not have been able to garner or apply information from part (a) to use in answering part (b) correctly.	Student demonstrated the ability to multiply the polynomials (expressing the product as a sum of monomials with like terms combined) and to apply the structure from part (a) to solve part (b). There may have been minor errors.	Student demonstrated the ability to multiply the polynomials (expressing the product as a sum of monomials with like terms combined) and to apply the structure from part (a) to solve part (b) as $91(232)$.
11	**a** **A-REI.C.6**	Student was unable to demonstrate the understanding that two equations with $(3, 4)$ as a solution are needed.	Student provided two equations that have $(3, 4)$ as a solution (or attempted to provide such equations) but made one or more errors. Student may have provided an equation with a negative slope.	Student showed one minor error in the answer but attempted to provide two equations both containing $(3, 4)$ as a solution and both with positive slope.	Student provided two equations both containing $(3, 4)$ as a solution and both with positive slope.
	b **A-REI.C.6**	Student was unable to identify the multiple correctly.	Student identified the multiple as 3.	N/A	Student correctly identified the multiple as 2.
	c **A-REI.C.6**	Student was unable to demonstrate even a partial understanding of how to find the solution to the system.	Student showed some reasoning required to find the solution but made multiple errors.	Student made a minor error in finding the solution point.	Student successfully identified the solution point as $(3, 4)$.

	d **A-REI.C.5** **A-REI.C.6** **A-REI.D.10**	Student was unable to answer or to support the answer with any solid reasoning.	Student concluded yes or no but was only able to express limited reasoning in support of the answer.	Student correctly explained that all the systems would have the solution point $(3, 4)$ but incorrectly assumed this is true for all cases of m.	Student correctly explained that while in most cases this is true, if $m = 1$, the two lines are coinciding lines, resulting in a solution set consisting of all the points on the line.
12	**a** **MP.2** **A-REI.D.12**	Student was unable to articulate any sound reasons.	Student was only able to articulate one sound reason.	Student provided two sound reasons but made minor errors in the expression of reasoning.	Student was able to articulate at least two valid reasons. Valid reasons include the following: the graph assumes x could be less than zero, the graph assumes y could be less than zero, the graph assumes a and b could be non-whole numbers, the graph assumes 160 children could attend with no adults.
	b **A-CED.A.2** **A-REI.D.10** **A-REI.D.12**	Student was unable to communicate a relevant requirement of the solution set.	Student provided a verbal description that lacked precision and accuracy but demonstrated some reasoning about the solution within the context of the problem.	Student made minor errors in communicating the idea that both (a) and (b) must be whole numbers whose sum is less than or equal to 160.	Student communicated effectively that both (a) and (b) must be whole numbers whose sum is less than or equal to 160.
	c **A-CED.A.2** **A-REI.C.6**	Student was unable to demonstrate any substantive understanding in how to create the equations and solve the system of equations.	Student made multiple errors in the equations and/or solving process but demonstrated some understanding of how to create equations to represent a context and/or solve the system of equations.	Student made minor errors in the equations but solved the system accurately, or the student created the correct equations but made a minor error in solving the system of equations.	Student correctly wrote the equations to represent the system. Student solved the system accurately and summarized by defining or describing the values of the variable in the context of the problem (i.e., that there were 100 adult tickets and 44 child tickets sold.

Name ____________________________ Date ______________

1. Solve the following equations for x. Write your answer in set notation.

   a. $3x - 5 = 16$

   $3x = 21$ Solution set: $\{7\}$
   $x = 7$

   b. $3(x + 3) - 5 = 16$

   $3x + 9 - 5 = 16$ Solution set: $\{4\}$
   $3x = 12$
   $x = 4$

   c. $3(2x - 3) - 5 = 16$

   $6x - 9 - 5 = 16$ Solution set: $\{5\}$
   $6x - 14 = 16$
   $6x = 30$
   $x = 5$

   d. $6(x + 3) - 10 = 32$

   $6x + 18 - 10 = 32$ Solution set: $\{4\}$
   $6x = 24$
   $x = 4$

2. Which two equations above have the same solution set? Write a sentence explaining how the properties of equality can be used to determine the pair without having to find the solution set for each.

   Problems (b) and (d) have the same solution set. The expressions on each side of the equal sign for (d) are twice those for (b). So, if (left side) = (right side) is true for only some x-values, then 2(left side) = 2(right side) will be true for exactly the same x-values. Or simply, applying the multiplicative property of equality does not change the solution set.

2. Let c and d be real numbers.

   a. If $c = 42 + d$ is true, then which is greater: c or d or are you not able to tell? Explain how you know your choice is correct.

   c must be greater because c is always 42 more than d

   b. If $c = 42 - d$ is true, then which is greater: c or d or are you not able to tell? Explain how you know your choice is correct.

   There is no way to tell. We only know that the sum of c and d is 42. If d were 10, c would be 32 and, therefore, greater than d. But if d were 40, c would be 2 and, therefore, less than d.

3. If $a < 0$ and $c > b$, circle the expression that is greater:

   [✓ $a(b-c)$] or $a(c-b)$

   Use the properties of inequalities to explain your choice.

   Since $c > b$,
   It follows that $0 > b - c$
   And since $a < 0$, a is negative
   And the product of two negatives will be a positive.

   Since $c > b$,
   it follows that $c - b > 0$.
   so $(c - b)$ is positive. And since a is negative, the product of
   $a \cdot (c - b) < a \cdot (b - c)$.

4. Solve for x in each of the equations or inequalities below and name the property and/or properties used:

a. $\frac{3}{4}x = 9$

$x = 9 \cdot \left(\frac{4}{3}\right)$ Multiplication Property of Equality
$x = 12$

b. $10 + 3x = 5x$

$10 = 2x$ Addition Property of Equality
$5 = x$ Multiplication Property of Equality

c. $a + x = b$

$x = b - a$ Addition Property of Equality

d. $cx = d$

$x = \frac{d}{c},\ c \neq 0$ Multiplication Property of Equality

e. $\frac{1}{2}x - g < m$

$\frac{1}{2}x < m + g$ Addition Property of Equality
$x < 2 \cdot (m + g)$ Multiplication Property of Equality

f. $q + 5x = 7x - r$

$q + r = 2x$ Addition Property of Equality
$\frac{(q+r)}{2} = x$ Multiplication Property of Equality

g. $\frac{3}{4}(x+2) = 6(x+12)$

$3\cdot(x+2)=24\cdot(x+12)$	*Multiplication Property of Equality*
$3x+6=24x+288$	*Distributive Property*
$-\frac{282}{21}=x$	*Addition Property of Equality and Multiplication*
$-\frac{94}{7}=x$	*Property of Equality*
$-\frac{94}{7}=x$	

h. $3(5 - 5x) > 5x$

$15-15x>5x$	*Distributive Property*
$15>20x$	*Addition Property of Inequality*
$\frac{3}{4}>x$	*Multiplication Property of Equality*

5. The equation, $3x + 4 = 5x - 4$, has the solution set $\{4\}$.

   a. Explain why the equation, $(3x + 4) + 4 = (5x - 4) + 4$, also has the solution set $\{4\}$.

   Since the new equation can be created by applying the additive property of equality, the solution set does not change.

   OR

   Each side of this equation is 4 more than the sides of the original equation. Whatever value(s) make $3x + 4 = 5x - 4$ true would also make 4 more than $3x + 4$ equal to 4 more than $5x - 4$.

   b. In part (a), the expression $(3x + 4) + 4$ is equivalent to the expression $3x + 8$. What is the definition of equivalent algebraic expressions? Describe why changing an expression on one side of an equation to an equivalent expression leaves the solution set unchanged?

   Algebraic expressions are equivalent if (possibly repeated) use of the distributive, associative, and commutative properties, and/or the properties of rational exponents can be applied to one expression to convert it to the other expression.
   When two expressions are equivalent, assigning the same value to x in both expressions will give an equivalent numerical expression, which then evaluates to the same number. Therefore, changing the expression to something equivalent will not change the truth value of the equation once values are assigned to x.

c. When we square both sides of the original equation, we get the following new equation:

$$(3x+4)^2 = (5x-4)^2.$$

Show that 4 is still a solution to the new equation. Show that 0 is also a solution to the new equation but is not a solution to the original equation. Write a sentence that describes how the solution set to an equation may change when both sides of the equation are squared.

$(3 \cdot 4 + 4)^2 = (5 \cdot 4 - 4)^2$ gives $16^2 = 16^6$, which is true.
$(3 \cdot 0 + 4)^2 = (5 \cdot 0 - 4)^2$ gives $4^2 = (-4)^2$, which is true.
But, $(3 \cdot 0 + 4) = (5 \cdot 0 - 4)$ gives $4 = -4$, which is false.
When both sides are squared, you might introduce new numbers to the solution set because statements like $4 = -4$ are false, but statements like $4^2 = (-4)^2$ are true.

d. When we replace x by x^2 in the original equation, we get the following new equation:

$$3x^2 + 4 = 5x^2 - 4.$$

Use the fact that the solution set to the original equation is {4} to find the solution set to this new equation.

Since the original equation $3x + 4 = 5x - 4$ was true when $x = 4$, the new equation $3x^2 + 4 = 5x^2 - 4$ should be true when $x^2 = 4$. And, $x^2 = 4$ when $x = 2$, so the solution set to the new equation is $\{-2, 2\}$.

6. The Zonda Information and Telephone Company (ZI&T) calculates a customer's total monthly cell phone charge using the formula,

$$C = (b + rm)(1 + t),$$

where C is the total cell phone charge, b is a basic monthly fee, r is the rate per minute, m is the number of minutes used that month, and t is the tax rate.

Solve for m, the number of minutes the customer used that month.

$$C = b + bt + rm + rmt$$
$$C - b - bt = m \cdot (r + rt)$$
$$\frac{C - b - bt}{r + rt} = m \qquad t \neq -1$$
$$r \neq 0$$

7. Students and adults purchased tickets for a recent basketball playoff game. All tickets were sold at the ticket booth—season passes, discounts, etc. were not allowed.

   Student tickets cost $\$5$ each, and adult tickets cost $\$10$ each. A total of $\$4{,}500$ was collected. 700 tickets were sold.

   a. Write a system of equations that can be used to find the number of student tickets, s, and the number of adult tickets, a, that were sold at the playoff game.

$$\begin{cases} 5s + 10a = 4500 \\ s + a = 700 \end{cases}$$

   b. Assuming that the number of students and adults attending would not change, how much more money could have been collected at the playoff game if the ticket booth charged students and adults the same price of $\$10$ per ticket?

$700 \times \$10 = \$7000$

$\$7000 - \$4500 = \$2500$ more

   c. Assuming that the number of students and adults attending would not change, how much more money could have been collected at the playoff game if the student price was kept at $\$5$ per ticket and adults were charged $\$15$ per ticket instead of $\$10$?

First solve for a & s

$$\begin{aligned} 5s + 10a &= 4500 \\ -5s - 5a &= -3500 \\ 5a &= 1000 \\ a &= 200 \\ s &= 500 \end{aligned}$$

$\$5 \cdot (500) + \$15 \cdot (200) = \$5500$

$\$1000$ more

OR

$\$5$ more per adult ticket $(200 \cdot \$5 = \$1000$ more$)$

8. Alexus is modeling the growth of bacteria for an experiement in science. She assumes that there are B bacteria in a Petri Dish at 12:00 noon. In reality, each bacterium in the Petri dish subdivides into two new bacteria *approximately* every 20 minutes. However, for the purposes of the model, Alexus assumes that each bacterium subdivides into two new bacteria *exactly* every 20 minutes.

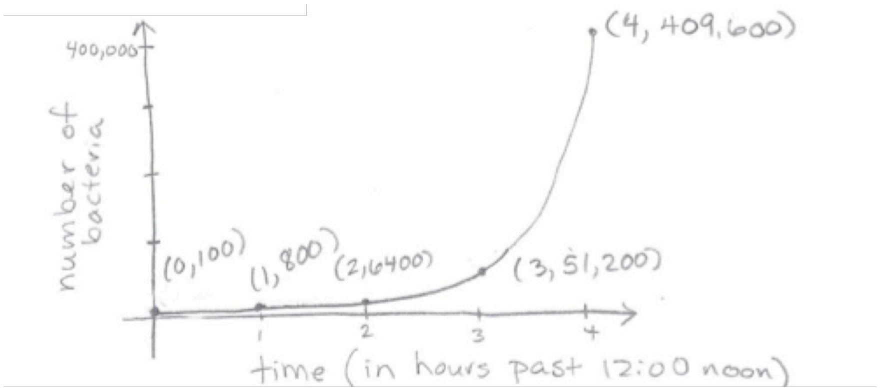
a. Create a table that shows the total number of bacteria in the Petri dish at $\frac{1}{3}$ hour intervals for 2 hours starting with time 0 to represent 12:00 noon.

Time	Number of Bacteria
0	B
1/3 hour	2B
2/3 hour	4B
1 hour	8B
1 1/3 hour	16B
1 2/3 hour	32B
2 hour	64B

b. Write an equation that describes the relationship between total number of bacteria *T* and time *h* in hours, assuming there are B bacteria in the Petri dish at *h*=0.

$$T = B \cdot (2)^{3h} \text{ or } T = B \cdot 8^{h}$$

c. If Alexus starts with 100 bacteria in the Petri dish, draw a graph that displays the total number of bacteria with respect to time from 12:00 noon ($h = 0$) to 4:00 p.m. ($h = 4$). Label points on your graph at time $h = 0,1,2,3,4$.



d. For her experiment, Alexus plans to add an anti-bacterial chemical to the Petri dish at 4:00 p.m. that is supposed to kill 99.9% of the bacteria instantaneously. If she started with 100 bacteria at 12:00 noon, how many live bacteria might Alexus expect to find in the Petri dish right after she adds the anti-bacterial chemical?

$$(1 - 0.999) \cdot 409{,}600 = 409.6$$

about 410 live bacteria

9. Jack is 27 years older than Susan. In 5 years time he will be 4 times as old as she is.

a. Find the present ages of Jack and Susan.

$$\begin{cases} J = S + 27 \\ J + 5 = 4 \cdot (S + 5) \end{cases}$$

$$\begin{aligned} S + 27 + 5 &= 4S + 20 \\ S + 32 &= 4S + 20 \\ 12 &= 3S \\ S &= 4 \end{aligned}$$

$$\begin{aligned} J &= 4 + 27 \\ J &= 31 \end{aligned}$$

Jack is 31 and Susan is 4.

b. What calculations would you do to check if your answer is correct?

Is Jack's age – Susan's age = 27?

Add 5 years to Jack's and Susan's ages and see if that makes Jack 4 times as old as Susan.

10.

a. Find the product: $(x^2 - x + 1)(2x^2 + 3x + 2)$

$$2x^4 + 3x^3 + 2x^2 - 2x^3 - 3x^2 - 2x + 2x^2 + 3x + 2$$
$$2x^4 + x^3 + x^2 + x + 2$$

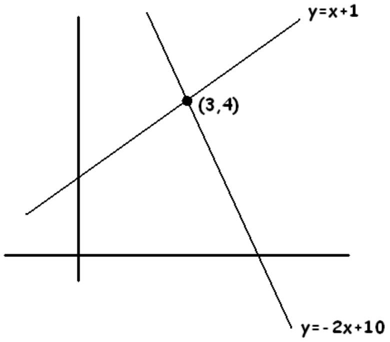
b. Use the results of part (a) to factor 21,112 as a product of a two-digit number and a three-digit number.

$$(100 - 10 + 1) \cdot (200 + 30 + 2)$$
$$(91) \cdot (232)$$

11. Consider the following system of equations with the solution $x = 3, y = 4$.

Equation A1: $y = x + 1$

Equation A2: $y = -2x + 10$



a. Write a unique system of two linear equations with the same solution set. This time make both linear equations have positive slope.

Equation B1: $y = \frac{4}{3}x$

Equation B2: $y = x + 1$

b. The following system of equations was obtained from the original system by adding a multiple of equation A2 to equation A1.

Equation C1: $y = x + 1$

Equation C2: $3y = -3x + 21$

What multiple of A2 was added to A1?

2 times A2 was added to A1.

c. What is the solution to the system given in part (b)?

(3,4)

d. For any real number m, the line $y = m(x - 3) + 4$ passes through the point $(3,4)$.

Is it certain, then, that the system of equations:

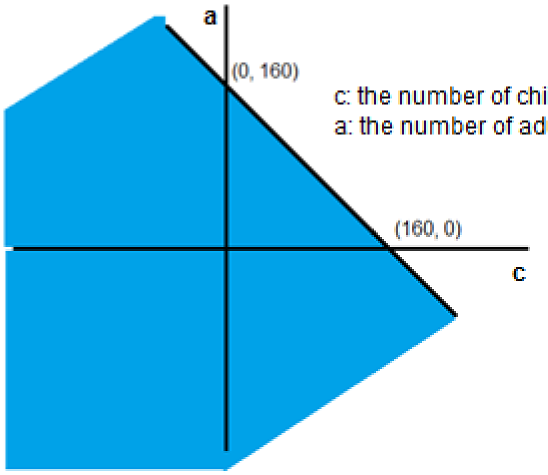
Equation D1: $y = x + 1$

Equation D2: $y = m(x - 3) + 4$

has only the solution $x = 3$, $y = 4$? Explain.

No. If m = 1, then the two lines have the same slope. Since both lines pass through the point (3,4), and the lines are parallel; therefore, they coincide. There are infinite solutions. The solution set is all the points on the line. Any other non-zero value of m would create a system with the only solution of (3,4).

12. The local theater in Jamie's home town has a maximum capacity of 160 people. Jamie shared with Venus the following graph and said that the shaded region represented all the possible combinations of adult and child tickets that could be sold for one show.



c: the number of child tickets sold
a: the number of adult tickets sold

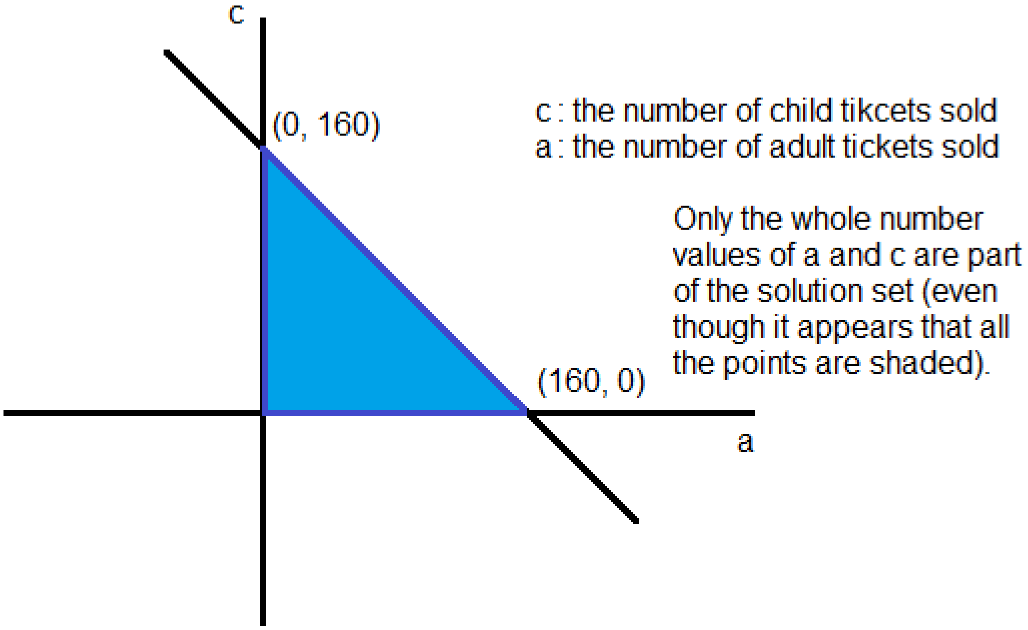
a. Venus objected and said there was more than one reason that Jamie's thinking was flawed. What reasons could Venus be thinking of?

1. The graph implies that the number of tickets sold could be a fractional amount, but really it only makes sense to sell whole number tickets. x and y must be whole numbers.

2. The graph also shows that negative ticket amounts could be sold which does not make sense.

b. Use equations, inequalities, graphs, and/or words to describe for Jamie the set of all possible combinations of adult and child tickets that could be sold for one show.

The system would be $\begin{cases} a + c \le 160 \\ a \ge 0 \\ c \ge 0 \end{cases}$ where a and c are whole numbers.



c. The theater charges \$9 for each adult ticket and \$6 for each child ticket. The theater sold 144 tickets for the first showing of the new release. The total money collected from ticket sales for that show was \$1,164. Write a system of equations that could be used to find the number of child tickets and the number of adult tickets sold, and solve the system algebraically. Summarize your findings using the context of the problem.

a: the number of adult tickets sold (must be a whole number)

c: the number of child tickets sold (must be a whole number)

$$\begin{cases} 9a + 6c = 1164 \\ a + c = 144 \end{cases}$$

$$\begin{aligned} 9a + 6c &= 1164 \\ -6a - 6c &= -864 \\ 3a &= 300 \\ a = 100,& \quad c = 44 \end{aligned}$$

In all, 100 adult tickets and 44 child tickets were sold.